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Space Physics and Global Geophysics
— Supplementary Material —

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Spherical harmonics

Solutions of the Laplace equation $\nabla^2\Phi = 0$ in spherical coordinates are called *spherical harmonics*. Since the Earth as a planet is approximately spherical, they are the most important function family in global geophysics.

If the Laplace operator is written in spherical coordinates (r, ϑ, λ), then

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\lambda^2} \;=\; 0\;.$$

This is a partial differential equation (PDE) of elliptic type which can be uniquely solved if suitable boundary conditions are given. We seek solutions through separation of variables which yields three ordinary differential equations (ODEs).

Separation of variables, step 1

We first write the solution Φ (which is called a *solid* spherical harmonic) as a product of a radial function $R = R(r)$ and a second function that depends only on the polar angle ϑ and the azimuth λ:

$$\Phi(r, \vartheta, \lambda) \;=\; R(r)\cdot\Psi(\vartheta, \lambda)\;.$$

Inserting this ansatz into the Laplace equation and multiplying by r^2/Φ yields

$$\frac{r^2\nabla^2\Phi}{\Phi} \;=\; \underbrace{\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)}_{=\text{const}=n(n+1)} + \underbrace{\frac{1}{\Psi\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Psi}{\partial\vartheta}\right) + \frac{1}{\Psi\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\lambda^2}}_{=-n(n+1)} \;=\; 0\;.$$

Note that the first underbraced expression depends only on the radial coordinate r, and the second one only shows angular dependences. This implies that each expression must be a constant which is written as $\pm n(n+1)$ for later convenience. The equation for the radial function R reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) \;=\; n(n+1)\,R$$

which can be solved by the ansatz $R \propto r^\alpha$ to yield two independent solutions r^n and $r^{-(n+1)}$. This yields the general *radial function of degree* n:

$$\boxed{R(r) \;=\; Ar^n + Br^{-(n+1)}}$$

(A and B are constants). The differential equation for the angular function reads

$$\boxed{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Psi}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\lambda^2} + n(n+1)\Psi \;=\; 0}$$

Solutions of this PDE are called *surface spherical harmonics* $\Psi_n^m(\vartheta,\lambda)$.

Separation of variables, step 2

The angular function Ψ is now assumed to be a product of two functions $\Theta = \Theta(\vartheta)$ and $\Lambda = \Lambda(\lambda)$:

$$\Psi(\vartheta,\lambda) \;=\; \Theta(\vartheta)\cdot\Lambda(\lambda)\ .$$

Inserting this ansatz into the PDE for the surface spherical harmonics and multiplying by $\sin^2\vartheta/\Psi$ yields

$$\underbrace{\frac{\sin\vartheta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\frac{\mathrm{d}\Theta}{\mathrm{d}\vartheta}\right) + n(n+1)\sin^2\vartheta}_{=\text{const}=m^2} + \underbrace{\frac{1}{\Lambda}\frac{\mathrm{d}^2\Lambda}{\mathrm{d}\lambda^2}}_{=-m^2} \;=\; 0\ .$$

Since the first term only depends on the polar angle ϑ and the second one only on the azimuth λ, we can conclude by the same argument as before that each expression must be a constant that we write as $\pm m^2$ for notational convenience. The azimuthal function $\Lambda(\lambda)$ satisfies the equation

$$\frac{\mathrm{d}^2\Lambda}{\mathrm{d}\lambda^2} + m^2\Lambda \;=\; 0$$

which has the general solution

$$\boxed{\Lambda(\lambda) \;=\; C\cos m\lambda + D\sin m\lambda}$$

(C and D are constants). Since we did not exclude $m^2 < 0$, the parameter m could in general be imaginary and in such a case we would effectively deal with hyperbolic functions. In fact, due to the uniqueness requirement $\Lambda(\lambda) \;=\; \Lambda(\lambda+2\pi)$ and $\Lambda'(\lambda) \;=\; \Lambda'(\lambda+2\pi)$ the number m must be a real integer and, without loss of generality, we can even assume that m is non-negative which implies that $m = 0, 1, 2, \ldots$

The ODE for $\Theta = \Theta(\vartheta)$ is more complicated than the previous ones:

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\frac{\mathrm{d}\Theta}{\mathrm{d}\vartheta}\right) + \left\{n(n+1)\sin^2\vartheta - m^2\right\}\Theta \;=\; 0\ .$$

Substituting $\mu = \cos\vartheta$, $-1 \le \mu \le 1$, and $M(\mu) = \Theta(\vartheta)$ we finally obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}M}{\mathrm{d}\mu}\right] + \left\{n(n+1)\sin^2\vartheta - \frac{m^2}{1-\mu^2}\right\}M \;=\; 0}$$

The solutions of this ODE are called *Legendre polynomials* if the azimuthal wavenumber $m = 0$, and *associated Legendre functions* if $m \neq 0$. The parameter n is called the *degree* of the spherical harmonic, and m is the *order*. It can be shown that n takes only non-negative integer values (just as m), and that $m \leq n$.

Legendre polynomials and associated Legendre functions

Legendre functions can be written in several different ways. Perhaps the most popular approach (although not the most practical one) is *Rodriguez' formula*:

$$\boxed{P_n(\mu) \;=\; \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left(\mu^2 - 1\right)^n}$$

The associated Legendre functions can be computed from the Legendre polynomials using

$$\boxed{P_n^m(\mu) \;=\; p_n^m (1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_n(\mu)}$$

Here p_n^m denotes a normalization factor:

- The straightforward choice
$$p_n^m \;=\; 1$$
is frequently used, e.g., in geodesy and quantum mechanics. It is sometimes referred to as the *full normalization* (motivated by the fact that the resulting functions P_n^m satisfy $\int_0^1 \mathrm{d}\mu [P_n^m(\mu)]^2$) although this can be misleading if surface integrals over the whole sphere are considered.
- The so-called *Schmidt normalization* is $p_n^0 = 1$ (i.e., $P_n \equiv P_n^0$) and
$$p_n^m \;=\; \sqrt{\frac{2(n-m)!}{(n+m)!}}\;,\; m \neq 0 \;.$$
This convention is standard in geophysics and geomagnetism, and it will also be used in this lecture.

There are further normalization conventions which all have their merits and shortcomings. For details see the subsection at the end of this document.

The first five Legendre polynomials are

$$\begin{aligned}
P_0 &= 1\;, \\
P_1 &= \mu \;=\; \cos\vartheta\;, \\
P_2 &= \frac{1}{2}(3\mu^2 - 1) \;=\; \frac{1}{2}(3\cos^2\vartheta - 1)\;, \\
P_3 &= \frac{1}{2}(5\mu^3 - 3\mu)\;, \\
P_4 &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)\;.
\end{aligned}$$

Associated Legendre functions (in Schmidt normalization) of degrees $n = 1$ and $n = 2$ are

$$\begin{aligned} P_1^1 &= (1-\mu^2)^{1/2} \;=\; \sin\vartheta \;, \\ P_2^1 &= \sqrt{3}(1-\mu^2)^{1/2}\mu \;=\; \sqrt{3}\sin\vartheta\cos\vartheta \;, \\ P_2^2 &= \frac{\sqrt{3}}{2}(1-\mu^2) \;=\; \frac{\sqrt{3}}{2}\sin^2\vartheta \;. \end{aligned}$$

Legendre functions have a number of useful properties.

- $P_n(\mu)$ is a polynomial of degree n with n zeroes in $]-1,1[$.
- $P_n^m(\mu)$ has $(n-m)$ zeroes in $]-1,1[$.
- The *generating function* of the Legendre polynomials is

$$\boxed{\frac{1}{\sqrt{1-2\mu h+h^2}} \;=\; \sum_{n=0}^{\infty} h^n P_n(\mu) \;,\; |h|<1}$$

- $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for all $n = 0, 1, 2, \ldots$
- *Orthogonality*:

$$\int_{-1}^{+1} P_k(\mu)P_n(\mu)\,\mathrm{d}\mu \;=\; \frac{2}{2n+1}\delta_{nk} \;=\; \begin{cases} \frac{2}{2n+1} & , \quad n = k \;, \\ 0 & , \quad n \neq k \;. \end{cases}$$

- *Recursion formula*:

$$\boxed{(n+1)P_{n+1} \;=\; (2n+1)\mu P_n - nP_{n-1}}$$

- *Addition theorem*: Let ϑ and λ denote the co-latitude and the azimuth of a point P, and $\tilde{\vartheta}$ and $\tilde{\lambda}$ the corresponding angles of a point $\tilde{P}$. Both points lie on the unit sphere, and χ denotes the angle between P and $\tilde{P}$. Then

$$\cos\chi \;=\; \cos\vartheta\cos\tilde{\vartheta} + \sin\vartheta\sin\tilde{\vartheta}\cos(\lambda-\tilde{\lambda})$$

  and

$$\boxed{P_n(\cos\chi) \;=\; \sum_{m=0}^{n} P_n^m(\cos\vartheta)P_n^m(\cos\tilde{\vartheta})\cos m(\lambda-\tilde{\lambda})}$$

Elementary surface spherical harmonics

The *elementary surface spherical harmonics* are given by

$$\Psi_n^{m\sigma}(\vartheta,\lambda) \;=\; P_n^m(\cos\vartheta)\cdot\left\{\begin{array}{lll} \cos m\lambda & , & \sigma = c \\ \sin m\lambda & , & \sigma = s \end{array}\right\}$$

for $0 \leq m \leq n$. Note that $\Psi_n^{0s} \equiv 0$.

These functions are orthogonal and complete in the following sense.

- *Orthogonality*:

$$\int_\Omega \Psi_n^{m\sigma} \Psi_{n'}^{m'\sigma'} \mathrm{d}\Omega \;=\; \begin{cases} \frac{4\pi}{2n+1} & : \quad n=n',\; n=n',\; \sigma=\sigma' = \begin{cases} s,c & : \quad m \neq 0 \;, \\ c & : \quad m=0 \end{cases} \\ 0 & : \quad \text{else} \;. \end{cases}$$

  Here $\int_\Omega \ldots \mathrm{d}\Omega$ denotes integration over the unit sphere Ω (spherical surface), and $\mathrm{d}\Omega = \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\lambda$.

- *Completeness*: The family of surface spherical harmonics is *complete* on the unit sphere Ω in the sense that every function $f(\vartheta, \lambda)$ can be represented as a convergent series of elementary surface spherical harmonics, i.e.,

$$f(\vartheta,\lambda) \;=\; \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=c,s} A_n^{m\sigma} \Psi_n^{m\sigma}(\vartheta,\lambda)$$

  with the coefficients

$$A_n^{m\sigma} \;=\; \frac{2n+1}{4\pi} \int_\Omega f(\vartheta,\lambda)\, \Psi_n^{m\sigma}(\vartheta,\lambda)\, \mathrm{d}\Omega \;.$$

  This result is important for the solution of boundary value problems in spherical geometry.

Terminology: the functions $\Psi_n^{m\sigma}$ are called

- *zonal* surface spherical harmonics if $m = 0$,
- *sectorial* surface spherical harmonics if $m = n$,
- *tesseral* surface spherical harmonics if $0 \neq m \neq n$.

Normalization conventions

The different normalization conventions for Legendre functions can be characterized through the factors K_n and K_n^m in

$$\begin{aligned} m=0 \;:\quad P_n(\mu) \;&=\; K_n \cdot \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}\mu^n} (\mu^2-1)^n \quad , \quad P_n^0 \equiv P_n \;, \\ m \neq 0 \;:\quad P_n^m(\mu) \;&=\; K_n^m \cdot (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu) \;. \end{aligned}$$

(1) No specific normalization: $K_n = K_n^m = 1$ (mathematics, physics, geodesy). In this case we obtain

$$\int_{-1}^{+1} [P_n(\mu)]^2 \, \mathrm{d}\mu \;=\; \int_{-1}^{+1} [P_n^m(\mu)]^2 \, \mathrm{d}\mu \;=\; \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \;.$$

(2) "Full/complete normalization I":

$$K_n = \sqrt{\frac{2n+1}{2}} \quad , \quad K_n^m = \sqrt{\frac{2n+1}{2}} \sqrt{\frac{(n-m)!}{(n+m)!}} \ .$$

This kind of normalization is useful if the Legendre functions are used as orthogonal functions on the interval $[-1, 1]$ and not as part of the surface spherical harmonics. One obtains

$$\int_{-1}^{+1} [P_n(\mu)]^2 \, \mathrm{d}\mu = \int_{-1}^{+1} [P_n^m(\mu)]^2 \, \mathrm{d}\mu = 1 \ .$$

(3) Schmidt normalization:

$$K_n = 1 \quad , \quad K_n^m = \sqrt{2 \frac{(n-m)!}{(n+m)!}} \ .$$

This normalization yields

$$\int_0^\pi \sin\vartheta \, \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\lambda \, [P_n^m(\cos\vartheta) \cos m\lambda]^2 = \frac{4\pi}{2n+1}$$

which is independent of m. In Schmidt normalization, the addition theorem holds without any additional factor in front of the sum.

(4) "Full/complete normalization II":

$$K_n = \sqrt{2n+1} \quad , \quad K_n^m = \sqrt{2n+1} \sqrt{2 \frac{(n-m)!}{(n+m)!}} \ .$$

The integral

$$\frac{1}{4\pi} \int_0^\pi \sin\vartheta \, \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\lambda \, [P_n^m(\cos\vartheta) \cos m\lambda]^2 = 1$$

is independent of m. This normalization is used, e.g., in satellite geodesy (Gaposhkin).