

Earth Science Applications of Space Based Geodesy

DES-7355

Tu-Th 9:40-11:05

Seminar Room in 3892 Central Ave. (Long building)

Bob Smalley

Office: 3892 Central Ave, Room 103

678-4929

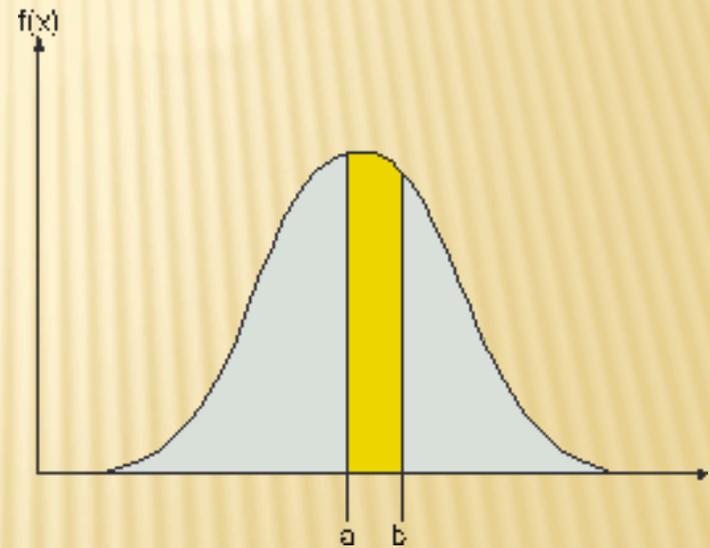
Office Hours – Wed 14:00-16:00 or if I'm in my office.

http://www.ceri.memphis.edu/people/smalley/ESCI7355/ESCI_7355_Applications_of_Space_Based_Geodesy.html

Class 8

If X is a continuous random variable, then the probability density function, pdf, of X , is a function $f(x)$ such that for two numbers, a and b with $a \leq b$

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$



That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the density function from a to b .

The probability density function for the Gaussian distribution is defined as:

$$P_G(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]$$

For the Gaussian PDF, the probability for the random variable x to be found between $\mu \pm z\sigma$,
Where z is the dimensionless range $z = |x - \mu|/\sigma$ is:

$$A_G(x, \mu, \sigma) = \int_{\mu - z\sigma}^{\mu + z\sigma} P_G(x, \mu, \sigma) dx = \frac{1}{\sqrt{2\pi}} \int_{-z}^z \exp\left[-\frac{1}{2} x^2\right] dx$$

$$A_G(z = \infty) = 1$$

The cumulative distribution function, cdf,
is a function $F(x)$ of a random variable, X , and
is defined for a number x by:

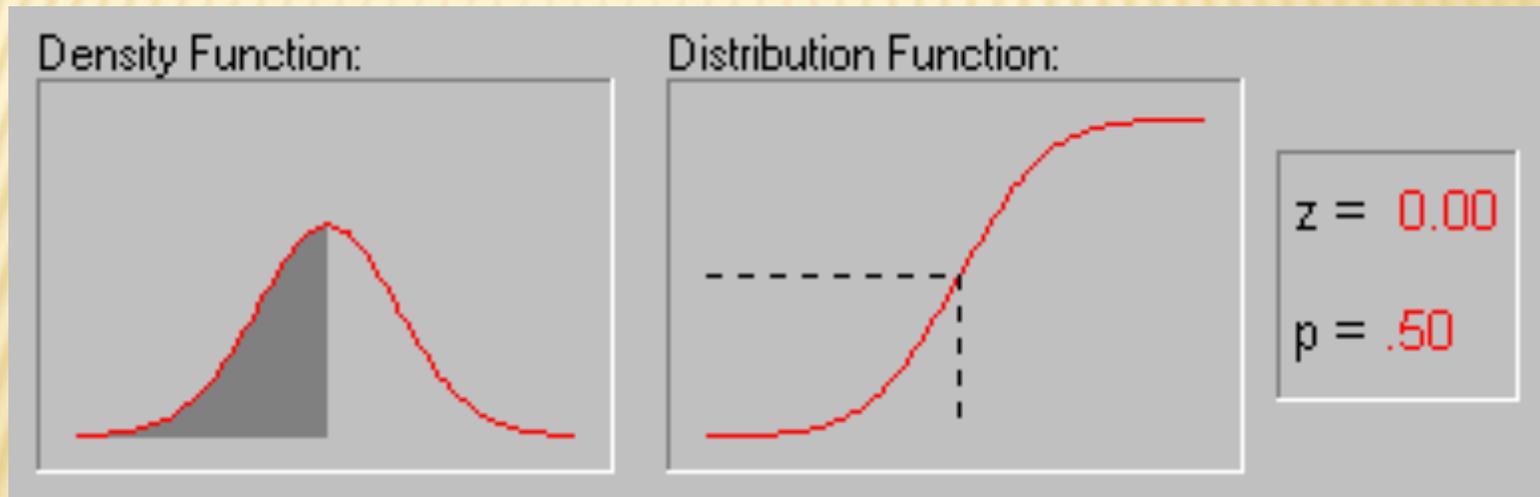
$$F(x) = P(X \leq x) = \int_{0, \infty}^x f(s) ds$$

That is, for a given value x , $F(x)$ is the probability that the observed value of X will be at most x . (note lower limit shows domain of s , integral goes from 0 to $x < \infty$)

Relationship between PDF and CDF

Density vs. Distribution Functions for Gaussian

<- derivative <-



-> integral ->

Multiple random variables

Expected value or mean of sum of two random variables is sum of the means.

known as additive law of expectation.

$$E(x + y) = E(x) + E(y)$$

covariance

$$\sigma_{xy}^2 = COV(x, y) = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - E(x))(y_i - E(y))$$

(variance is covariance of variable with itself)

(more general with) individual probabilities

$$\sigma_{xy}^2 = COV(x, y) = \sum_{i=1}^n p_{xy_i} (x_i - E(x))(y_i - E(y))$$

Covariance matrix

$$\delta x_i = x_i - \bar{x}$$

$$\delta y_i = y_i - \bar{y}$$

$$C = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{(n-1)} \sum_{i=1}^n \delta x_i^2 & \frac{1}{(n-1)} \sum_{i=1}^n \delta x_i \delta y_i \\ \frac{1}{(n-1)} \sum_{i=1}^n \delta x_i \delta y_i & \frac{1}{(n-1)} \sum_{i=1}^n \delta y_i^2 \end{pmatrix}$$

Covariance matrix defines error ellipse.

Eigenvalues are squares of semimajor and semiminor axes (σ_1 and σ_2)

Eigenvectors give orientation of error ellipse

(or given σ_x and σ_y , correlation gives “fatness” and “angle”)

Distance Root Mean Square (DRMS, 2-D extension of RMS)

$$RMS = \sqrt{\frac{1}{n} \sum_{i=1}^n x_i^2} \quad \Rightarrow$$

$$DRMS = \left(\sigma_x^2 + \sigma_y^2 \right)^{\frac{1}{2}}$$

For a scalar random variable or measurement with a Normal (Gaussian) distribution, the probability of being within the 1- σ ellipse about the mean is 68.3%

Etc for 3-D

Use of variance, covariance – in Weighted Least Squares

common practice to use the reciprocal of the variance as
the weight

variance of the sum of two random variables

$$VAR(x + y) = VAR(x) + 2COV(x, y) + VAR(y)$$

The variance of the sum of two random variables is equal to the sum of each of their variances only when the random variables are independent

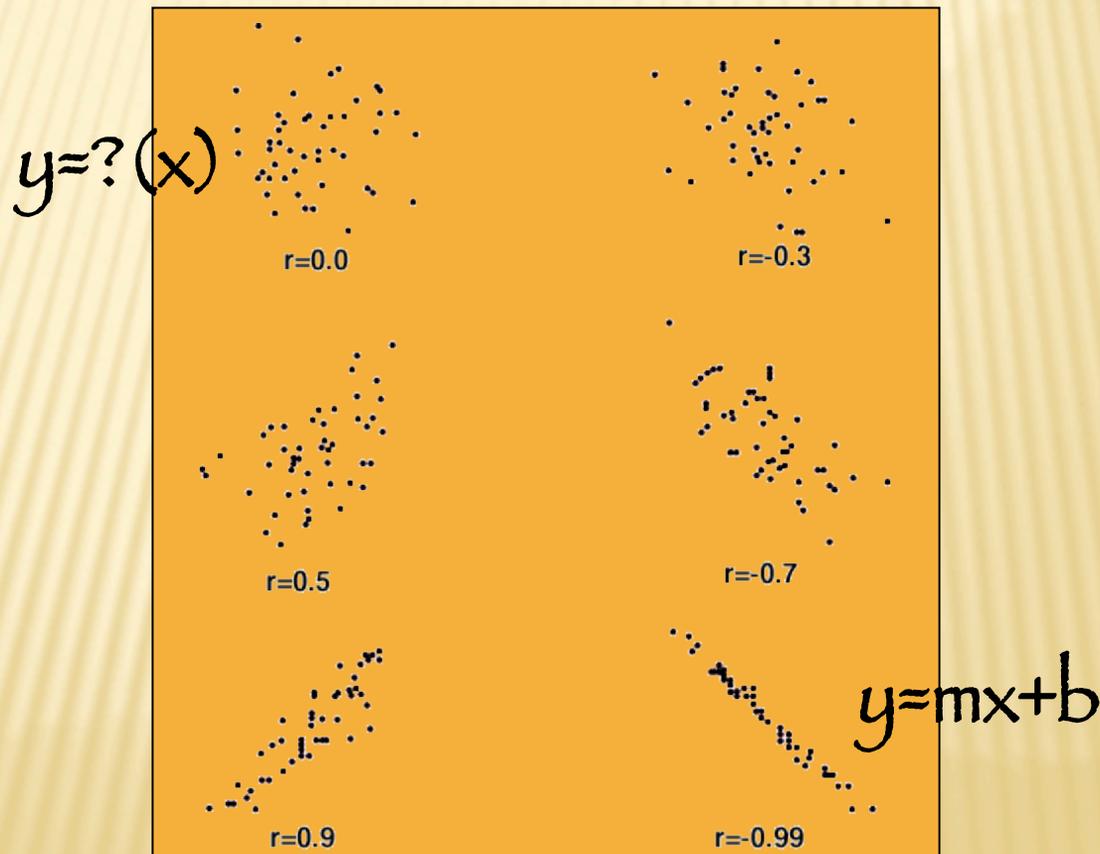
(The covariance of two independent random variables is zero, $cov(x, y) = 0$).

Multiplying a random variable by a constant increases the variance by the square of the constant.

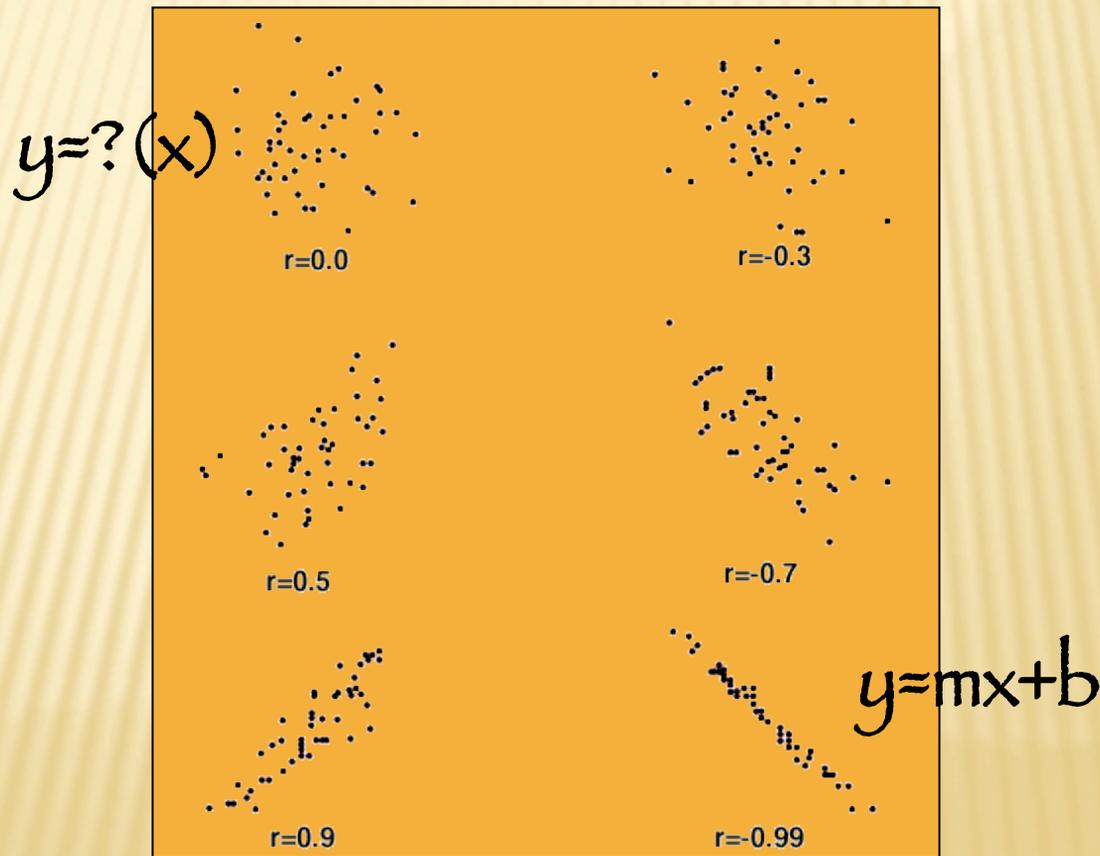
$$\sigma_{cx}^2 = E(cx) = c^2 E(x)$$

Correlation

The more tightly the points are clustered together the higher the correlation between the two variables and the higher the ability to predict one variable from another

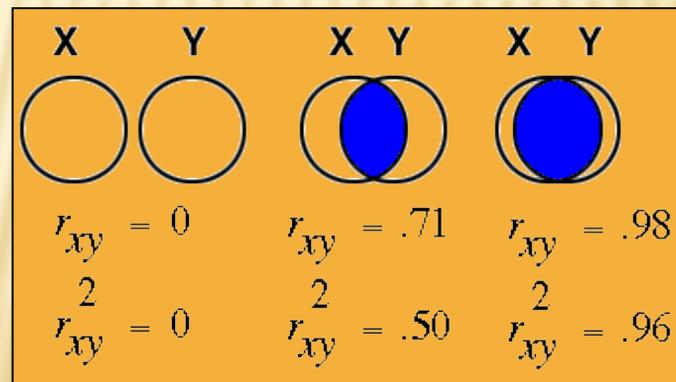


Correlation coefficients are between -1 and +1,
+ and - 1 represent perfect correlations,
and zero representing no relationship, between the
variables.

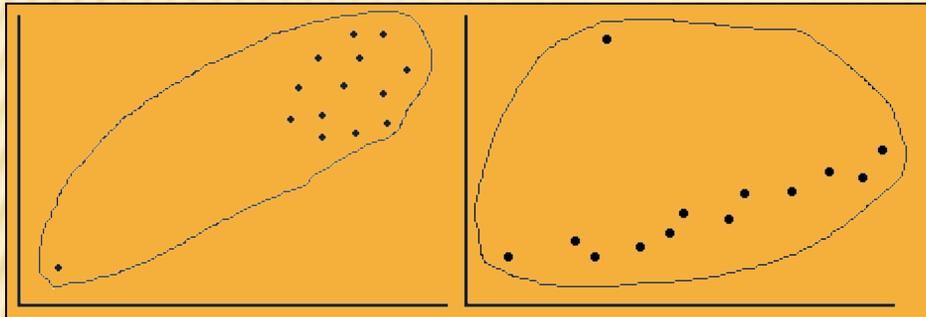


Correlations are interpreted by squaring the value of the correlation coefficient.

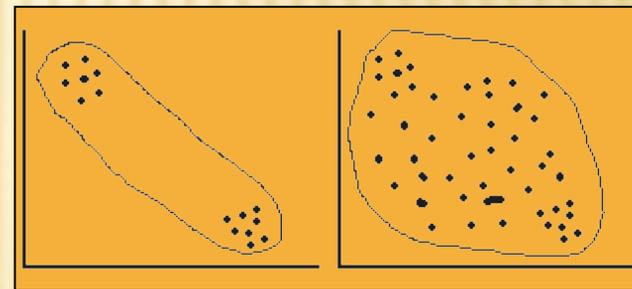
The squared value represents the proportion of variance of one variable that is shared with the other variable, in other words, the proportion of the variance of one variable that can be predicted from the other variable.



Sources of misleading correlation (and problems with least squares inversion)

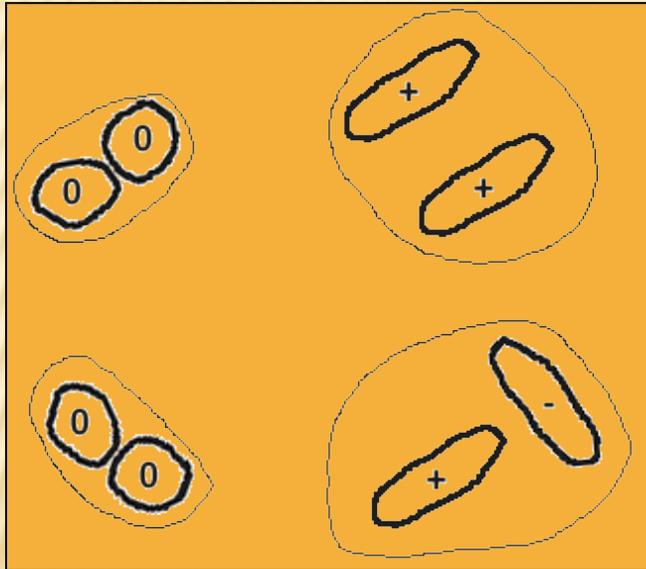


outliers

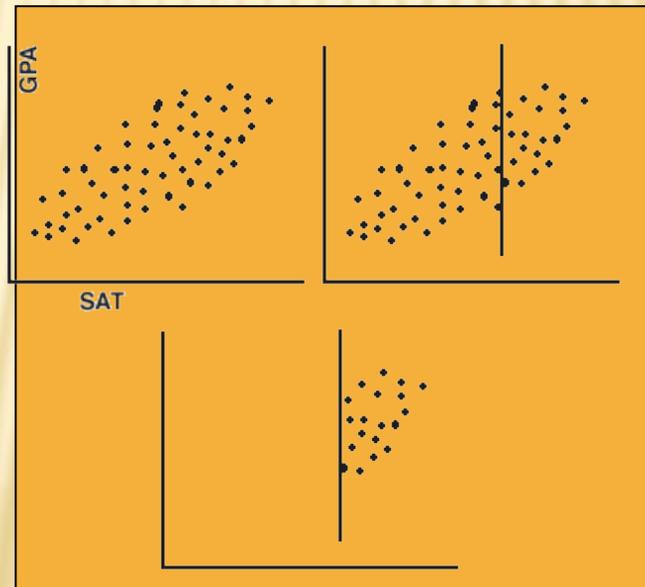


Bimodal No
distribution relation

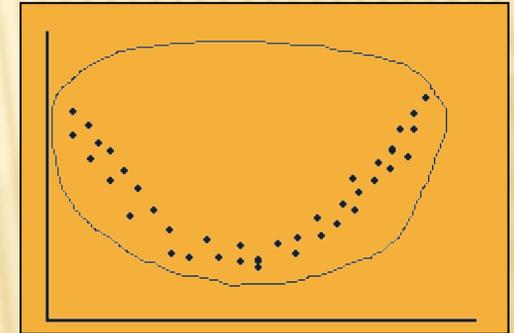
Sources of misleading correlation (and problems with least squares inversion)



Combining
groups



Restriction of range



curvilinearity

rule of thumb for interpreting correlation coefficients:

Corr	Interpretation
0 to .1	trivial
.1 to .3	small
.3 to .5	moderate
.5 to .7	large
.7 to .9	very large

Correlations express the inter-dependence between variables.

For two variables x and y in a linear relationship, the correlation between them is defined as

$$\chi_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

High correlation does not mean that the variations of one are caused by the variations of the others, although it may be the case.

In many cases, external influences may be affecting both variables in a similar fashion.

two types of correlation

physical correlation and mathematical correlation

Physical correlation refers to the correlations between the actual field observations.

It arises from the nature of the observations as well as their method of collection.

If different observations or sets of observation are affected by common external influences, they are said to be physically correlated.

Hence all observations made at the same time at a site may be considered physically correlated because similar atmospheric conditions and clock errors influence the measurements.

Mathematical correlation is related to the parameters in the mathematical model.

It can therefore be partitioned into two further classes which correspond to the two components of the mathematical adjustment model:

Functional correlation

Stochastic correlation

Functional Correlation:

The physical correlations can be taken into account by introducing appropriate terms into the *functional model* of the observations.

That is, functionally correlated quantities share the same parameter in the observation model.

An example is the clock error parameter in the one-way GPS observation model, used to account for the physical correlation introduced into the measurements by the receiver clock and/or satellite clock errors.

Stochastic Correlation:

Stochastic correlation (or statistical correlation) occurs between observations when non-zero off-diagonal elements are present in the variance-covariance (VCV) matrix of the observations.

Also appears when functions of the observations are considered (eg. differencing), due to the Law of Propagation of Variances.

However, even if the VCV matrix of the observations is diagonal (no stochastic correlation), the VCV matrix of the resultant LS estimates of the parameters will generally be full matrices, and therefore exhibit stochastic correlation.

Covariance and Cofactor matrix in GPS

If observations had no errors and the model was perfect then the estimations from

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Would be perfect

Errors, \mathbf{v} , in the original observations \mathbf{b} will map into errors \mathbf{v}_x in the estimates of \mathbf{x} and this mapping will take the same form as the estimation

$$\left(\hat{\mathbf{x}} + \mathbf{v}_x\right) = \left(A^T A\right)^{-1} A^T \left(\vec{\mathbf{b}} + \vec{\mathbf{v}}\right)$$

$$\mathbf{v}_x = \left(A^T A\right)^{-1} A^T \vec{\mathbf{v}}$$

If we have an expected (a priori) value for the error in the data, σ , we can compute the expected error in the parameters

Consider the covariance matrix

and for this discussion suppose that the observations are uncorrelated (covariance matrix is therefore diagonal)

$$C_{ii} = E(v_i^2) = \sigma_i^2$$

$$C = E(vv^T)$$

Assume further that we can characterize the error in the observations by a single number, σ .

$$C_{ii} = \sigma^2 I$$

then

$$C_x = E(\mathbf{v}_x \mathbf{v}_x^T) = E\left(\left(\left(A^T A\right)^{-1} A^T \vec{v}\right)\left(\left(A^T A\right)^{-1} A^T \vec{v}\right)^T\right)$$

$$C_x = E\left(\left(A^T A\right)^{-1} A^T \vec{v} \mathbf{v}^T A \left(A^T A\right)^{-1}\right)$$

$$C_x = \left(A^T A\right)^{-1} A^T E\left(\vec{v} \mathbf{v}^T\right) A \left(A^T A\right)^{-1}$$

$$C_x = \left(A^T A\right)^{-1} A^T \sigma^2 I A \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \left(A^T A\right)^{-1} \left(A^T A\right) \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 (A^T A)^{-1}$$

Expected covariance is σ^2 (a number) times cofactor matrix, same form as

$$\vec{v}_x = (A^T A)^{-1} A^T \vec{v}$$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Covariance or cofactor matrix

$$(A^T A)^{-1}$$

Interpretation of covariance

$$C_x = \sigma^2 (A^T A)^{-1}$$

Variance of measurements

Measurement errors may be independent (our assumption – why we could factor out constant σ^2)

We saw before that A is dependent on the “direction” from antenna to satellite – so it is a function of the geometry

But total effect, after Least Squares, can be non-diagonal.

Since A is function of geometry only, the cofactor matrix is also a function of geometry only.

$$(A^T A)^{-1}$$

Can use cofactor matrix to quantify the relative strength of the geometry.

Also relates measurement errors to expected errors in position estimations

In the old days,
before the full constellation of satellites was flying,
one had to plan – design – the GPS surveying sessions
based on the (changing) geometry.

A is therefore called the “design” matrix

Don't have to worry about this anymore
(most of the time).

Look at full covariance matrix

$$C_x = \sigma^2 (A^T A)^{-1}$$

$$C_x = \sigma^2 \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{x\tau} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} & \sigma_{y\tau} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 & \sigma_{z\tau} \\ \sigma_{\tau x} & \sigma_{\tau y} & \sigma_{\tau z} & \sigma_\tau^2 \end{pmatrix}$$

$$\sigma_{ij} = \sigma_{ji}$$

Off diagonal elements indicate degree of correlation between parameters.

Correlation coefficient

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}$$

Depends only on cofactor matrix

Independent of observation variance (the σ^2 's cancel out)

+1 perfect correlation – what does it mean – the two parameters behave practically identically (and not independent?!).

0 – no correlation, independent

-1 perfect anti-correlation – practically opposites (and not independent)

So far all well and good –
but Cartesian coordinates are not the most useful.

We usually need estimates of horizontal and vertical positions on earth (ellipsoid?).

We also need error estimates on the position.

Since the errors are a function of the geometry only, one might expect that the vertical errors are larger than the horizontal errors.

How do we find the covariance / cofactor matrices in the local (north, east, up) coordinate system?

Have to transform the matrix
from its representation in one coordinate system
to its representation in another
using the rules of error propagation.

First how do we transform a small relative vector in
Cartesian coordinates (u,v,w)
to local topocentric coordinates (n,e,u) ?

$$\Delta \vec{L} = G \Delta \vec{X}$$

$$\begin{pmatrix} \Delta n \\ \Delta e \\ \Delta u \end{pmatrix} = \begin{pmatrix} -\sin \varphi \cos \lambda & -\sin \varphi \sin \lambda & \cos \varphi \\ -\sin \lambda & \cos \lambda & 0 \\ \cos \varphi \cos \lambda & \cos \varphi \sin \lambda & \sin \varphi \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Where φ and λ are the lat and long of the location
(usually on the surface of the earth) respectively

Errors (small magnitude vectors) transform the same way

$$\vec{v}_L = G\vec{v}_x$$

Why?

$$\Delta\vec{L} = G\Delta\vec{X}$$

$$A(\Delta\vec{X} + \vec{v}_x) = G\Delta\vec{X} + G\vec{v}_x = \Delta\vec{L} + G\vec{v}_x = \Delta\vec{L} + \vec{v}_L$$

linearity

Errors (small magnitude vectors) transform the same way

$$\vec{v}_L = G\vec{v}_x$$

Now – how does the covariance

$$C = E(\vec{v}\vec{v}^T)$$

Transform?

Plug in – get

“law of propagation of errors”

law of propagation of errors

$$C_L = E(\vec{v}_L \vec{v}_L^T)$$

$$C_L = E(G \vec{v}_x (G \vec{v}_x)^T)$$

$$C_L = E(G \vec{v}_x \vec{v}_x^T G^T)$$

$$C_L = G E(\vec{v}_x \vec{v}_x^T) G^T$$

$$C_L = G C_x G^T$$

(does this look familiar?)

law of propagation of errors

$$C_L = E(\vec{v}_L \vec{v}_L^T)$$

$$C_L = E(G \vec{v}_x (G \vec{v}_x)^T)$$

$$C_L = E(G \vec{v}_x \vec{v}_x^T G^T)$$

$$C_L = G E(\vec{v}_x \vec{v}_x^T) G^T$$

$$C_L = G C_x G^T$$

(does this look familiar?)

(transforming tensors!!)

This is a general result for affine transformations
(multiplication of a column vector by any rectangular
matrix)

An affine transformation is any transformation that preserves

collinearity

(i.e., all points lying on a line initially still lie on a line after transformation)

and ratios of distances

(e.g., the midpoint of a line segment remains the midpoint after transformation).

Geometric contraction, expansion, dilation, reflection, rotation, shear, similarity transformations, spiral similarities, and translation

are all affine transformations,
as are their combinations.



In general, an affine transformation is a composition of rotations, translations, dilations, and shears.

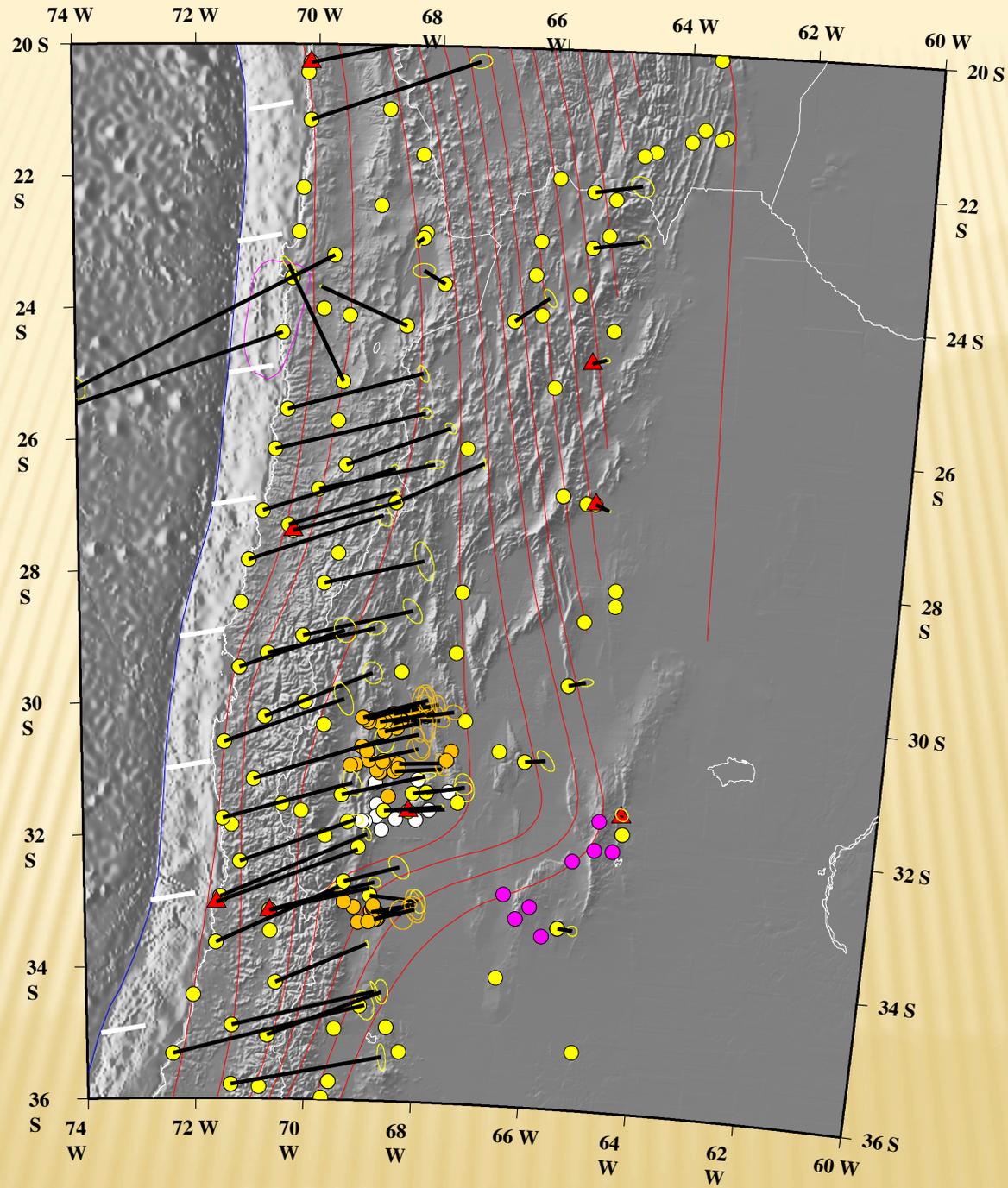
While an affine transformation preserves *proportions* on lines, it does not necessarily preserve angles or lengths.

Look at full covariance matrix
(actually only the spatial part)

$$C_x = \sigma^2 (A^T A)^{-1}$$

$$C_L = \sigma^2 \begin{pmatrix} \sigma_n^2 & \sigma_{ne} & \sigma_{nh} \\ \sigma_{en} & \sigma_e^2 & \sigma_{eh} \\ \sigma_{hn} & \sigma_{he} & \sigma_h^2 \end{pmatrix}$$

Can use this to plot error ellipses on a map (horizontal plane).



Error estimators –

Remember the expression for the RMS in 2-D from before

$$DRMS = \left(\sigma_x^2 + \sigma_y^2 \right)^{\frac{1}{2}}$$

We can now apply this to the covariance matrix

Error estimates called “dilution of Precision” – DOP – are defined in terms of the diagonal elements of the covariance matrix

$$GDOP = \left(\sigma_n^2 + \sigma_e^2 + \sigma_h^2 + \sigma_\tau^2 \right)^{\frac{1}{2}}$$

$$PDOP = \left(\sigma_n^2 + \sigma_e^2 + \sigma_h^2 \right)^{\frac{1}{2}}$$

$$HDOP = \left(\sigma_n^2 + \sigma_e^2 \right)^{\frac{1}{2}}$$

$$VDOP = \sigma_h$$

$$TDOP = \sigma_\tau$$

G – geometric, P – position, H – horizontal, V – vertical,

T – time

The DOPs map the errors of the observations
(represented/quantified statistically by the
standard deviations)

Into the parameter estimate errors

$$\sigma GDOP = \sigma \left(\sigma_n^2 + \sigma_e^2 + \sigma_h^2 + \sigma_\tau^2 \right)^{\frac{1}{2}}$$

$$\sigma PDOP = \sigma \left(\sigma_n^2 + \sigma_e^2 + \sigma_h^2 \right)^{\frac{1}{2}}$$

$$\sigma HDOP = \sigma \left(\sigma_n^2 + \sigma_e^2 \right)^{\frac{1}{2}}$$

$$\sigma VDOP = \sigma \sigma_h$$

$$\sigma TDOP = \sigma \sigma_\tau$$

So for a σ of 1 m and an χ DOP of 5, for example,
errors in the position χ
(where χ is one of G, P, H, V, T)
Would be $5\sigma=5$ m

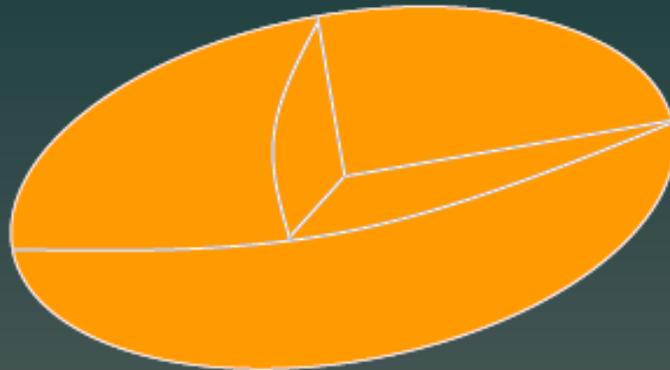
“Good” geometry gives ‘small’ DOP

“Bad” geometry gives ‘large’ DOP

(it is relative, but $PDOP > 5$ is considered poor)

3-D Accuracy Terms

- Error Ellipsoid



- 19.9% of positions will lie inside this “error ellipsoid”

In 2-D

There is a 40% chance of being inside the $1\text{-}\sigma$ error ellipse (compared to 68% in 1-D)

Normally show 95% confidence ellipses, is 2.54 s in 2-D

(is only 2σ in 1-D)
Can extend to 3-D

Another method of estimating location

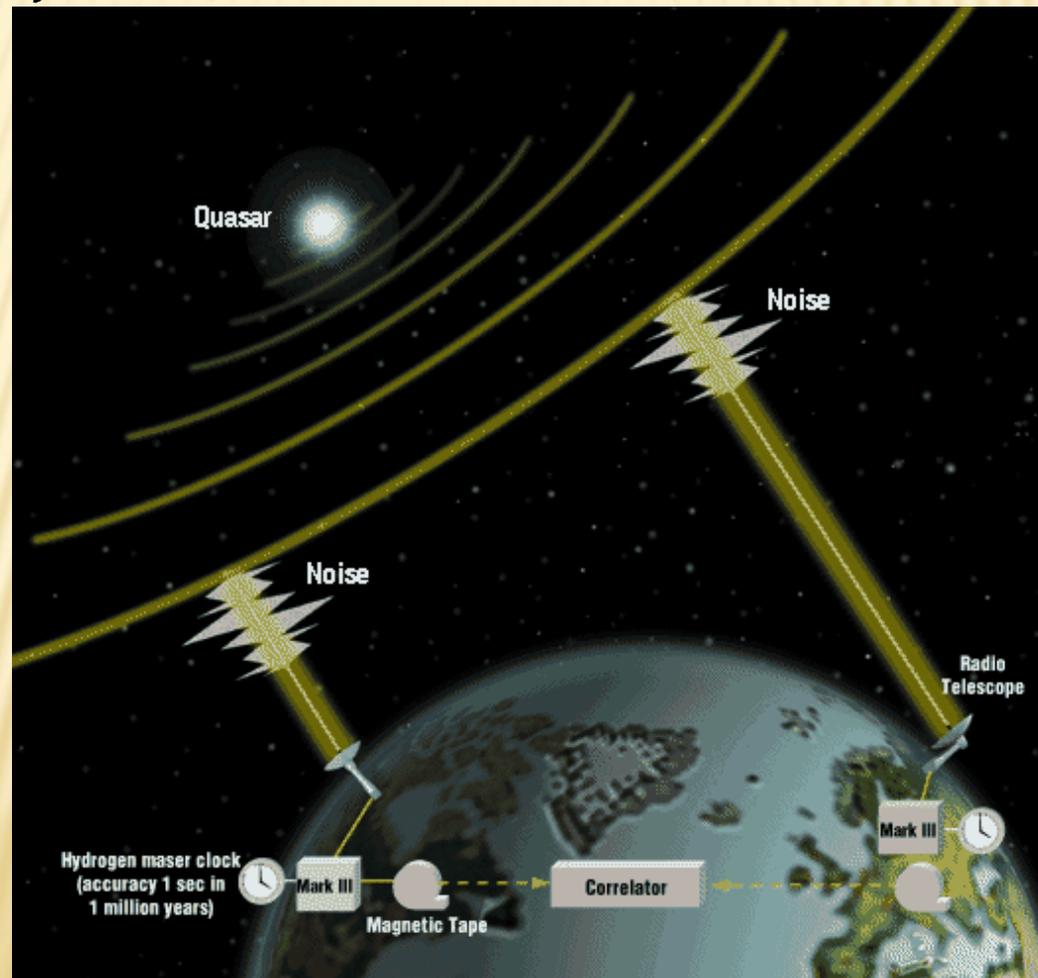
Phase comparison/Interferometer

- VLBI

- GPS-Carrier Phase Observable

VLBI

Uses techniques/physics similar to GPS but with natural sources
(in same frequency band and suffers from similar errors)



Correlate signal at two (or more) sites to find time shift

Need more than 1 receiver.

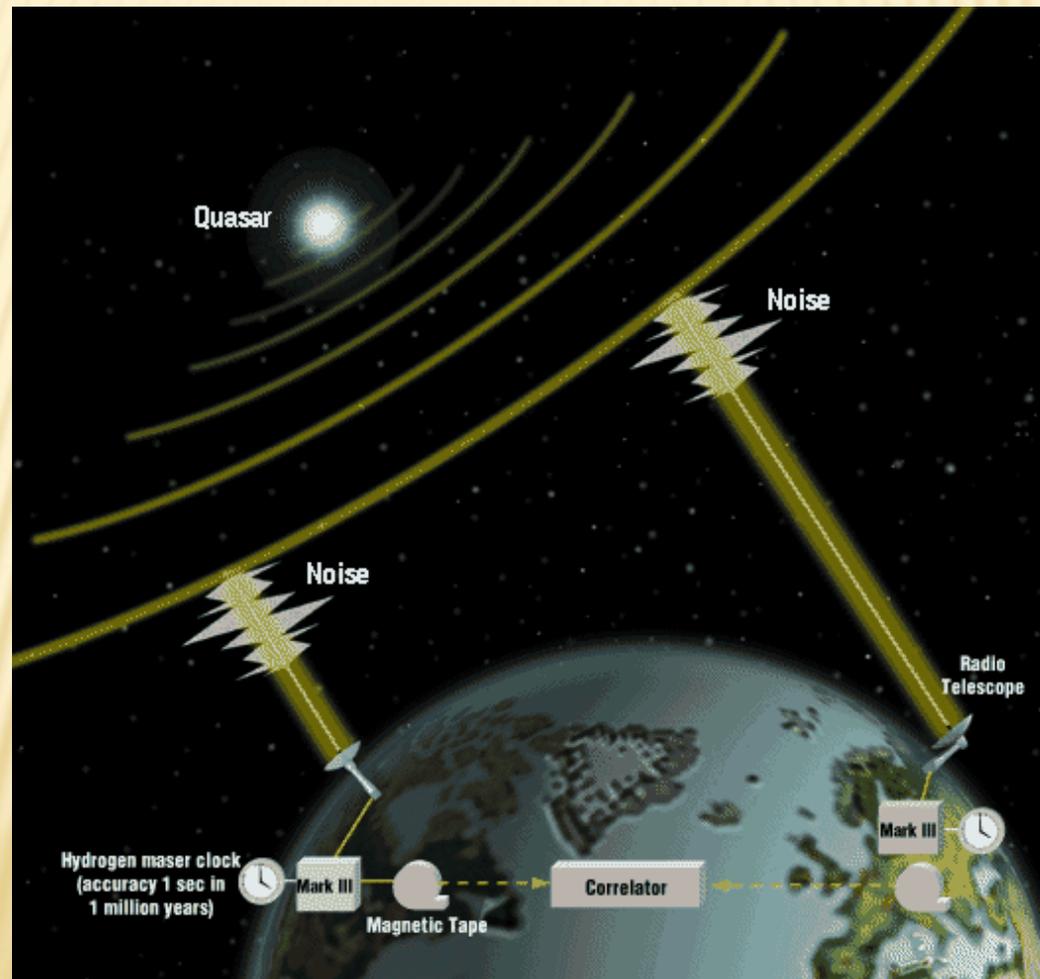
Differential (difference) method

(similar to PRN correlation with GPS codes

or

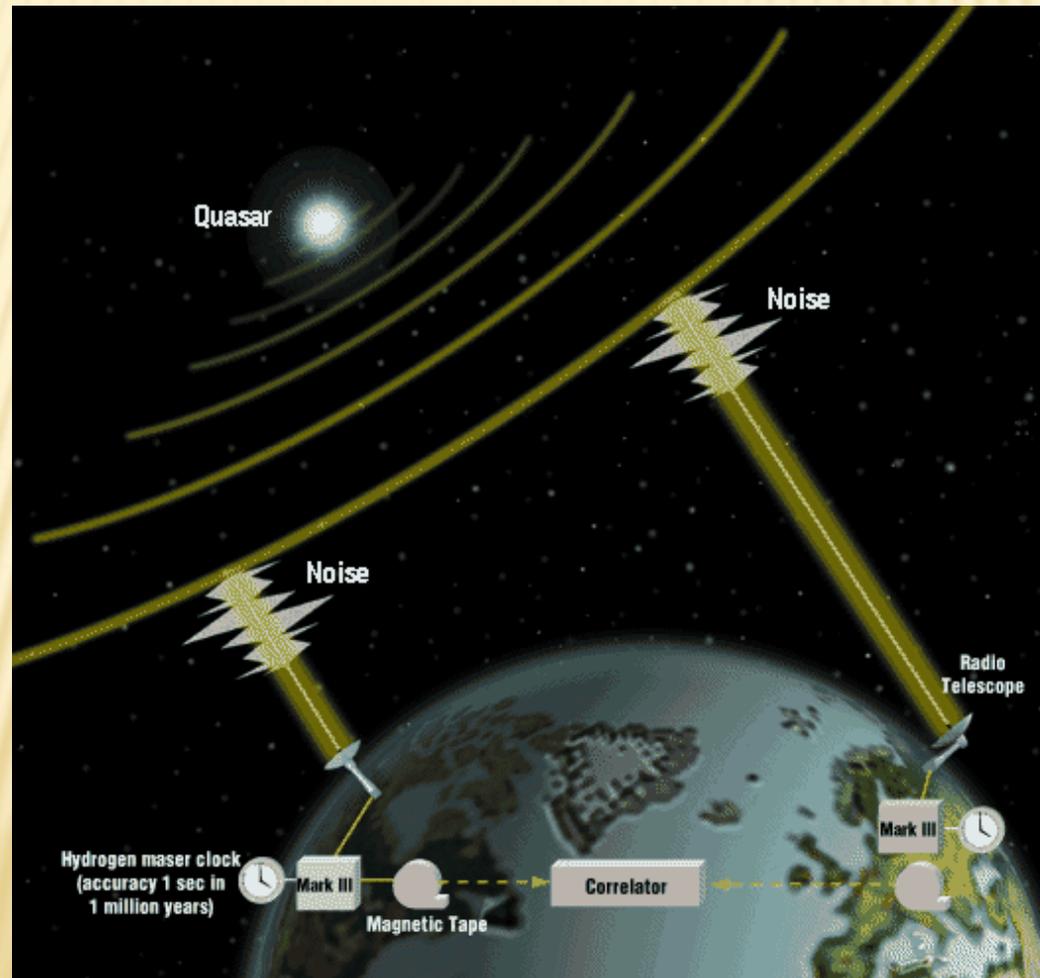
Aligning two seismograms that are almost same but have
time shift)

Assume you are receiving a plane wave from a distant quasar



Two radio antennas observe signal from quasar simultaneously.

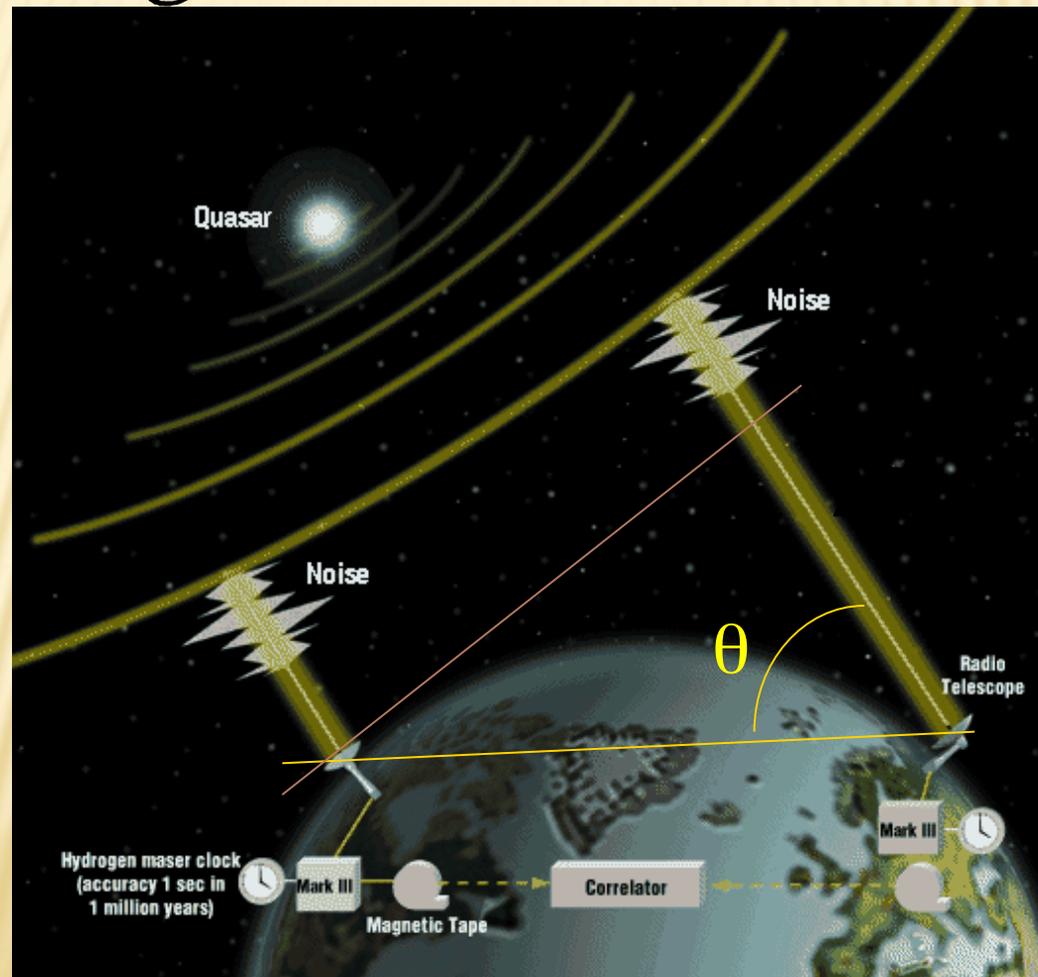
The signal arrives at the two antennae at different times



The distance or baseline length b between the two antennas can be defined as:

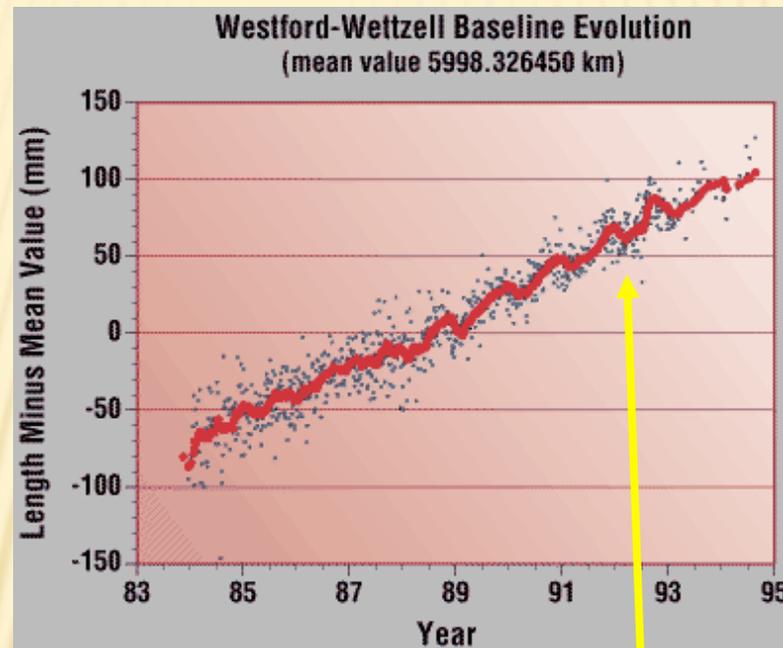
$$b * \cos(\theta) = c * \Delta T$$

where θ is the angle between the baseline and the quasar



Baseline length

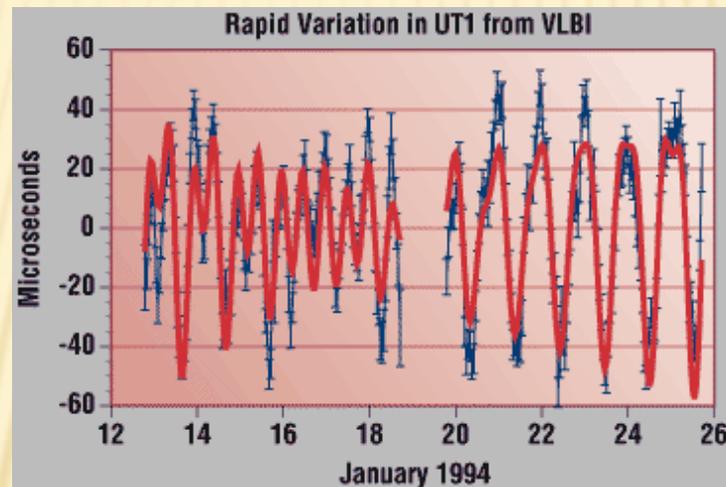
Massachusetts to Germany



What is this variation?

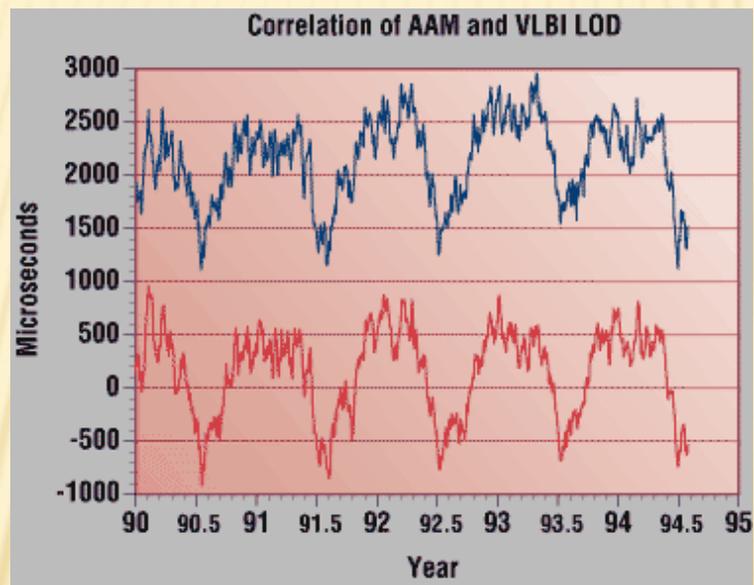
Seasonal variation, geophysical phenomena, modeling problems?

Short period (hours/days) variations in LOD

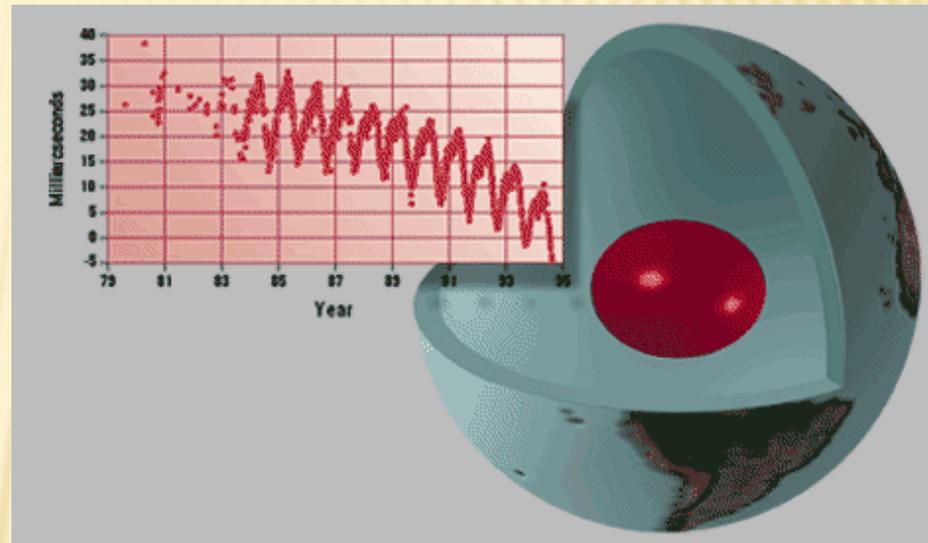
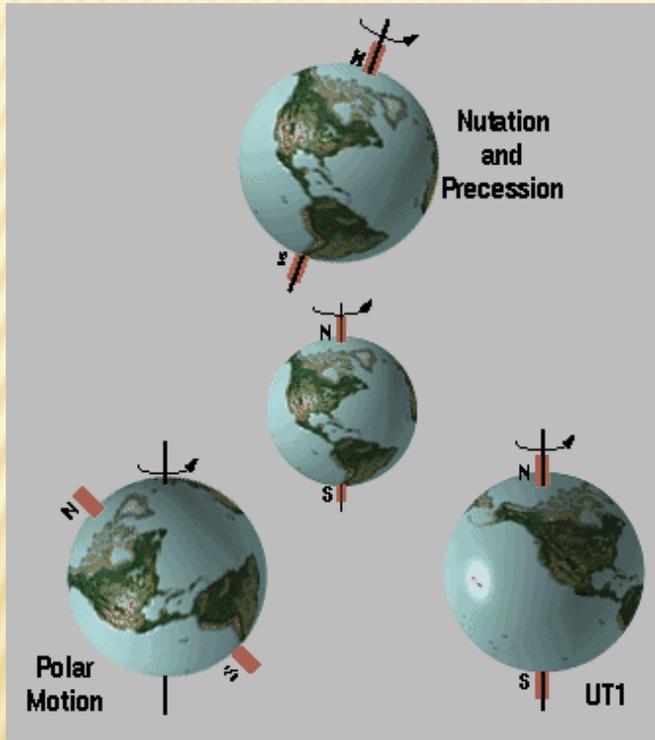


Mostly from ocean tides and currents

Correlation of Atmospheric Angular Momentum with (longer period – weeks/months) variations in LOD.

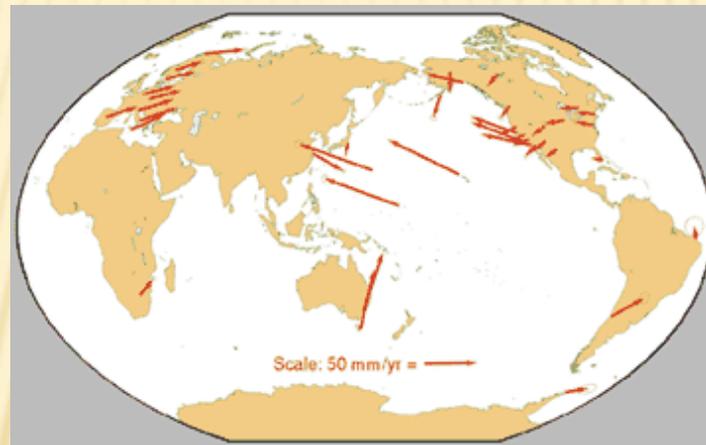


(longer term – months/years/...) changes in LOD and EOP



Exchange angular momentum between large earth structures (eg. core!) and Moon, Sun.

Plate velocities



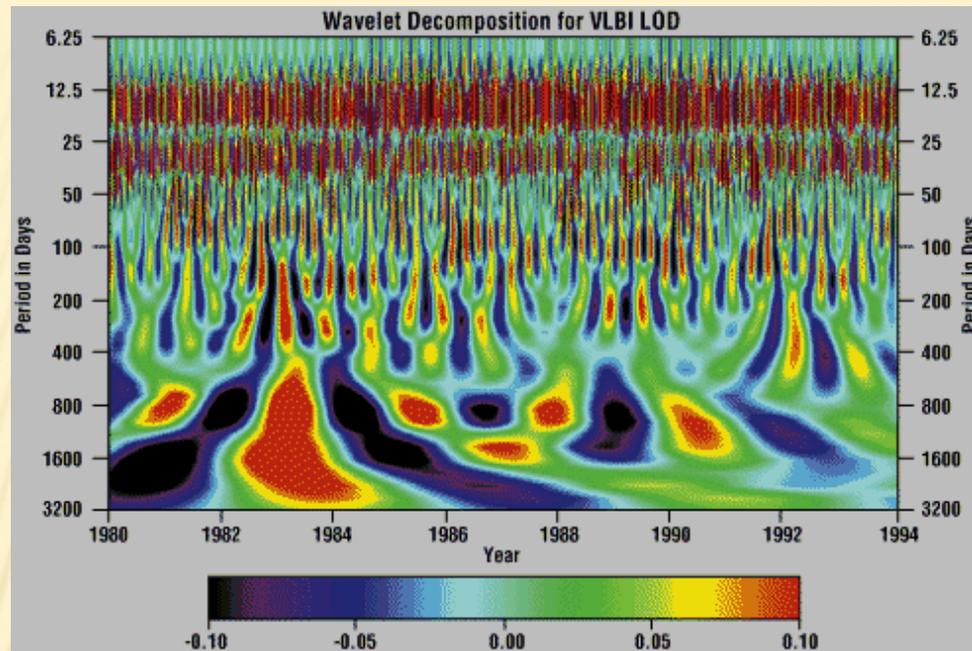


VLBI



Not the most portable or inexpensive system –
But best definition of inertial reference frame external to
earth.

Use to measure changes in LOD and EOP due to
gravitational forces and redistribution of angular
momentum.



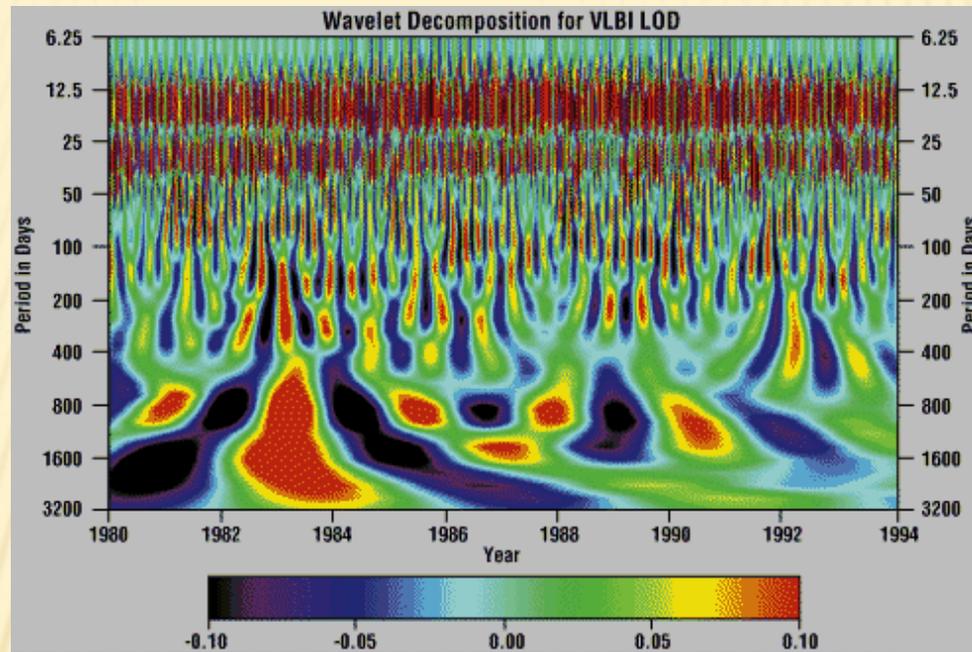
vertical cut - spectral decomposition LOD at that instant.

horizontal cut - how strength of component varies with time.

Combining both - 2-D view dynamic nature of LOD.

dark red -
peaks

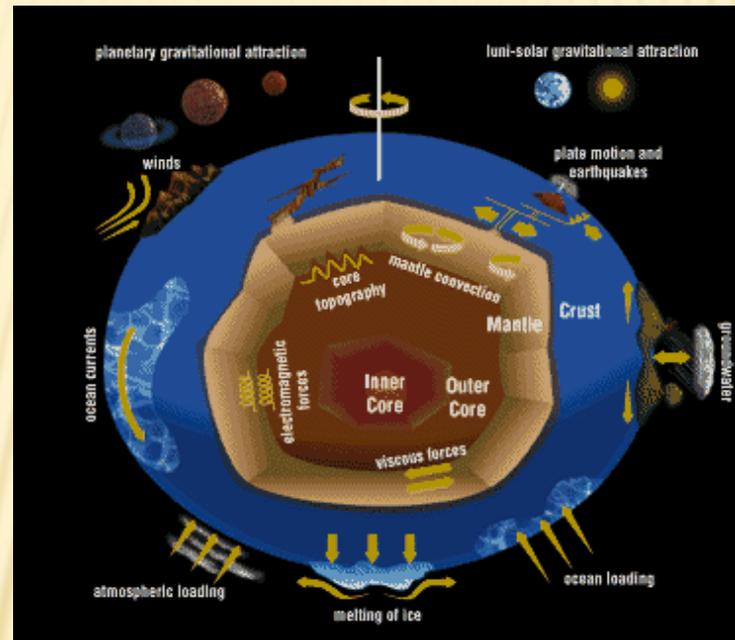
dark blue -
troughs.



dominant features

monthly and half-monthly lunar tides, the ~800-day quasi-biennial oscillation, and the ~1600-day El Nino (dark red structure in 1983). Yearly and half-yearly seasonal excitations caused by meteorological variations have been removed for clarity.

Factors affecting EOP



Great book –

Longitude, by Dava Sobel,

describes one of the first great scientific competitions--
to provide ship captains with their position at sea.

This was after the loss of two thousand men in 1707,
when British warships ran aground entering the English
Channel.

The competition was between the Astronomers Royal,
using distance to the moon and its angle with the stars,
and a man named John Harrison, who made clocks.

The accuracy demanded in the 18th century was a modest $1/2$ degree in longitude.

The earth rotates that much in two minutes.

For a six-week voyage this allows a clock error of three seconds per day.

Newton recommended the moon, and a German named Mayer won 3000 English pounds for his lunar tables. Even Euler got 300 for providing the right equations.

But lunar angles had to be measured, on a rolling ship at sea, within 1.5 minutes of arc.

The big prize was practically in sight, when Harrison came from nowhere and built clocks that could do better.

(You can see the clocks at Greenwich

Competing in the long trip to Jamaica, Harrison's clock lost only five seconds and eventually won the prize.

The modern version of this same competition was between VLBI and GPS.

Very Long Baseline Interferometry uses “God’s satellites,” the distant quasars.

The clock at the receiver has to be very accurate and expensive.

The equipment can be moved on a flatbed, but it is certainly not handheld.

There are valuable applications of VLBI, but it is GPS that will appear everywhere.

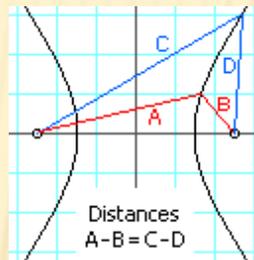
GPS is perhaps the second most important military contribution to civilian science, after the Internet.

The key is the atomic clock in the satellite, designed by university physicists to confirm Einstein's prediction of the gravitational red shift.

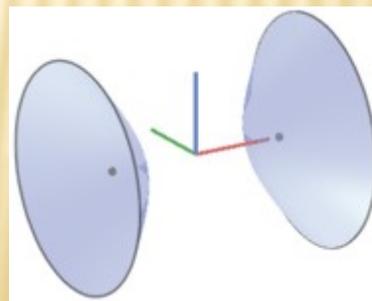
Using pseudo-range, the receiver solves a nonlinear problem in geometry.

What it knows is the difference d_{ij} between its distances to satellite i and to satellite j .

In a plane (2-D), when we know the difference d_{12} between the distances to two points, the receiver is located on a hyperbola.



In space (3-D) this becomes a hyperboloid.



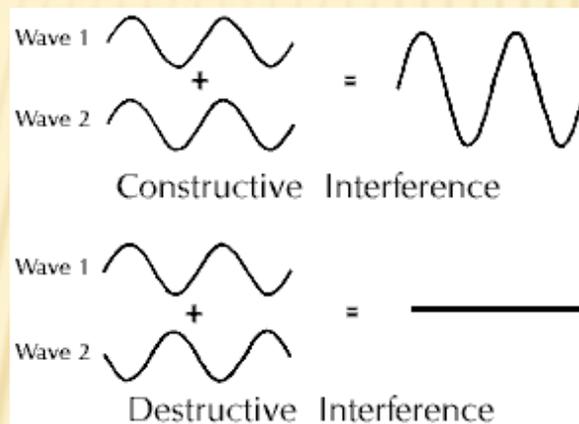
Then the receiver lies at the intersection of three hyperboloids, determined by d_{12} , d_{13} , and d_{14} .

Two hyperboloids are likely to intersect in a simple closed curve.

The third probably cuts that curve at two points. But again, one point is near the earth and the other is far away.

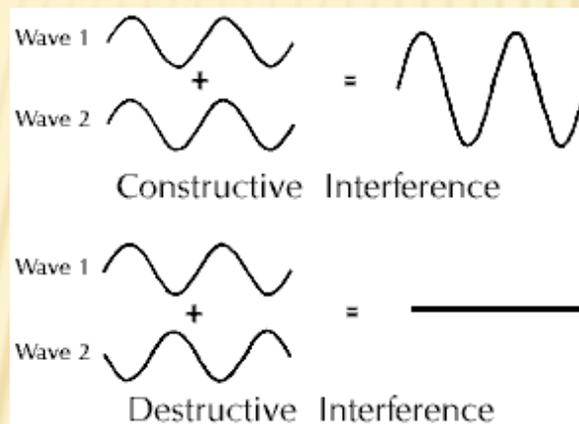
Interferometer

Based on interference of waves



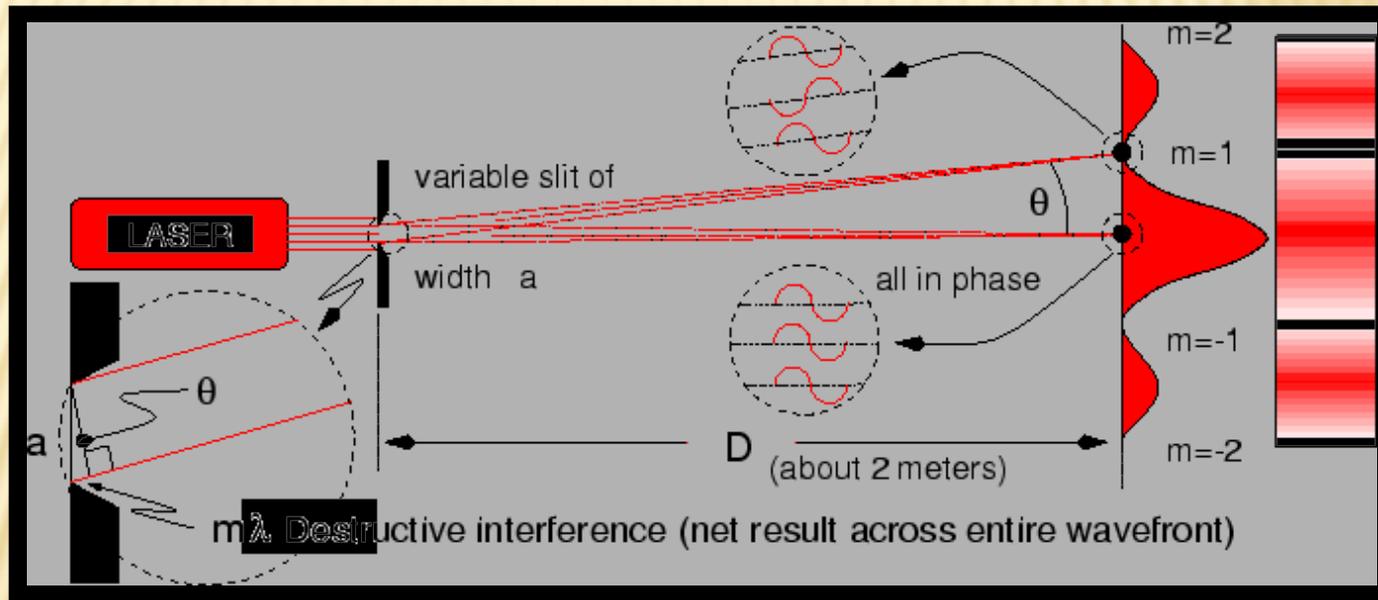
How to make principle of interference useful?

i.e. how does one get relative phase difference to vary,
so the interference varies?



Interference from single slit

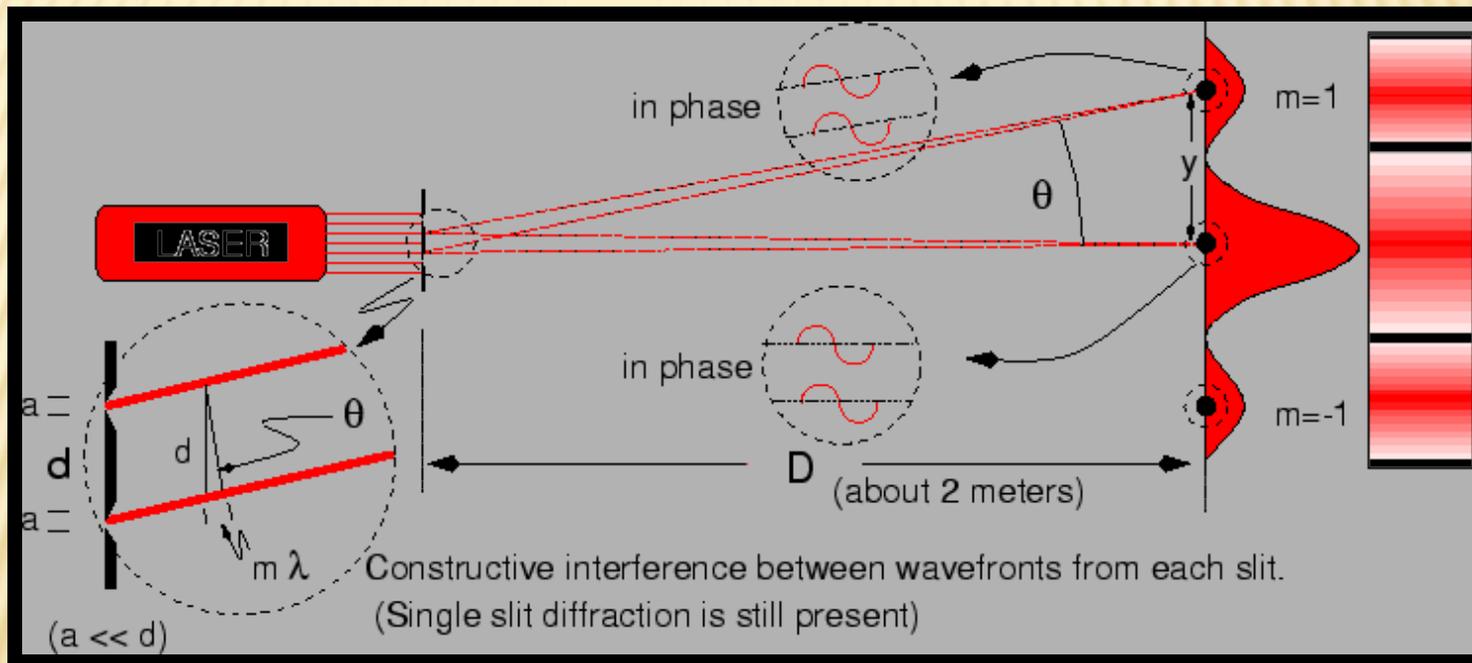
As move across screen get phase difference from different lengths of paths through slits



Makes "fringes"
As phase goes through change of 2π

Interference from double (multiple) slit

Similar for multi-slits, but now interference is between the waves leaving each slit



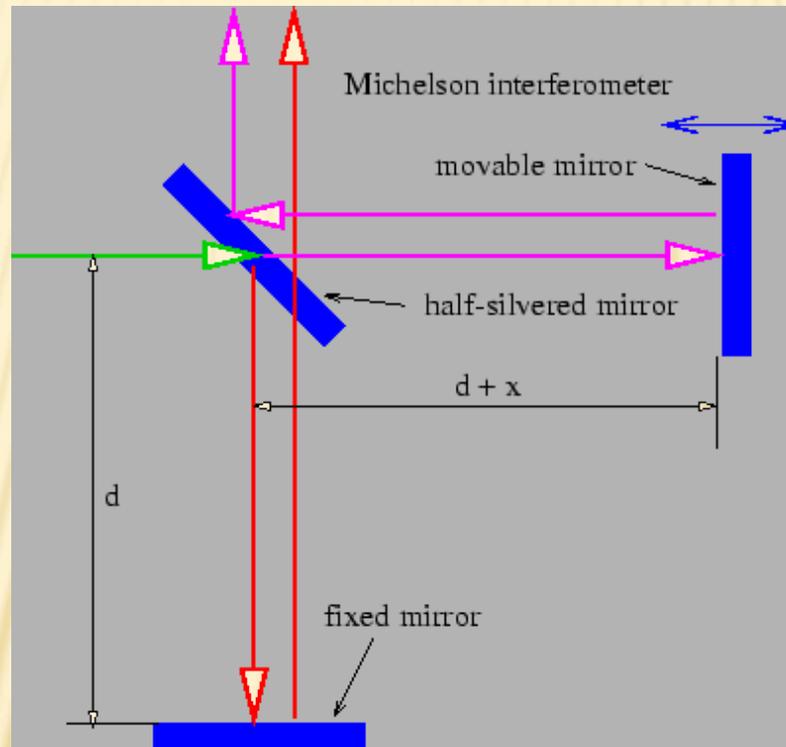
Light going through slits has to be “coherent”
(does not work with “white” light)

The phase change comes from
the change in geometric length
between the two “rays”

(change in length of $1/2$ wavelength causes π change in
phase – and destructive interference)

Michelson Interferometer

Make two paths from same source
(for coherence, can't do with white light)



Can change geometric path length with movable mirror.
Get interference fringes when recombine.