Earth Science Applications of Space Based Geodesy

DES-7355

Tu-Th 9:40-11:05

Seminar Room in 3892 Central Ave. (Long building)

Bob Smalley
Office: 3892 Central Ave, Room 103
678-4929

Office Hours – Wed 14:00-16:00 or if I'm in my office.

http://www.ceri.memphis.edu/people/smalley/ESCI7355/ESCI_7355_Applications_of_Space_Based_Geodesy.html

Class 6

Frequency Hopped Spread Spectrum

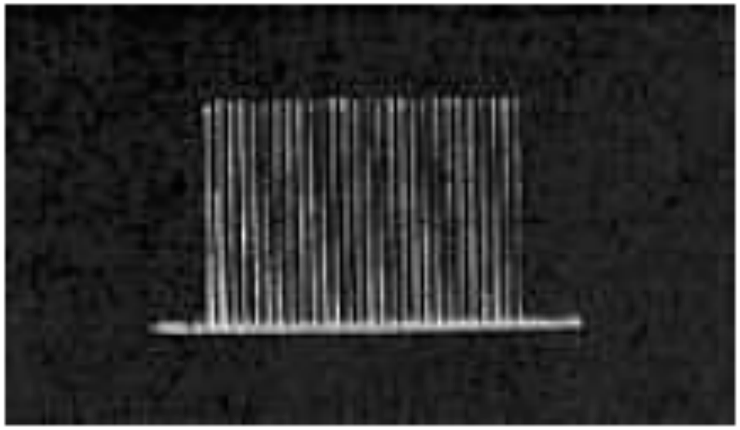
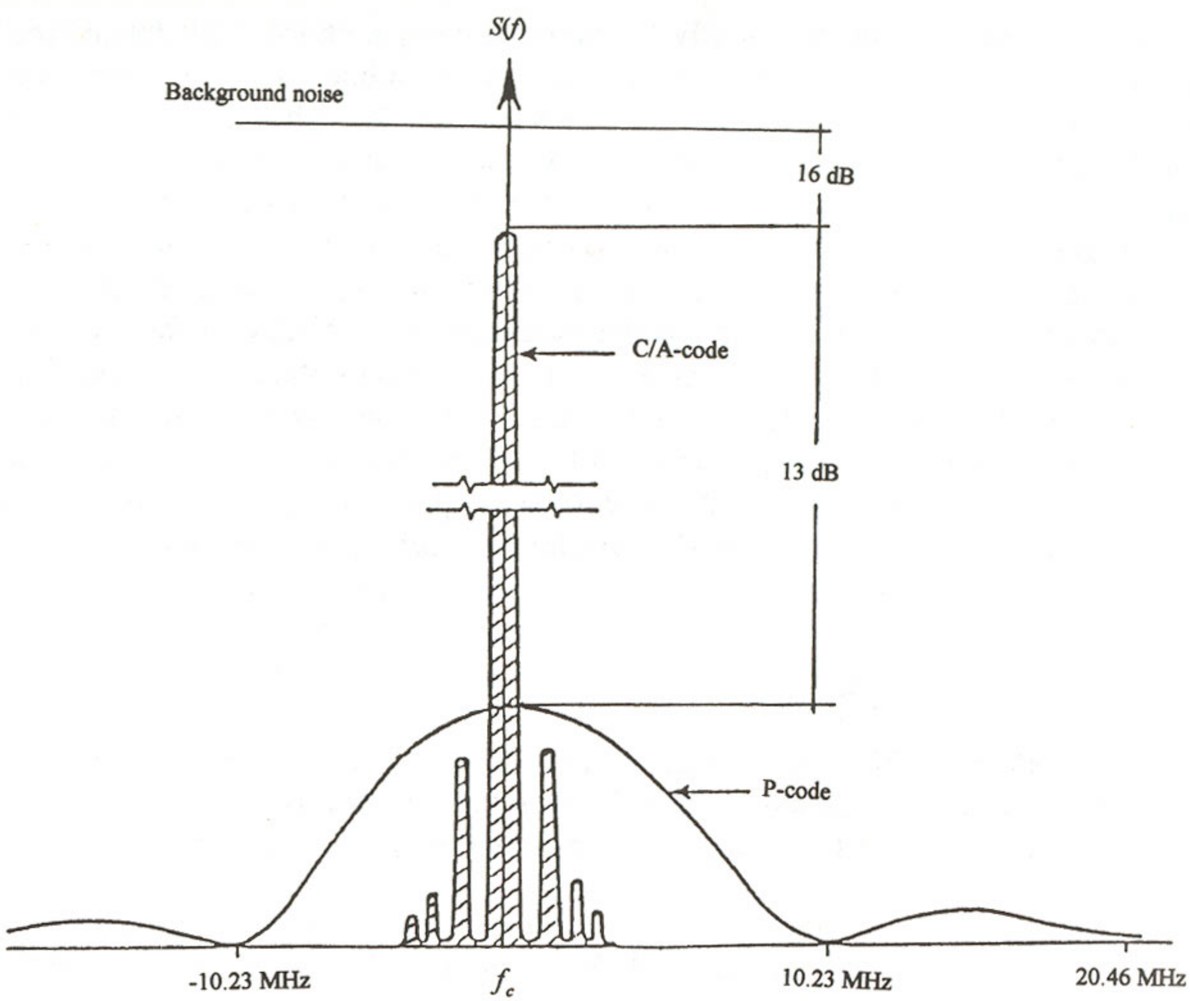


Fig 5: Spectrum of a FHSS Signal

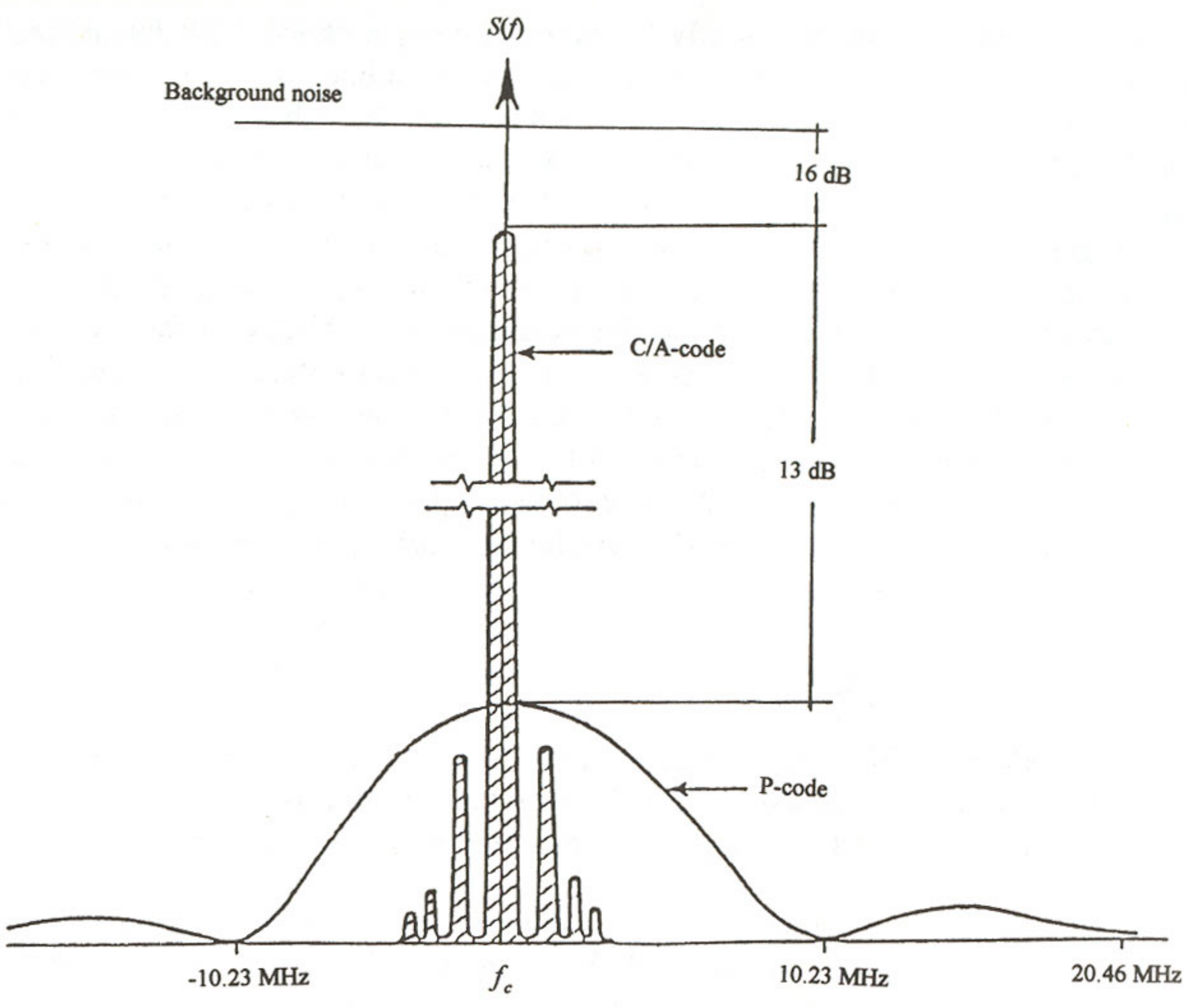
http://www.ieee.org/organizations/history_center/cht_papers/SpreadSpectrum.pdf

GPS signal strength - frequency domain



Note that C/A code is below noise level; signal is multiplied in the receiver by the internally calculated code to allow tracking. C/A-code chip is 1.023 Mhz, P-code chip is 10.23 Mhz

GPS signal strength – frequency domain



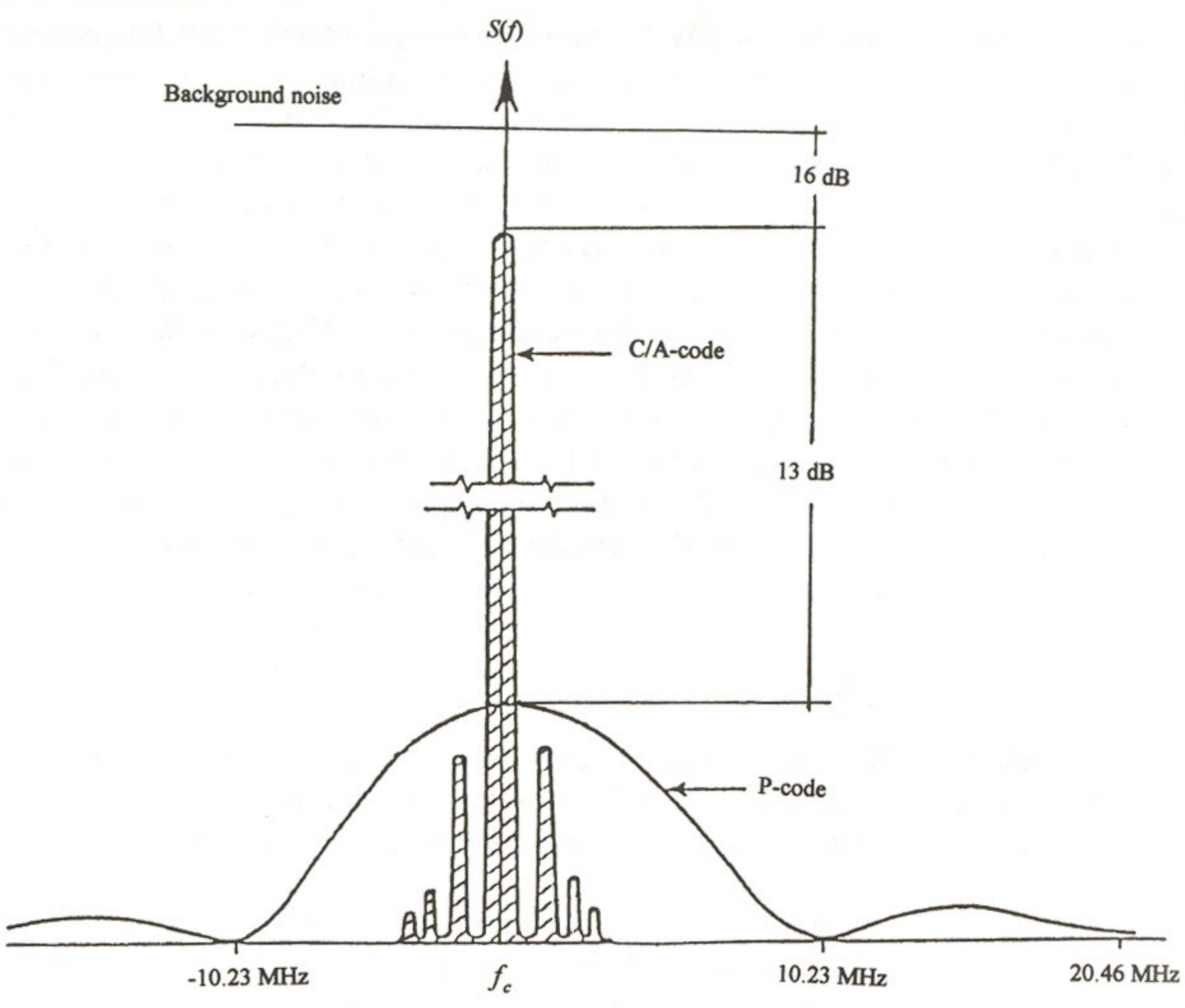
$$\text{Power} = P(t) = y^2(t)$$

$$\text{Bandwidth} \equiv B \approx \frac{1}{T}$$

where

$T \equiv$ is chip duration

GPS signal strength – frequency domain



The calculated power spectrum derives from the Fourier transform of a square wave of width 2π and unit amplitude.

Common function in DSP called the "sinc" function.

PRN Codes

GPS signals implement *PseudoRandom Noise* Codes

- Enables very low power (below background noise)
- A form of *direct-sequence spread-spectrum*
- Specifically a form of Code Division Multiple Access (CDMA), which permits frequency sharing

Pseudo random numbers/sequences

What are they?

Deterministic but “look” random

Example – digits of π

3.1415926535897932384626433832795028841971693993751 0

Looks like a random sequence of single digit numbers.

But you can compute it. Is perfectly deterministic.

Frequency of individual digits (first 10,000 digits)
This list excludes the 3 before the decimal point

Digit	Frequency
0	968
1	1026
2	1021
3	974
4	1012
5	1046
6	1021
7	970
8	948
9	1014
Total	10000

http://www.ex.ac.uk/cimt/general/pi10000.htm

PRN Codes

Codes are known "noise-like" sequences

Each bit (0/1) in the sequence is called a *chip*

Each GPS SV has an assigned code

Receiver generates equivalent sequences internally and matches signal to identify each SV

There are currently 32 reserved PRN's (so max 32 satellites)

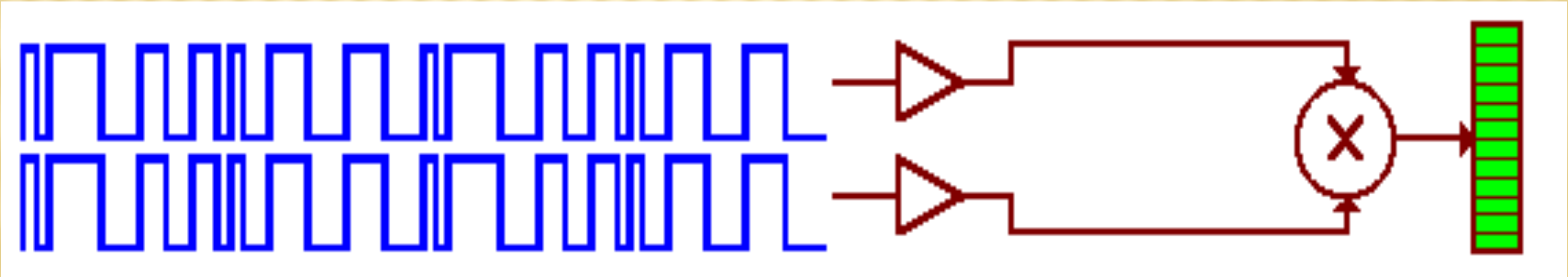
PRN Code matching

Receiver slews internally-generated code sequence until full "match" is achieved with received code

Time data in the nav message tells receiver when the transmitted code went out

Slew time = time delay incurred by SV-to-receiver range Minus clock bias and whole cycle ambiguities

Receiver/Signal Code Comparison



A. Ganse, U. Washington , http://staff.washington.edu/aganse/

Coarse Acquisition (C/A) Code

1023-bit Gold Code

Originally intended as simply an acquisition code for *P-code* receivers

Modulates L1 only

Chipping rate = 1.023 MHz
(290 meter "wavelength")

Coarse Acquisition (C/A) Code

Sequence Length = 1023 bits, thus Period = 1 millisec

~300 km range ambiguity: receiver must know range to better than this for position solution

Provides the data for *Standard Positioning Service* (SPS)

The usual position generated for most civilian receivers

Modulated by the Navigation/Timing Message code

Precise (P) Code

Generally encrypted by W code into the *Y-code*

Requires special chip to decode

Modulates both L1 & L2

Also modulated by Nav/Time data message

Chipping rate = 10.23 MHz

Precise (P) Code

Sequence Length = BIG (Period = 267 days)

Actually the sum of 2 sequences, X1 & X2, with sub-period of 1 week

Precise (P) Code

P-code rate is the fundamental frequency (provides the basis for all others)

P-Code (10.23 MHz) /10 = 1.023 MHz (C/A code)

P-Code (10.23 MHz) X 154 = 1575.42 MHz (L1).

P-Code (10.23 MHz) X 120 = 1227.60 MHz (L2).

Code Modulation

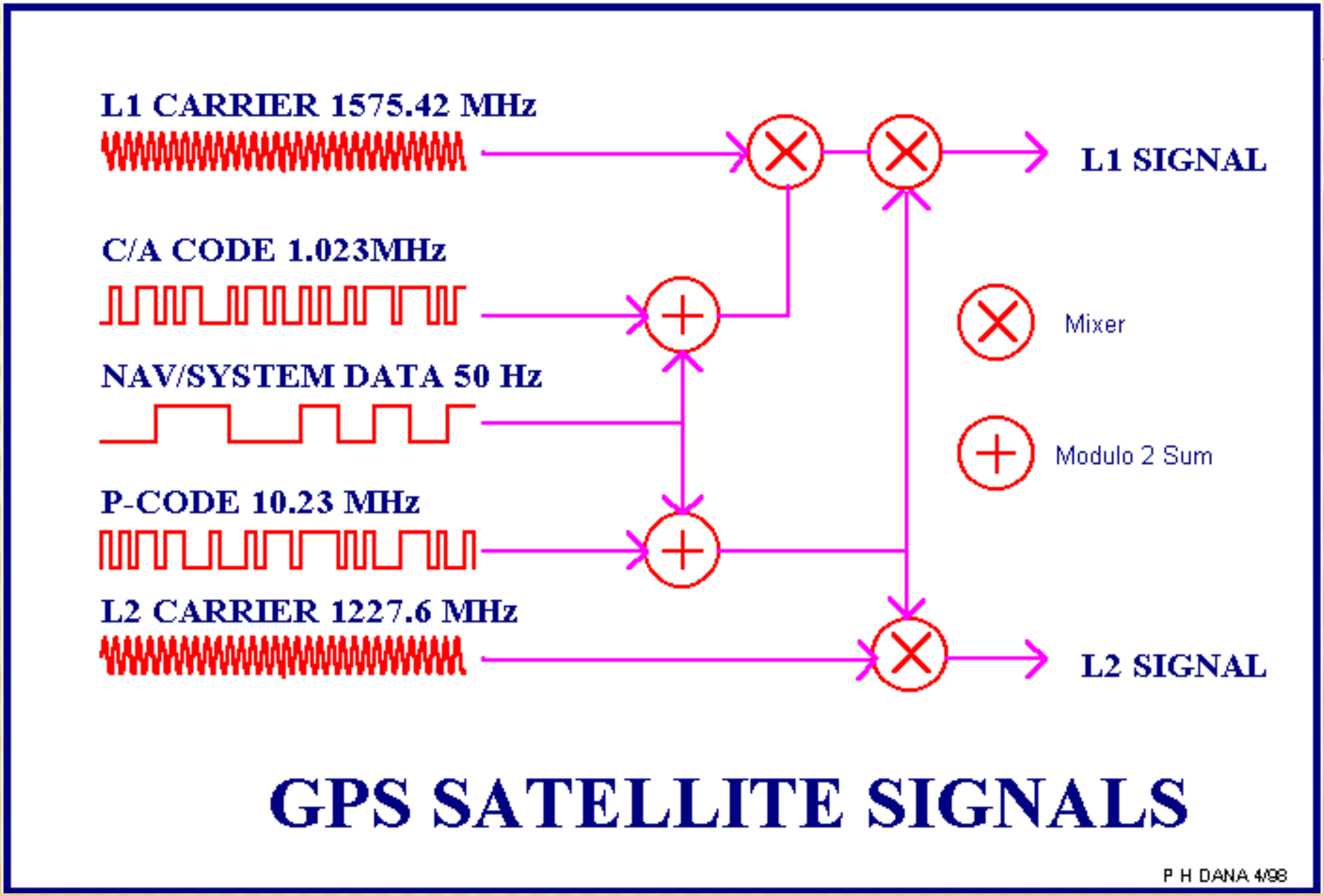
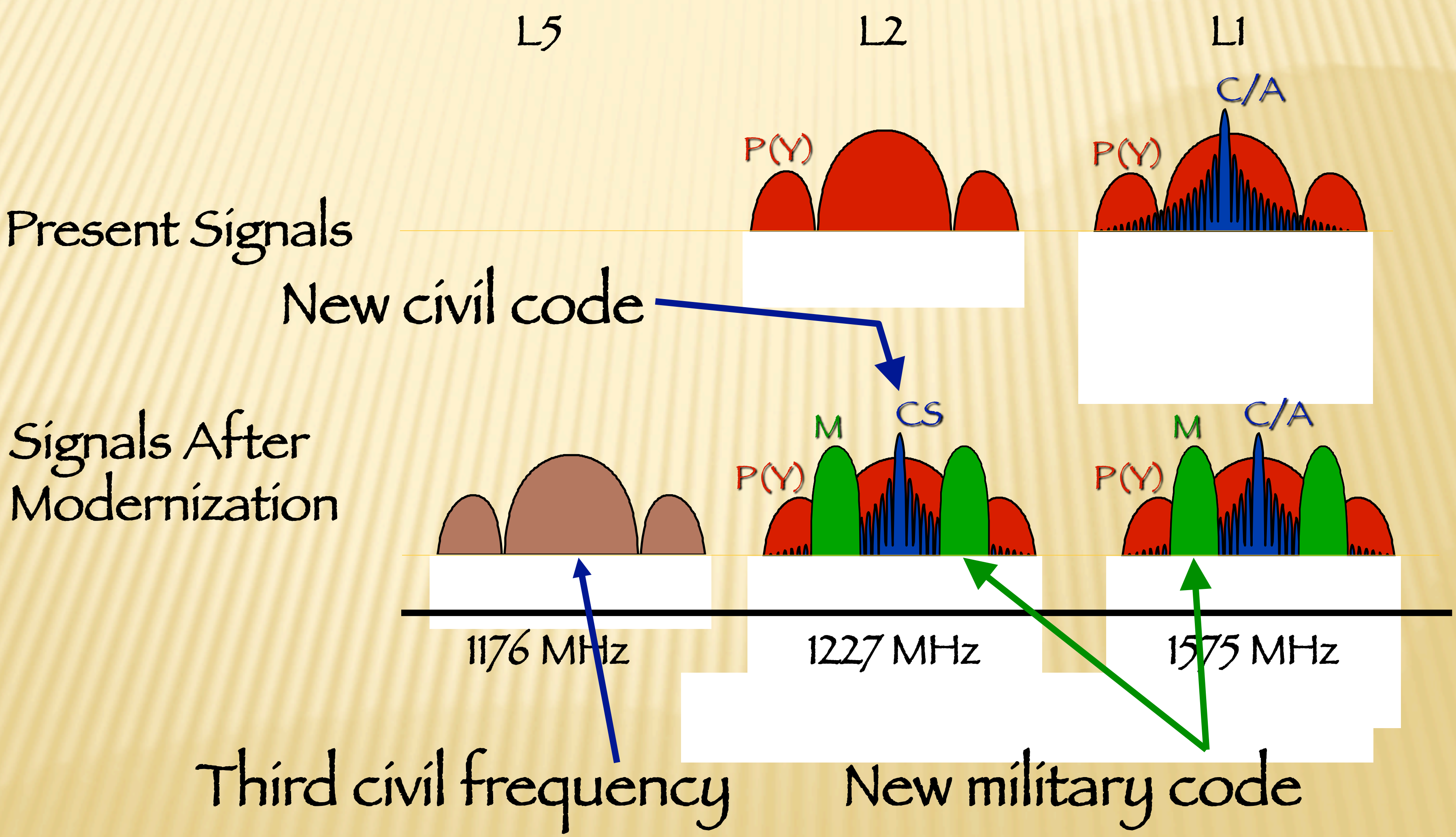


Image courtesy: Peter Dana, http://www.colorado.Edu/geography/gcraft/notes/gps/gps_f.html

A. Ganse, U. Washington , http://staff.washington.edu/aganse/

Modernized Signal Evolution



DGPS overview, www.edu-observatory.org/gps/BostonSection.ppt , Dierendonck

Why Modernize?

National policy - GPS is a vital dual-use system

For civil users, new signals/frequencies provide:

More robustness against interference, compensation for ionospheric delays and wide/tri-laning

For military users, new signals provide:

Enhanced ability to deny hostile GPS use, greater military anti-jam capability and greater security

For both civil/military, system improvements in accuracy, reliability, integrity, and availability

DGPS overview, www.edu-observatory.org/gps/BostonSection.ppt , Dierendonck

generation of code - satellite and receiver

```
Time
Seconds

00000000001111111111222222222233333333334444444444555555
01234567890123456789012345678901234567890123456789012345 6

(Genesis – sent by satellite 1 and generated in receiver)
In the beginning God created the heavens and thIn the beg

(Exodus – sent by satellite 2 and generated in receiver)
These are the names of the sons of Israel who wThese are

(Leviticus – sent by satellite 3 and generated in
receiver)
Yahweh called Moses, and from the Tent of MeetiYahweh cal
```

code

(repeats)

"chip"

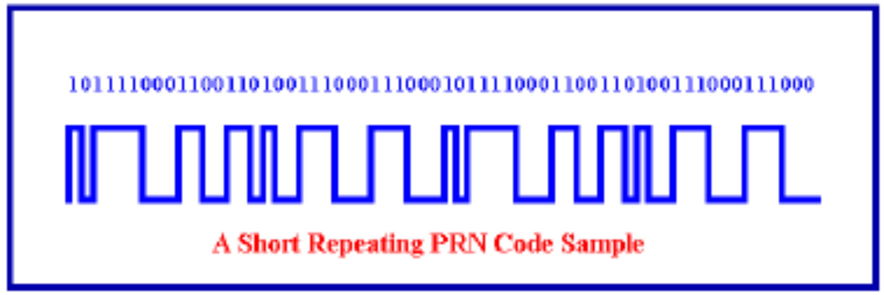
Reception of code in receiver

The time of the reception of the code is found by lining up the known and received signals

```
Time
Seconds

000000000011111111112222222222333333333344444444445555555
012345678901234567890123456789012345678901234567890123456

              In the beginning God created the heavens an
              ^14 seconds

     These are the names of the sons of Israel who wThese
     ^5 seconds
                      Yahweh called Moses, and from the T
                      ^22 seconds
```

GPS Signals (Code)

- Coarse/Acquisition (C/A) codes enable:
  - Simultaneous measurements from many satellites (CDMA)
  - Accurate absolute range (10-100m) using signal propagation delay



A Short Repeating PRN Code Sample

- Each satellite has a unique C/A code
  - One of a repeating sequence 1023 **chips** long
  - Rate of 1.023MHz (period of 1ms)
  - Appear random (pseudo random - PRN), but one of the deterministic “Gold Codes”

GPS Code Signal Acquisition

- Receiver replicates the C/A code to **correlate** with the measured signal

- What is **correlation**?
  - Mathematics:

$$\Psi_{ij}(\tau) = \frac{1}{T}\int_0^T \left[c_i(t)\tilde{c}_j(t-\tau)\right]dt$$

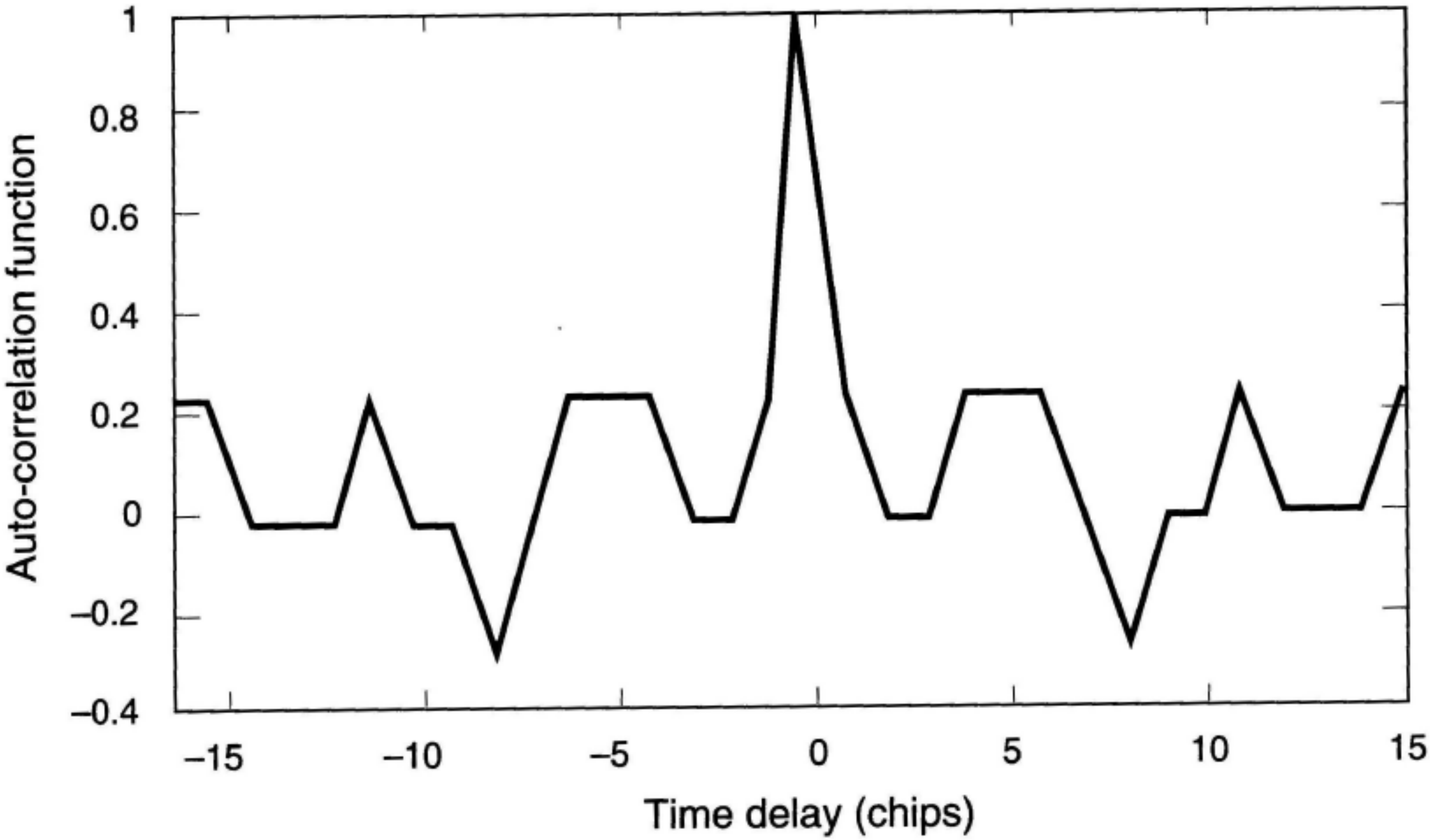
- Receiver slides code replica (τ) until strong correlation found with transmitted signal
  - With N "chips" in the signal period

    ➔ Correlation ≈ (# Agree - # Differ)/N

- Comparison with wrong code will produce very low agreement for all τ
  - Right code, wrong τ also produces low agreement

From J. HOW, MIT

Correlation Visualization



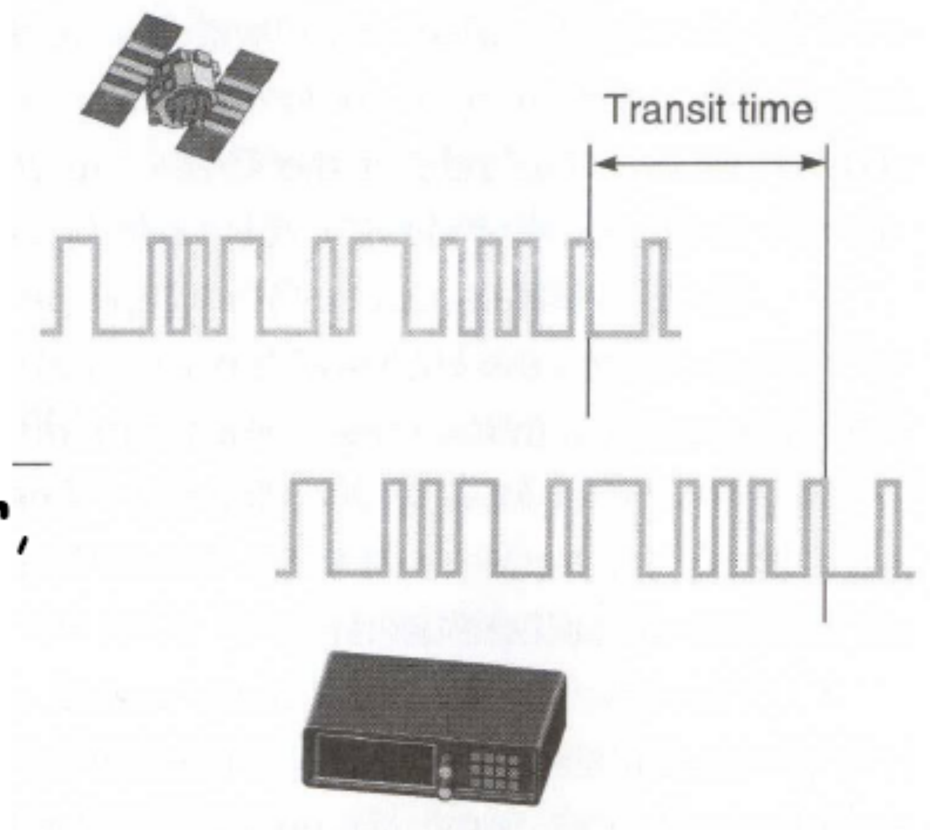
Full Correlation (Code-Phase Lock) of Receiver and Satellite PRN Codes

From J. HOW, MIT

GPS Signal Processing

- Correlation:
  - Allows receiver to pick up the very weak signal
  - Allows us to determine the time offset of the incoming signal

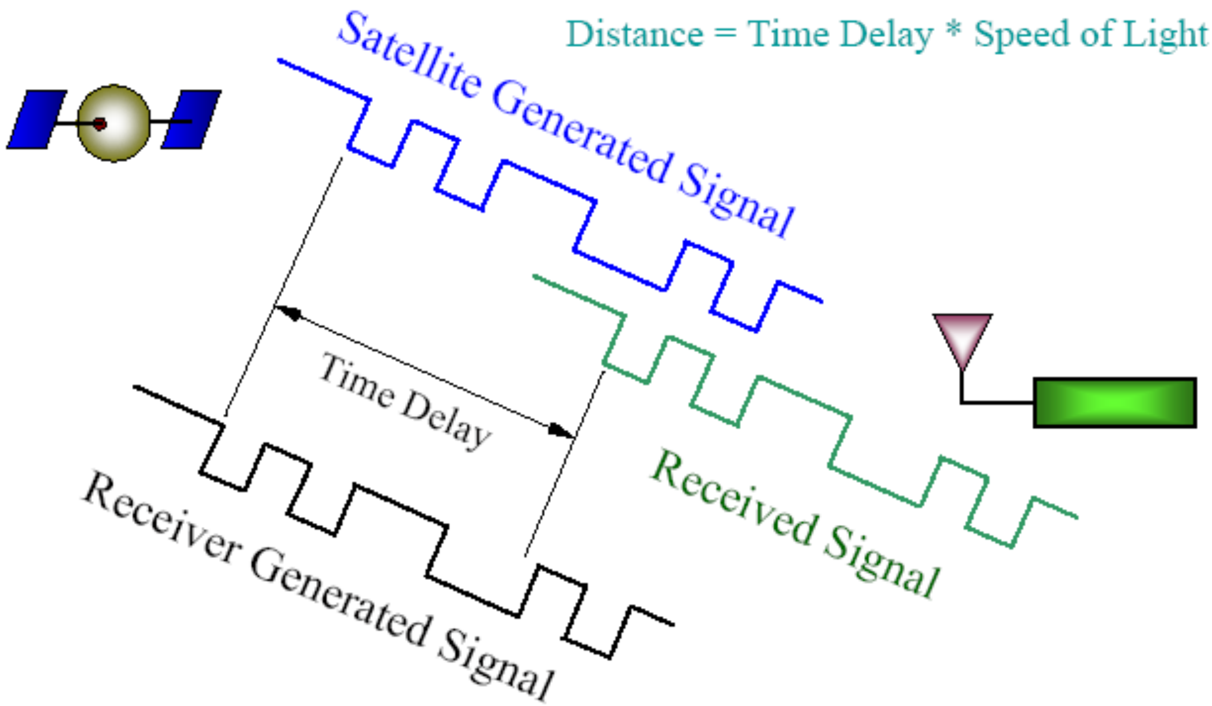
➔ **Transit Time**



- Transit time will be off _ by the **user clock error**, but we can solve for & remove that term

From J. HOW, MIT

Measuring Distance



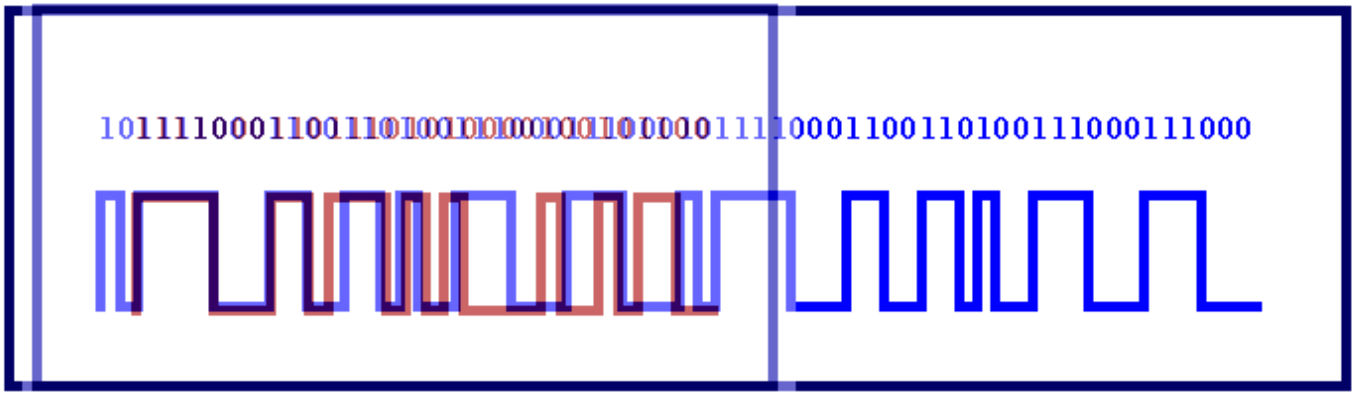
http://www.unav-micro.com/about_gps.htm

GPS Signal Acquisition Process

- Determine which satellites are visible
  - Approximate time/position, GPS almanac ➔ skyplot
- Determine approximate Doppler for each satellite
  - Reduces time-to-first-fix since defines frequency search space (estimate of receiver clock drift required)
- Must search **both** frequency and C/A code phase
  - Large velocity of GPS satellites, so the received signals can have a large Doppler shift (+/- 5 KHz) that can vary rapidly
  - Accurate frequency knowledge required to "strip off" the carrier part of the signal
    - Estimate frequency range to search using Doppler shift
    - Search frequency range using ~20 *frequency bins* of 500 Hz

From J. HOW, MIT

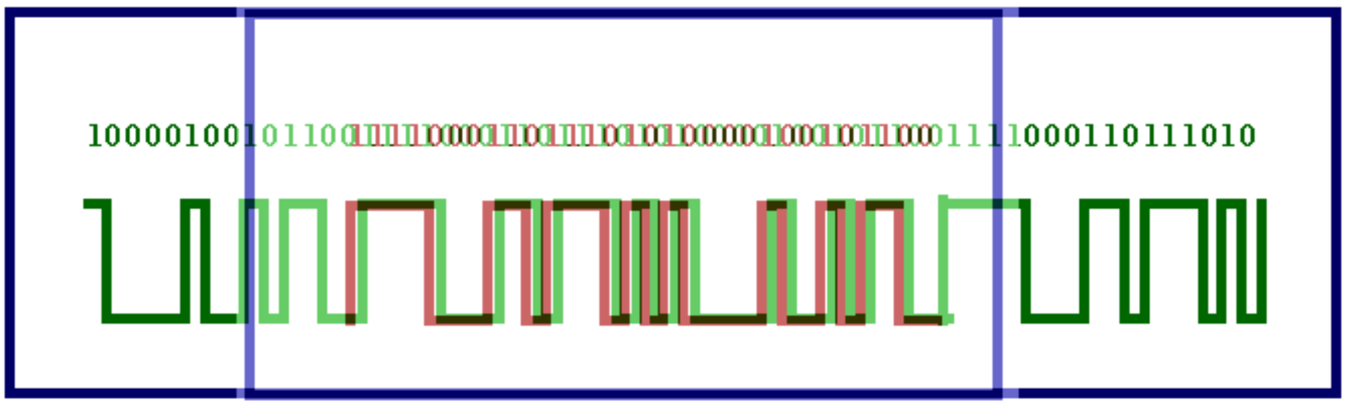
if receiver applies different PRN code to SV signal …no correlation



No Correlation with a Different PRN Code

Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

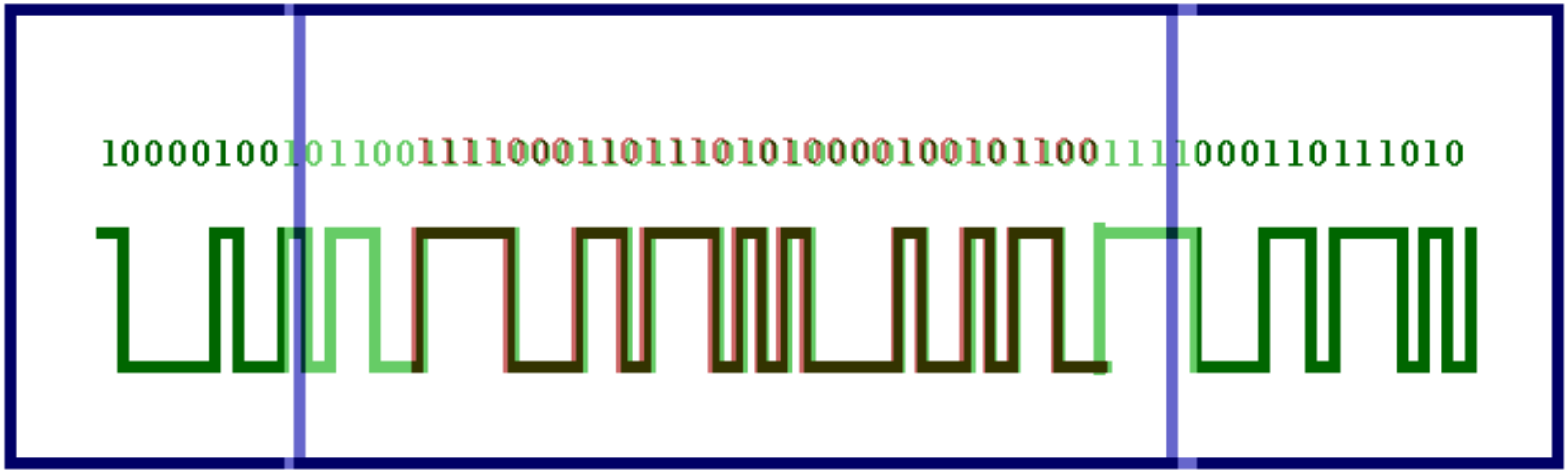
when receiver uses same code as SV and codes begin to align
…some signal power detected



Partial Correlation of Identical Receiver and Satellite PRN Codes

Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

when receiver and SV codes align completely
...full signal power detected



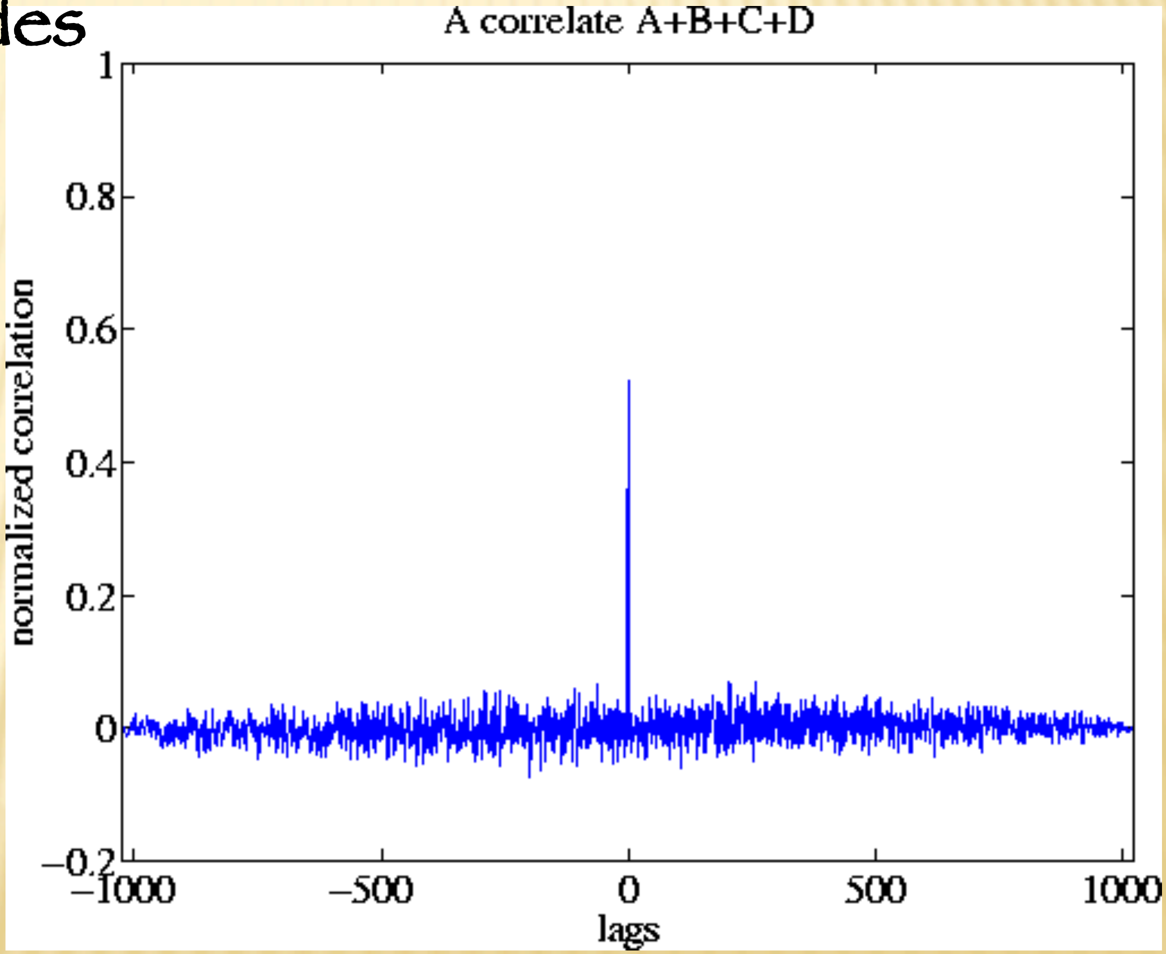
Full Correlation (Code-Phase Lock) of Receiver and Satellite PRN Codes

usually a late version of code is compared with early version
to insure that correlation peak is tracked

Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

PRN Cross-correlation

Correlation of receiver generated PRN code (A) with incoming data stream consisting of multiple (e.g. four, A, B, C, and D) codes



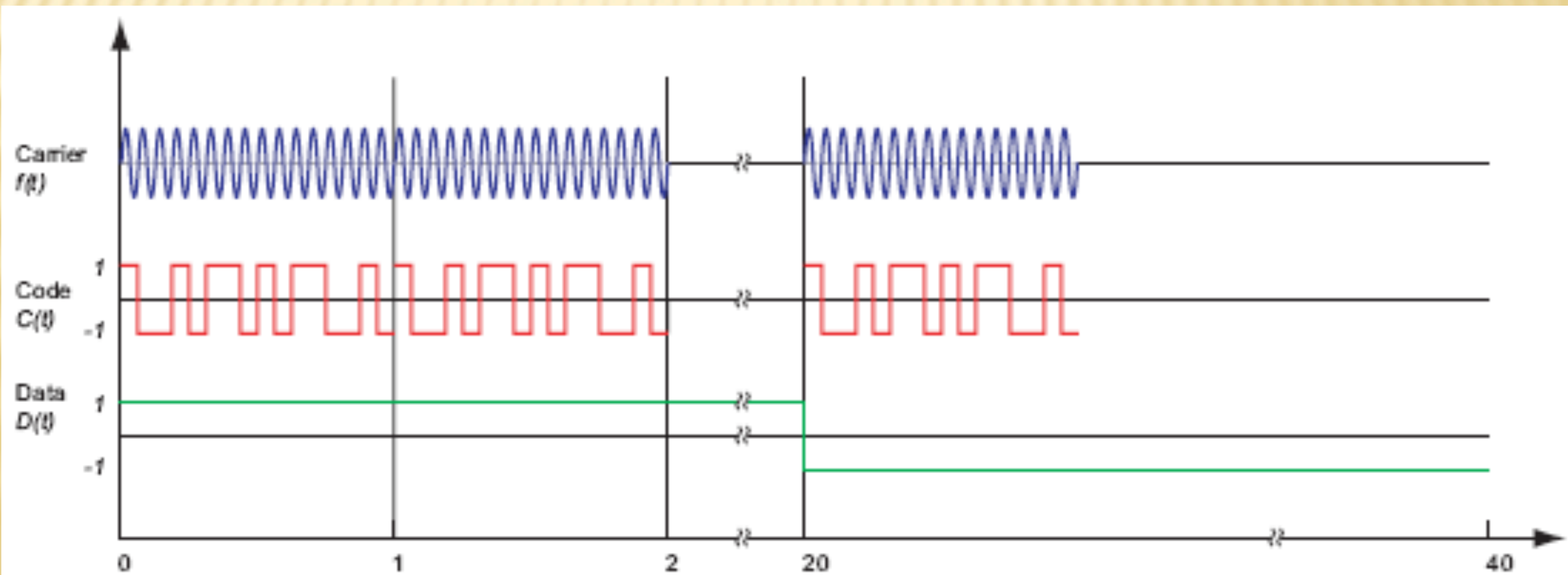
Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html

Construction of L1 signal

Carrier – blue

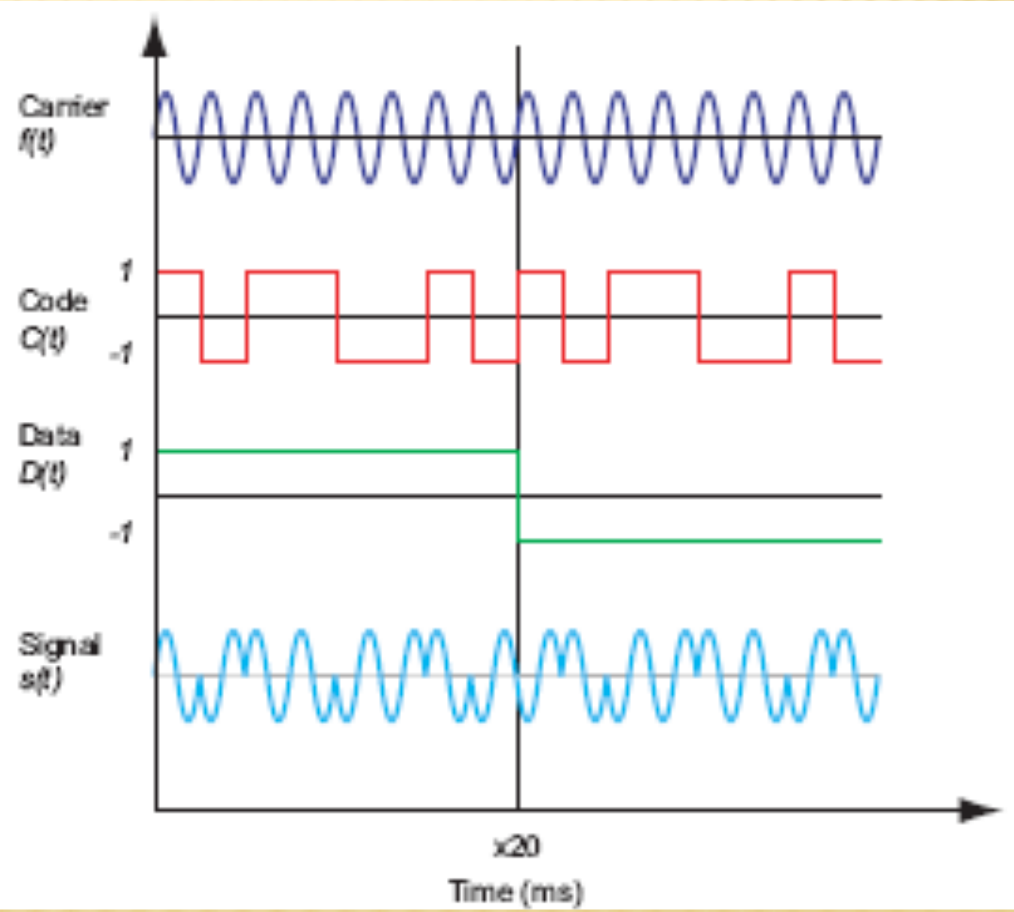
C/A code sequence – red, 1 bit lasts ~1 µsec, sequence of ~1000 bits repeats every 1 ms

Navigation data – green, one bit lasts 20 ms (20 C/A sequences)



Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Construction of L1 signal
BPSK modulation
(Carrier) x (C/A code) x (navigation message) = L1 signal



Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Digital Modulation Methods

Amplitude Modulation (AM) also known as amplitude-shift keying. This method requires changing the amplitude of the carrier phase between 0 and 1 to encode the digital signal.

Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

Digital Modulation Methods

Frequency Modulation (FM) also known as frequency-shift keying. Must alter the frequency of the carrier to correspond to 0 or 1.

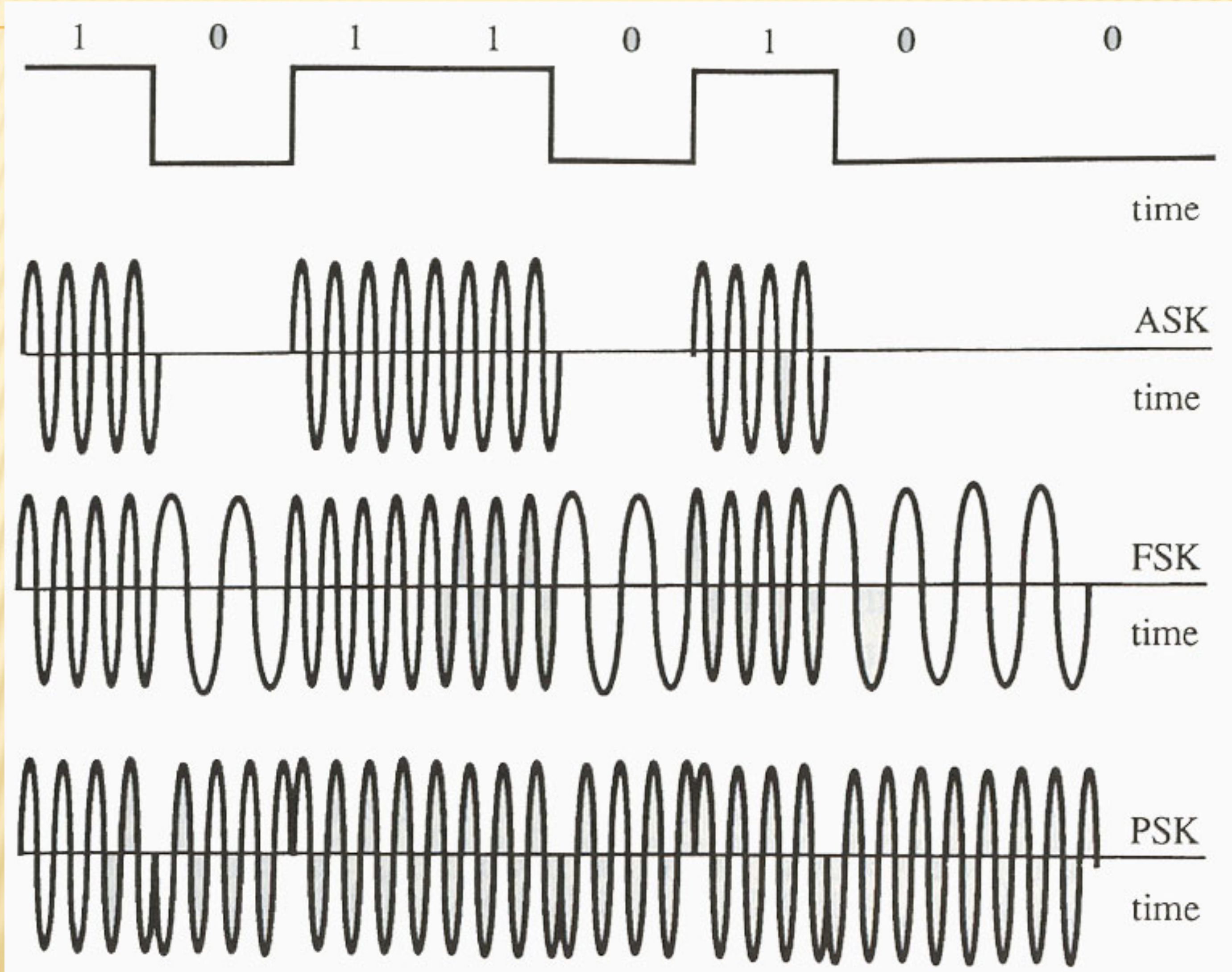
Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

Digital Modulation Methods

Phase Modulation (PM) also known as phase-shift keying. At each phase shift, the bit is flipped from 0 to 1 or vice versa. This is the method used in GPS.

Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

Modulation Schematics



Mattioli-http://comp.uark.edu/~mattioli/geol_4733.html and Dana

Nearly no cross-correlation.

C/A codes nearly uncorrelated with one another.

$$R_{ik}(n) = \sum_{l=0}^{1022} c^{i}(l)\, c^{k}(l+n) \approx 0,\ \forall n$$

Nearly no auto-correlation, except for zero lag

C/A codes nearly uncorrelated with themselves, except for zero lag.

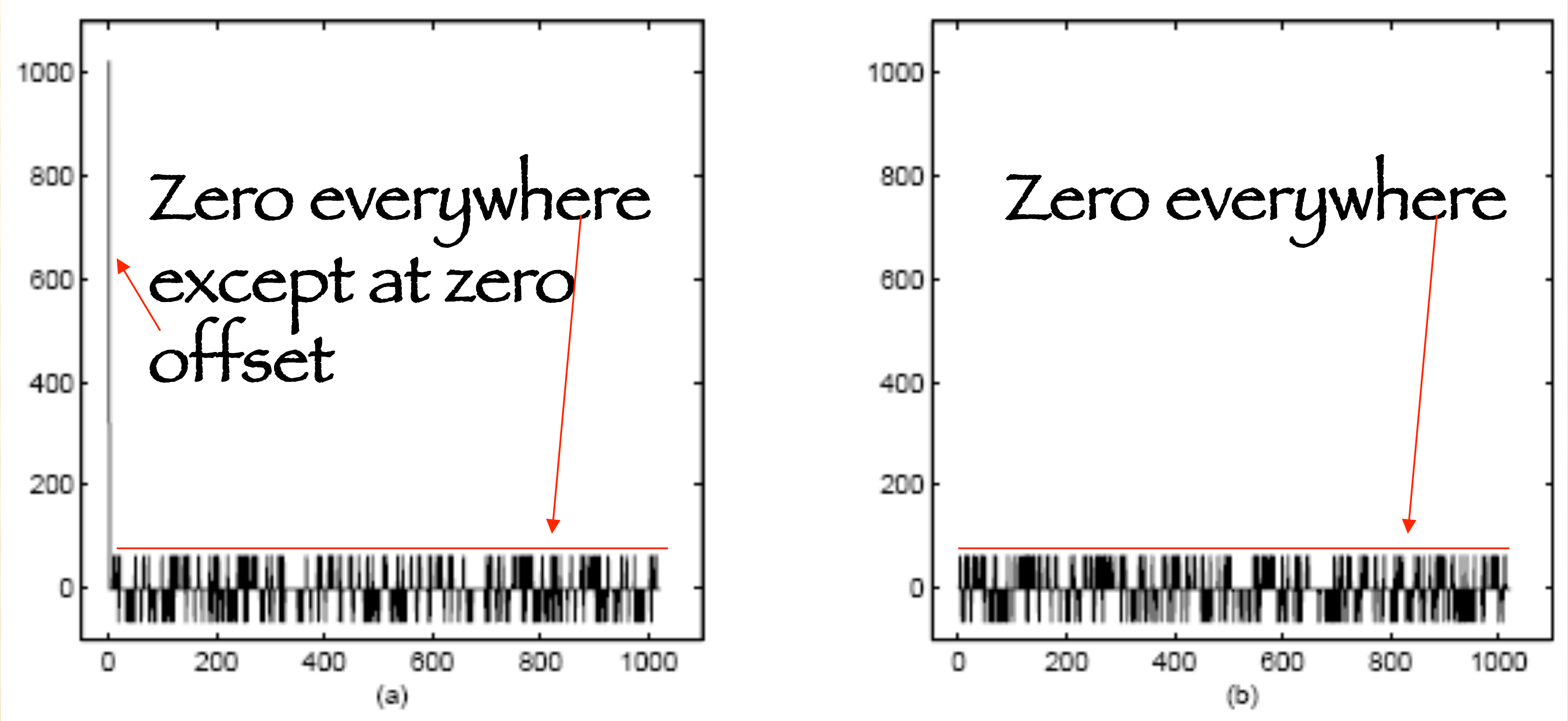
$$R_{kk}(n) = \sum_{l=0}^{1022} c^{k}(l)\, c^{k}(l+n) \approx 0,\ \forall |n| \geq 0$$

Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Gold Code correlation properties

Auto-correlation with itself (narrow peak, 1023)

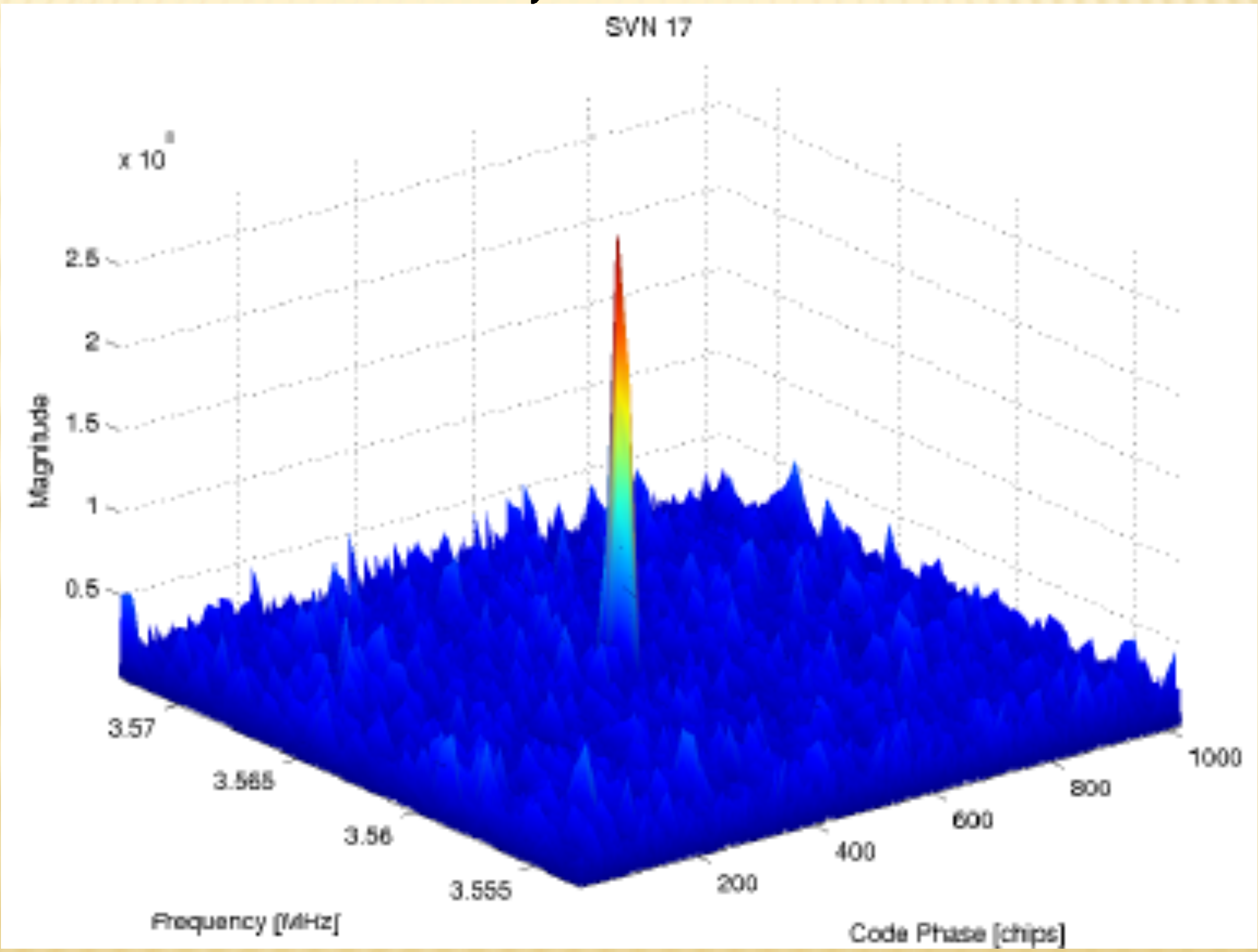
Cross-correlation with another code



Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Signal acquisition

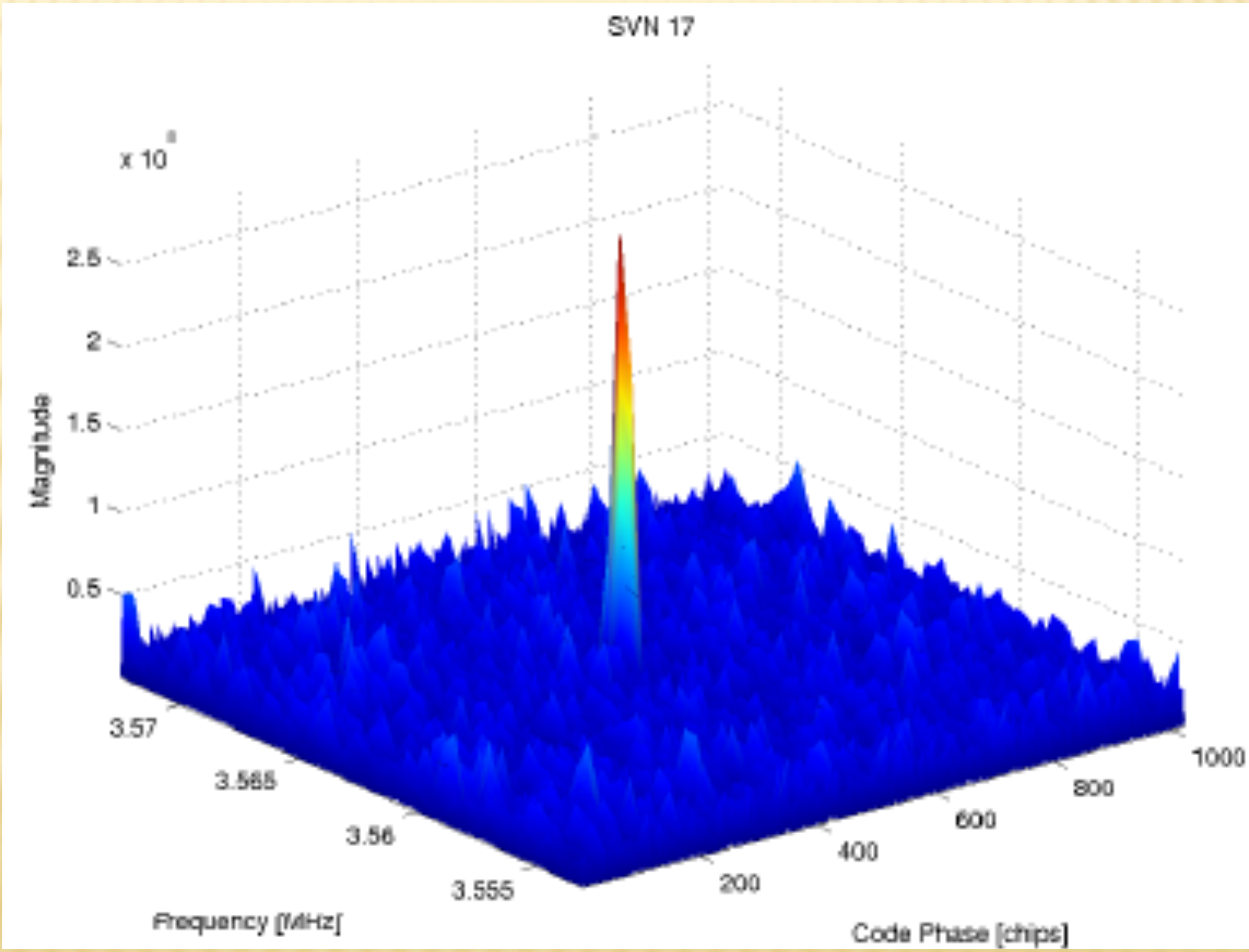
Is a search procedure over correlation by frequency and code phase shift



kom.aau.dk/~rinder/AAU_software_receiver.pdf

Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

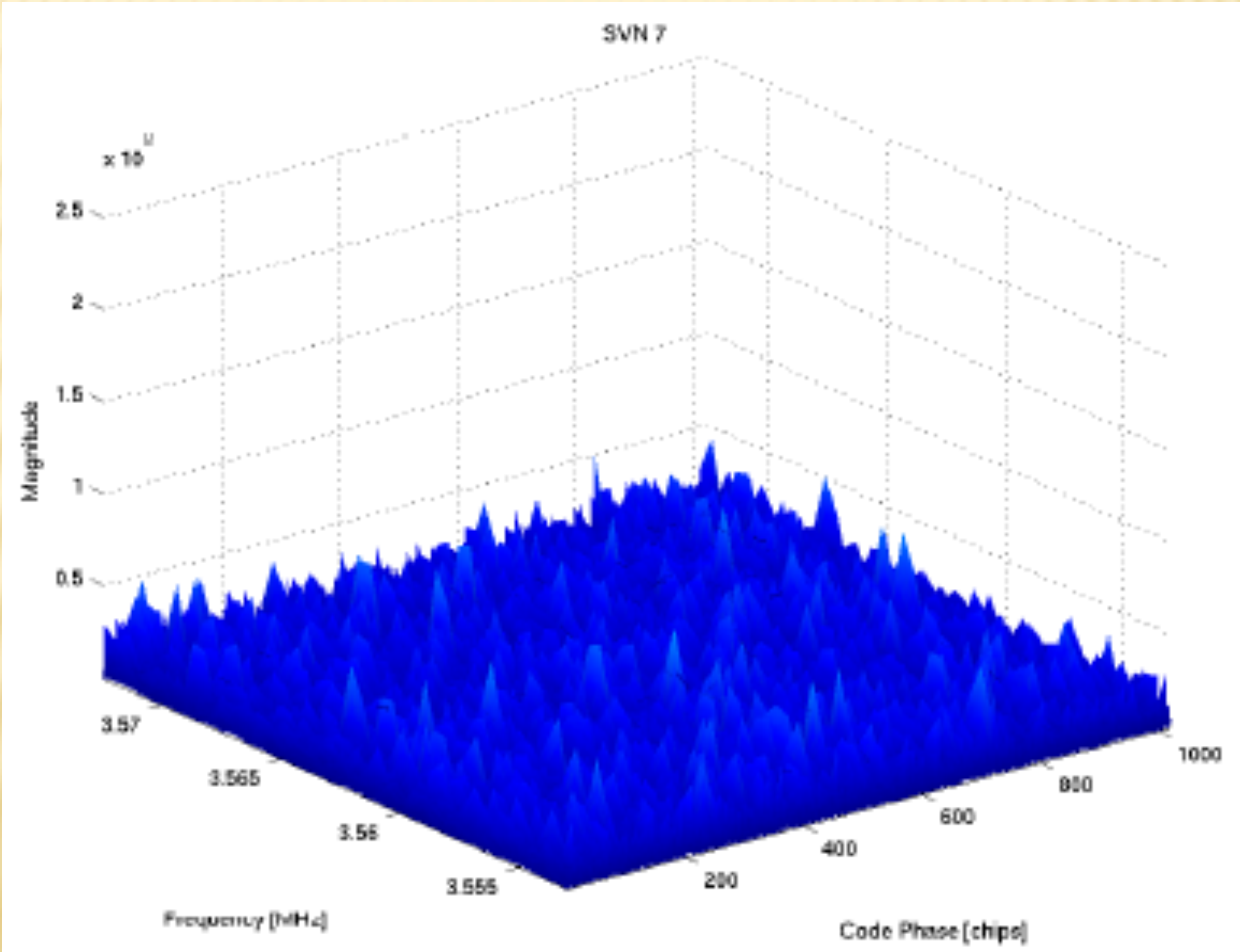
Search resulting grid of correlations for maximum, if above some threshold signal has been detected at some frequency and phase shift.



kom.aau.dk/~rinder/AAU_software_receiver.pdf

Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Search resulting grid of correlations for maximum, it it is small everywhere, below threshold, no signal has been detected.



Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

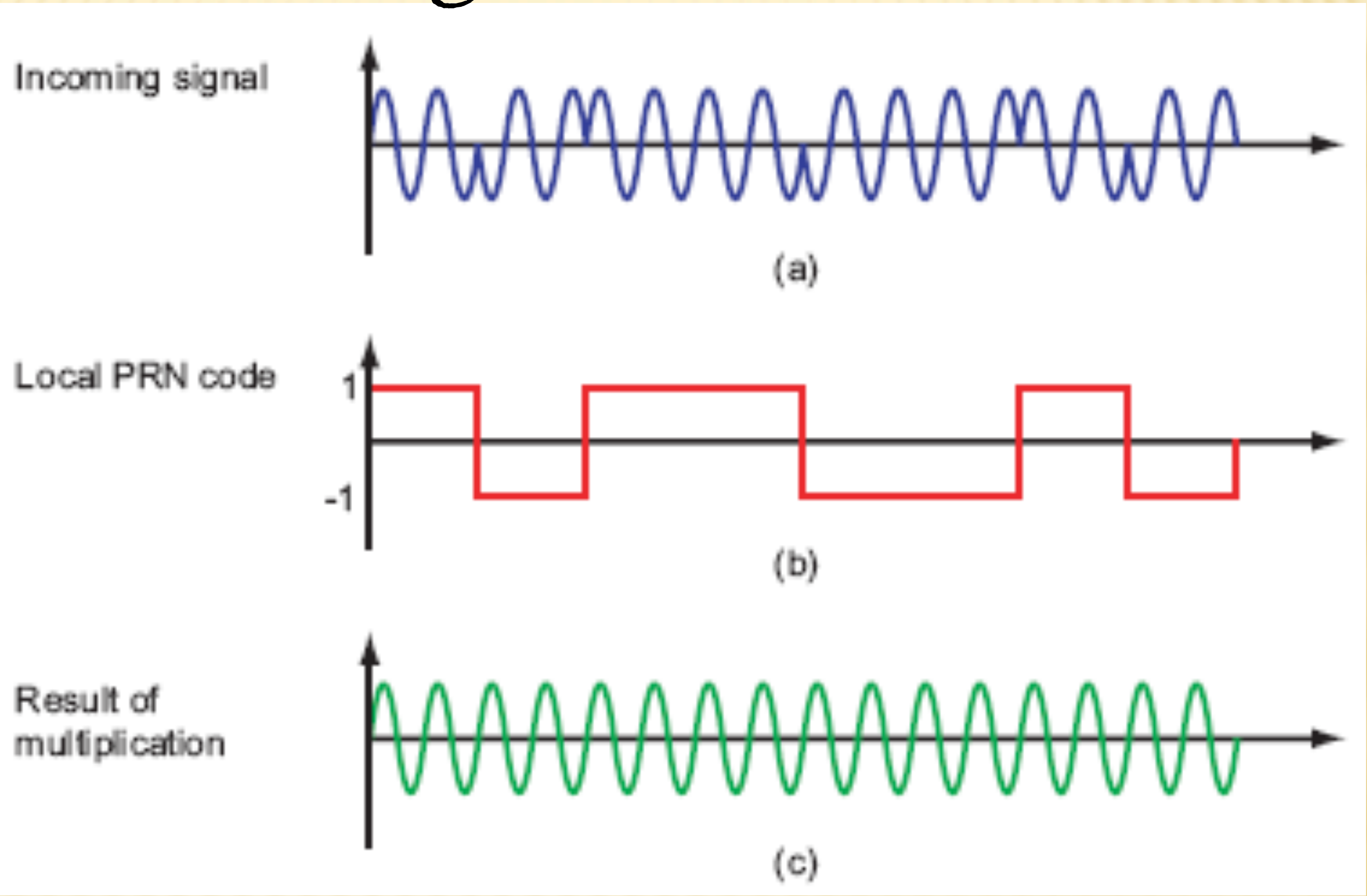
This method,

while correct and useful for illustration,

is too slow for practical use

Recovering the signal

What do we get if we multiply the L1 signal by a perfectly aligned C/A code?

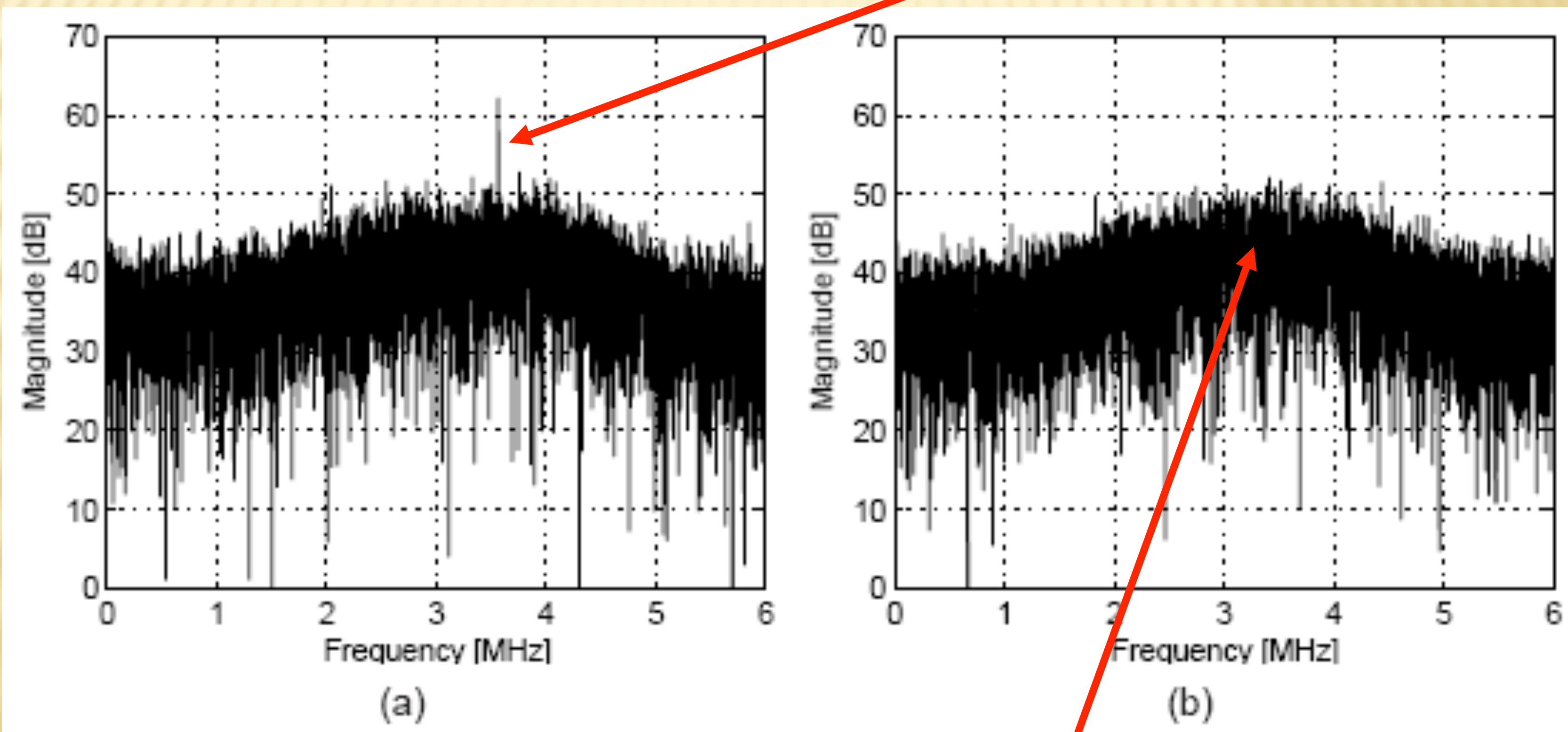


A sine wave!

Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Recovering the signal

Fourier analysis of this indicates the presence of the signal and identifies the frequency



No signal

Rinder and Bertelsen, kom.aau.dk/~rinder/AAU_software_receiver.pdf

Additional information included in GPS signal
Navigation Message

In order to solve the user position equations, one must know where the SV is.

The navigation and time code provides this

50 Hz signal modulated on L1 and L2

Navigation Message

The SV's own position information is transmitted in a 1500-bit data frame

Pseudo-Keplerian orbital elements, fit to 2-hour spans

Determined by control center via ground tracking

Receiver implements orbit-to-position algorithm

Navigation Message

Also includes clock data and satellite status

And ionospheric/tropospheric corrections

Additional information on GPS signal

The Almanac

In addition to its own nav data, each SV also broadcasts info about ALL the other SV's

In a reduced-accuracy format

Known as the *Almanac*

The Almanac

Permits receiver to predict, from a cold start, "where to look" for SV's when powered up

GPS orbits are so predictable, an almanac may be valid for months

Almanac data is large

12.5 minutes to transfer in entirety

Selective Availability (SA)

To deny high-accuracy realtime positioning to potential enemies, DoD reserves the right to deliberately degrade GPS performance

Only on the C/A code

By far the largest GPS error source

Selective Availability (SA)

Accomplished by:

1) “Dithering” the clock data

Results in erroneous pseudoranges

2) Truncating the navigation message data

Erroneous SV positions used to compute position

Selective Availability (SA)

Degrades SPS solution by a factor of 4 or more

Long-term averaging only effective SA compensator

FAA and Coast Guard pressured DoD to eliminate

ON 1 MAY 2000: SA WAS DISABLED BY PRESIDENTAL DIRECTIVE

How Accurate Is It?

Remember the 3 types of Lies:
Lies, Damn Lies, and Statistics…

Loosely Defined "2-Sigma" Repeatable Accuracies:

All depend on receiver quality

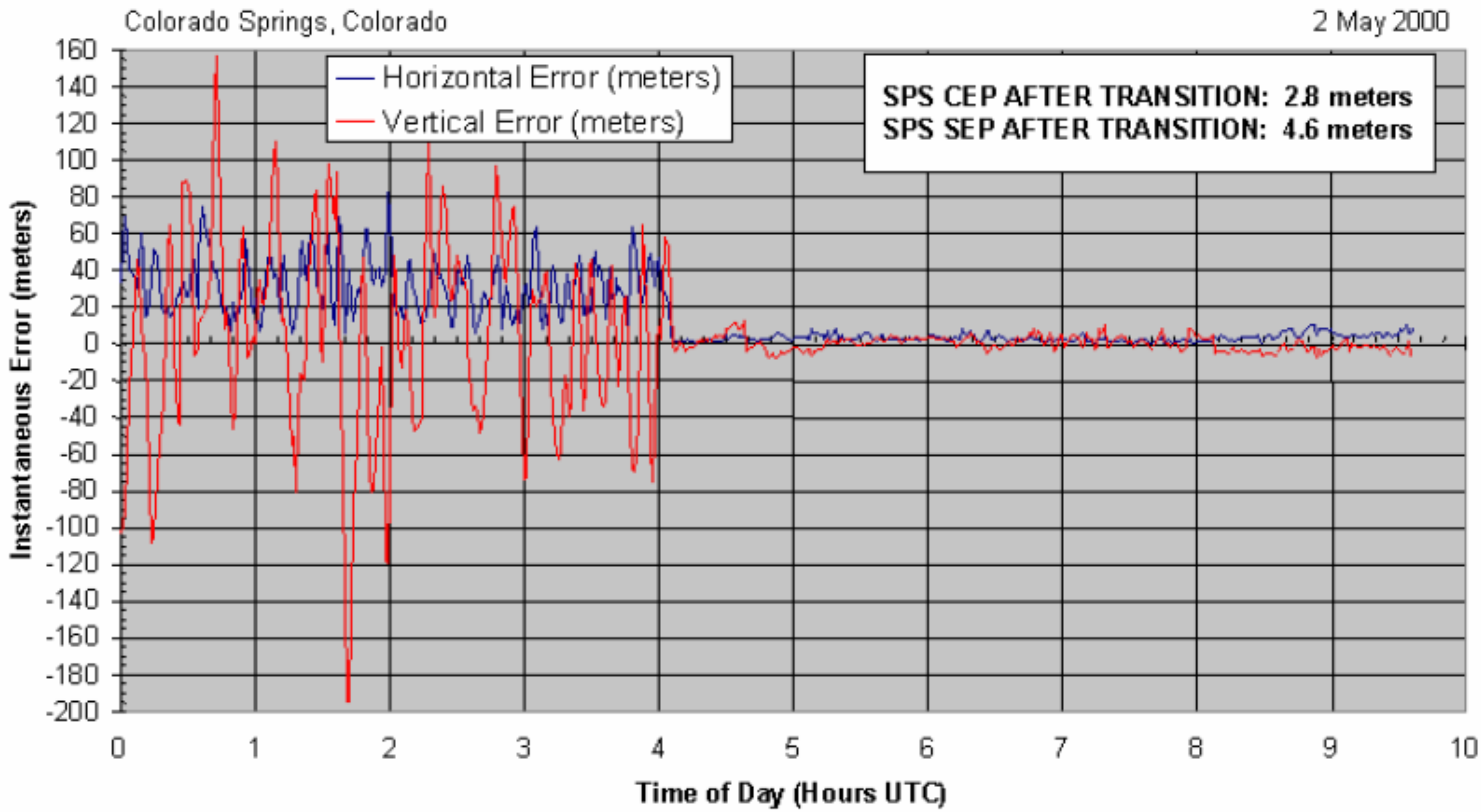
How Accurate Is It?

SPS (C/A Code Only)

S/A On:
Horizontal: 100 meters radial
Vertical: 156 meters
Time: 340 nanoseconds

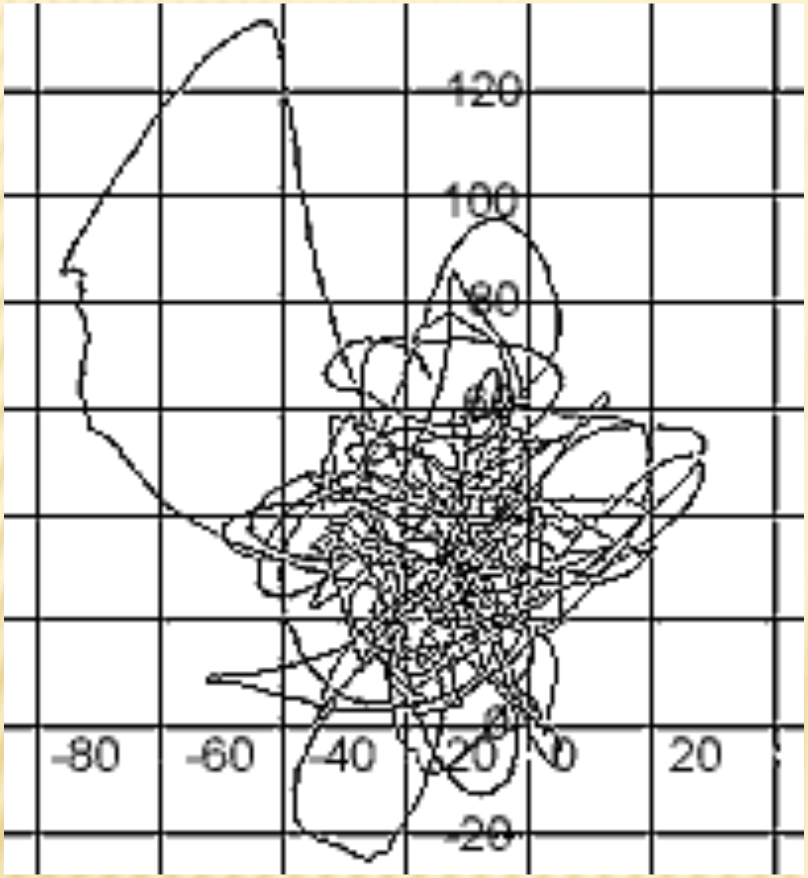
S/A Off:
Horizontal: 22 meters radial
Vertical: 28 meters
Time: 200 nanoseconds

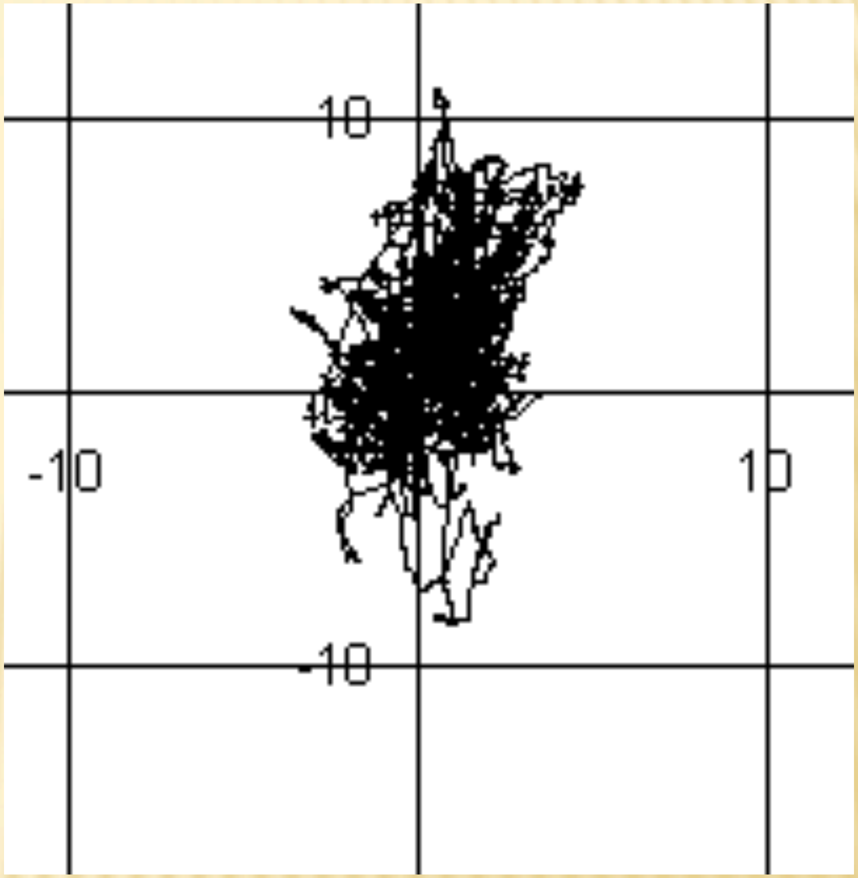
SA Transition -- 2 May 2000



From J. HOW, MIT

Position averages





5.5 hours S/A on

8 hours S/A off

Note scale difference

How Accurate Is It?

PPS (P-Code)

Slightly better than C/A Code w/o S/A (?)

Differential GPS

A reference station at a known location compares predicted pseudoranges to actual & broadcasts corrections: "Local Area" DGPS

Broadcast usually done on FM channel

Corrections only valid within a finite range of base
User receiver must see same SV's as reference
USCG has a number of DGPS stations operating (CORS network)

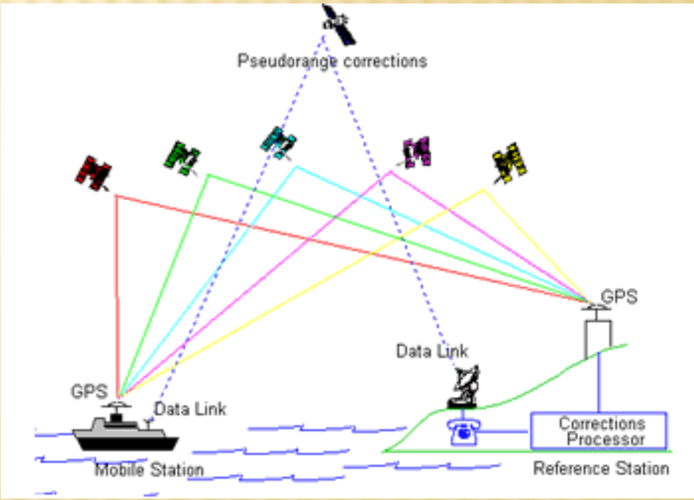
Differential GPS

Base stations worldwide collect pseudorange and SV ephemeris data and "solve-for" time and nav errors

"Wide Area" DGPS -- WAAS

Not yet globally available

DGPS can reduce errors to < 10 meters



A. Ganse, U. Washington , http://staff.washington.edu/aganse/

Carrier Phase Tracking

Used in high-precision survey work

Can generate sub-centimeter accuracy

The ~20 cm carrier is tracked by a reference receiver and a remote (user) receiver

The carrier is not subject to S/A and is a much more precise measurement than pseudoranges.

Carrier Phase Tracking

Requires bookeeping of cycles: subject to "slips" (loss of "lock" by the phase locked loop tracking each satellite)

Ionospheric delay differences must be small enough to prevent full slips

Requires remote receiver be within ~30km of base (for single frequency, short occupations)

Usually used in post-processed mode, but RealTime Kinematic (RTK) method is developing

Receivers

Basic 12 channel receivers start at $100

Usually includes track & waypoint entry

With built-in maps start at $150

Combination GPS receiver/cell phone ~$350

Receivers

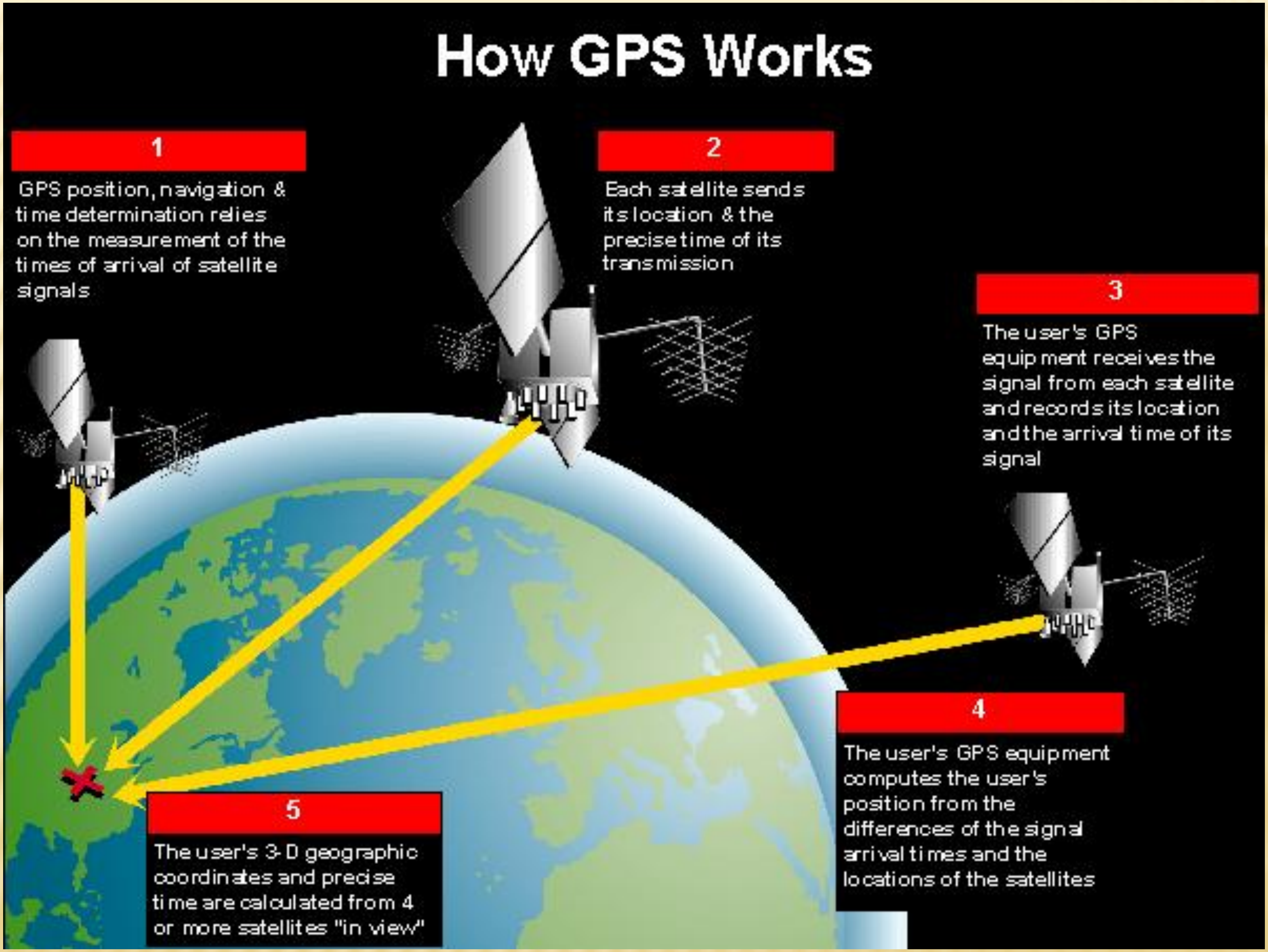
Survey-quality: $1000 and up

Carrier tracking

FM receiver for differential corrections

RS232 port to PC for realtime or post-processing

Point positioning with Psuedorange from code


How GPS Works
1
GPS position, navigation & time determination relies on the measurement of the times of arrival of satellite signals
2
Each satellite sends its location & the precise time of its transmission
3
The user's GPS equipment receives the signal from each satellite and records its location and the arrival time of its signal
4
The user's GPS equipment computes the user's position from the differences of the signal arrival times and the locations of the satellites
5
The user's 3-D geographic coordinates and precise time are calculated from 4 or more satellites "in view"

Code Point Positioning

- Write pseudorange as a function of
  - Spacecraft position X^k,...
  - Receiver position (ECEF) X_u,...
  - Clock errors of spacecraft and receiver

$$\tau_u^k = \left[\sqrt{(x_u - x^k)^2 + (y_u - y^k)^2 + (z_u - z^k)^2} + b_u - B^k\right] + \nu_u^k$$

  - Measure pseudorange from >= 4 satellites and you can solve for x_u, y_u, z_u, b_u

$$\tau_u^k = f^k(x_u, y_u, z_u, b_u) + \nu_u^k$$

From J. HOW, MIT, GPS-RJH5.pdf

Position Equations

$$P_1 = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2} + b$$
$$P_2 = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2} + b$$
$$P_3 = \sqrt{(X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2} + b$$
$$P_4 = \sqrt{(X - X_4)^2 + (Y - Y_4)^2 + (Z - Z_4)^2} + b$$

Where:

P_i = Measured PseudoRange to the i^{th} SV

X_i , Y_i , Z_i = Position of the i^{th} SV, Cartesian Coordinates

X , Y , Z = User position, Cartesian Coordinates, to be solved-for

b = User clock bias (in distance units), to be solved-for

The above nonlinear equations are solved iteratively using an initial estimate of the user position, XYZ, and b

Code Point Positioning

- Write pseudorange as a function of
  - Spacecraft position X^k,...
  - Receiver position (ECEF) X_u,...
  - Clock errors of spacecraft and receiver

$$\tau_u^k = \left[\sqrt{(x_u - x^k)^2 + (y_u - y^k)^2 + (z_u - z^k)^2} + b_u - B^k\right] + \nu_u^k$$

  - Measure pseudorange from >= 4 satellites and you can solve for x_u, y_u, z_u, b_u

$$\tau_u^k = f^k(x_u, y_u, z_u, b_u) + \nu_u^k$$

Linearization

Choose initial state estimate:

$$\bar{x}_0 = (x_{u0}, y_{u0}, z_{u0}, b_{u0})^T$$

Assume that actual state is given by

$$\bar{x} = \bar{x}_0 + \delta\bar{x}$$

Linearize the pseudorange measurement

$$\bar{\tau} = \bar{f}(\bar{x}) + \bar{\nu}_u \approx \bar{f}(\bar{x}_0) + \left.\frac{\partial \bar{f}}{\partial \bar{x}}\right|_{\bar{x}_0} (\bar{x} - \bar{x}_0) + \bar{\nu}_u$$

which can be rewritten as

$$\bar{\tau} - \bar{f}(\bar{x}_0) = \delta\bar{\tau} \approx G_{\bar{x}_0}\, \delta\bar{x} + \bar{\nu}_u$$

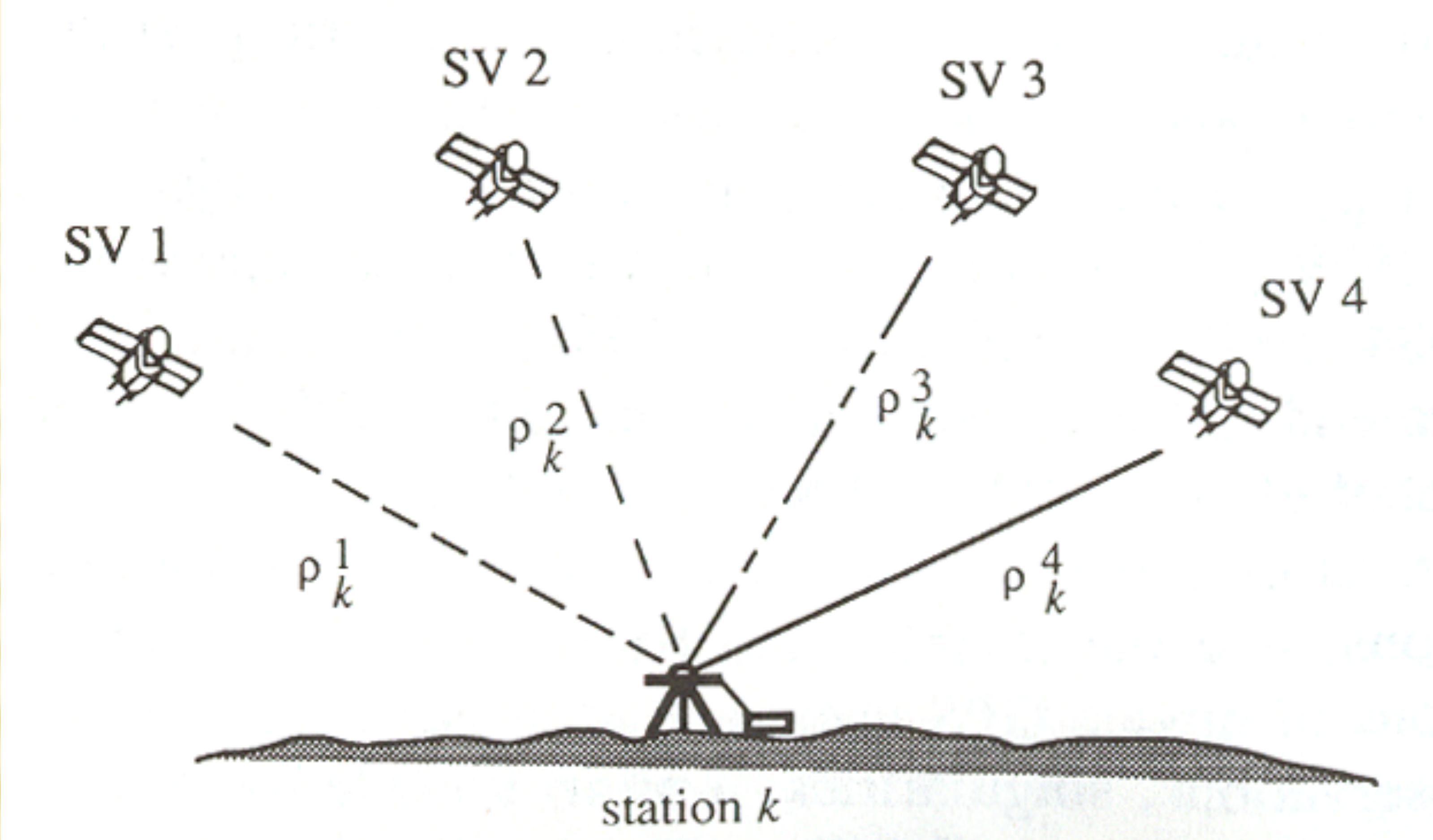
and then solved for $\delta\bar{x}$ to find the actual state

Pseudoranging

- Solution procedure fairly simple with 4 measurements - can just invert matrix G
- With more than 4 (normal case), must solve a least squares problem - "pseudo-inverse"

$$\delta\bar{\tau} = G\delta\bar{x} + \bar{\nu}_u \quad \Rightarrow \quad \delta\hat{\bar{x}} = (G^T G)^{-1} G^T \delta\bar{\tau}$$

- One complication is that linearization of G depends on our current best estimate of x
  - Which is (hopefully) improving → iteration might be required.



$$T^R = t^R + \tau^R \qquad T^S = t^S + \tau^S$$

$$P^{RS} = \left(\left(t^R + \tau^R\right) - \left(t^S + \tau^S\right)\right) c = \left(t^R - t^S\right) c + \left(\tau^R - \tau^S\right) c = \rho^{RS}\left(t^R, t^S\right) + \left(\tau^R - \tau^S\right) c$$

Pseudo range – measure <u>time</u>, not range.
Calculate range from *r=ct*

Blewitt, Basics of GPS in "Geodetic Applications of GPS"

From Pathagoras

$$\rho^{RS}\left(t^R,t^S\right)=\sqrt{\left(x^S\left(t^S\right)-x^R\left(t^R\right)\right)^2+\left(y^S\left(t^S\right)-y^R\left(t^R\right)\right)^2+\left(z^S\left(t^S\right)-z^R\left(t^R\right)\right)^2}$$

(x^S,y^S,z^S) and τ^S known from satellite navigation message

(x^R,y^R,z^R) and τ^R are 4 unknowns

Assume *c* constant along path, ignore relativity.

Complicating detail, satellite position has to be calculated at transmission time.

Satellite range can change by up to 60 m during the approximately 0.07 sec travel time from satellite to receiver.

Using receive time can result in 10's m error in range.

Calculating satellite transmit time

$$t^S(0) = t^R = \left(T^R - \tau^R\right)$$

$$t^S(1) = t^R - \frac{\rho^{SR}\left(t^R, t^S(0)\right)}{c}$$

$$t^S(2) = t^R - \frac{\rho^{SR}\left(t^R, t^S(1)\right)}{c}$$

$$\vdots$$

Start w/ receiver time, need receiver clock bias

(once receiver is operating clock bias is kept less than few milliseconds)

$$P^{R1}\left(t^R, t^1\right) = \sqrt{\left(x^1\left(t^1\right) - x^R\left(t^R\right)\right)^2 + \left(y^1\left(t^1\right) - y^R\left(t^R\right)\right)^2 + \left(z^1\left(t^1\right) - z^R\left(t^R\right)\right)^2} + \left(\tau^R - \tau^1\right) c$$

$$P^{R2}\left(t^R, t^2\right) = \sqrt{\left(x^2\left(t^2\right) - x^R\left(t^R\right)\right)^2 + \left(y^2\left(t^2\right) - y^R\left(t^R\right)\right)^2 + \left(z^2\left(t^2\right) - z^R\left(t^R\right)\right)^2} + \left(\tau^R - \tau^2\right) c$$

$$P^{R3}\left(t^R, t^3\right) = \sqrt{\left(x^3\left(t^3\right) - x^R\left(t^R\right)\right)^2 + \left(y^3\left(t^3\right) - y^R\left(t^R\right)\right)^2 + \left(z^3\left(t^3\right) - z^R\left(t^R\right)\right)^2} + \left(\tau^R - \tau^3\right) c$$

$$P^{R4}\left(t^R, t^4\right) = \sqrt{\left(x^4\left(t^4\right) - x^R\left(t^R\right)\right)^2 + \left(y^4\left(t^4\right) - y^R\left(t^R\right)\right)^2 + \left(z^4\left(t^4\right) - z^R\left(t^R\right)\right)^2} + \left(\tau^R - \tau^4\right) c$$

Note, have to keep track of which superscript is exponent and which is satellite or receiver (later we will have multiple receivers also) identification.

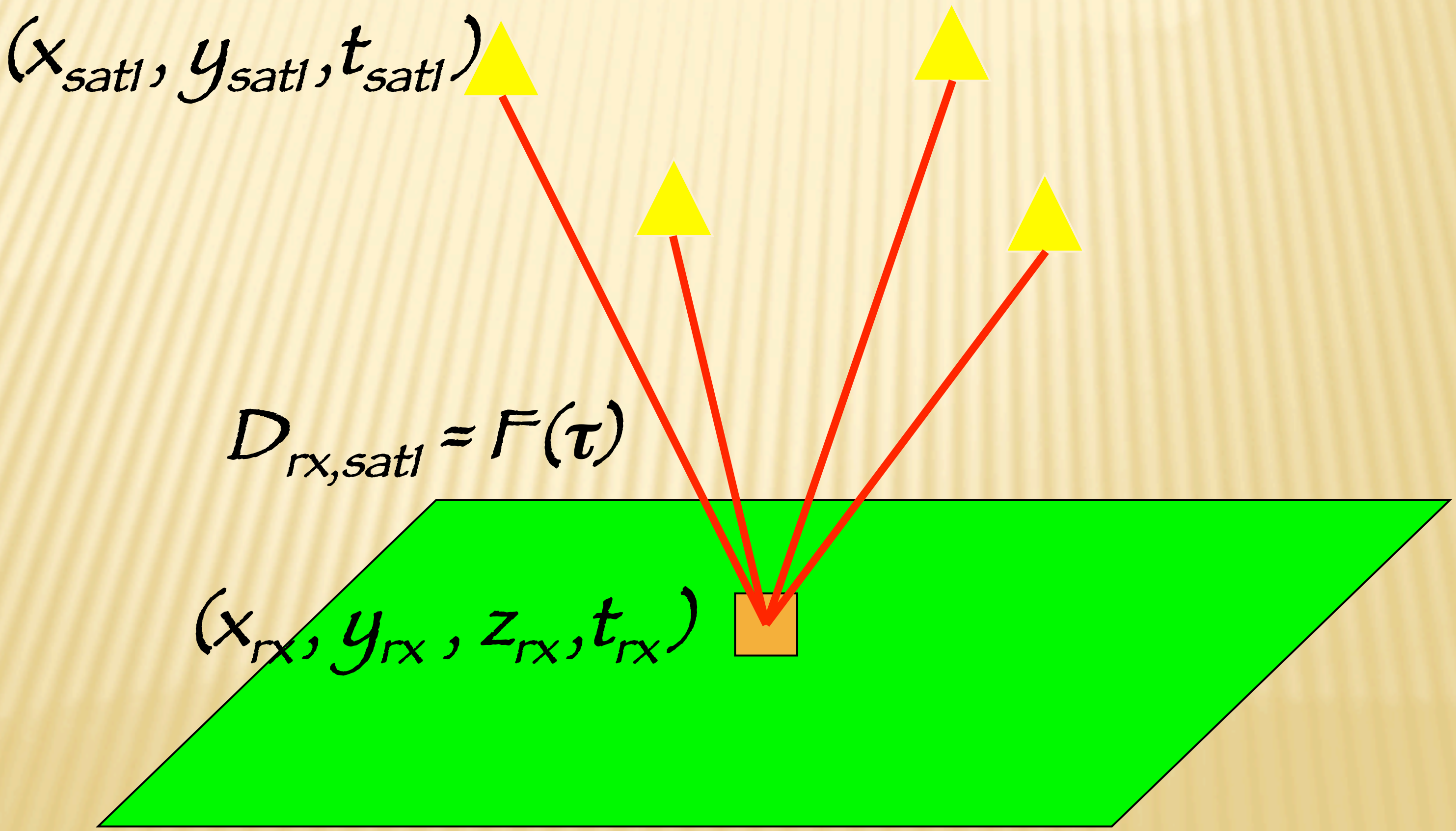
We have 4 unknowns (x^R, y^R, z^R and τ^R)
And 4 (nonlinear) equations
(later we will allow more satellites)
So we can solve for the unknowns

Blewitt, Basics of GPS in "Geodetic Applications of GPS"

GPS geometry

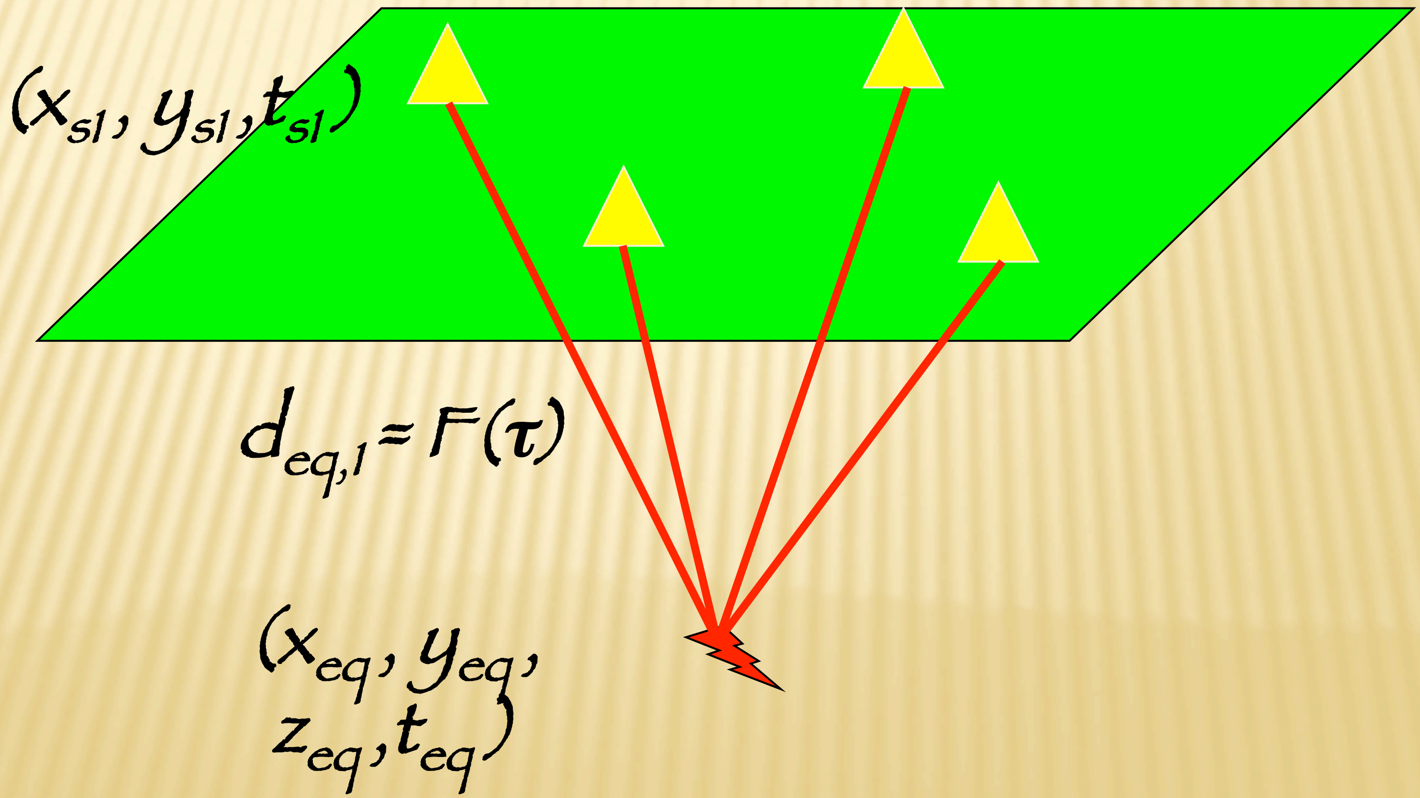
Raypaths (approximately) straight lines.

Really function of travel time (τ) but can change to psuedo-range.



Note that GPS location is almost exactly the same as the earthquake location problem.

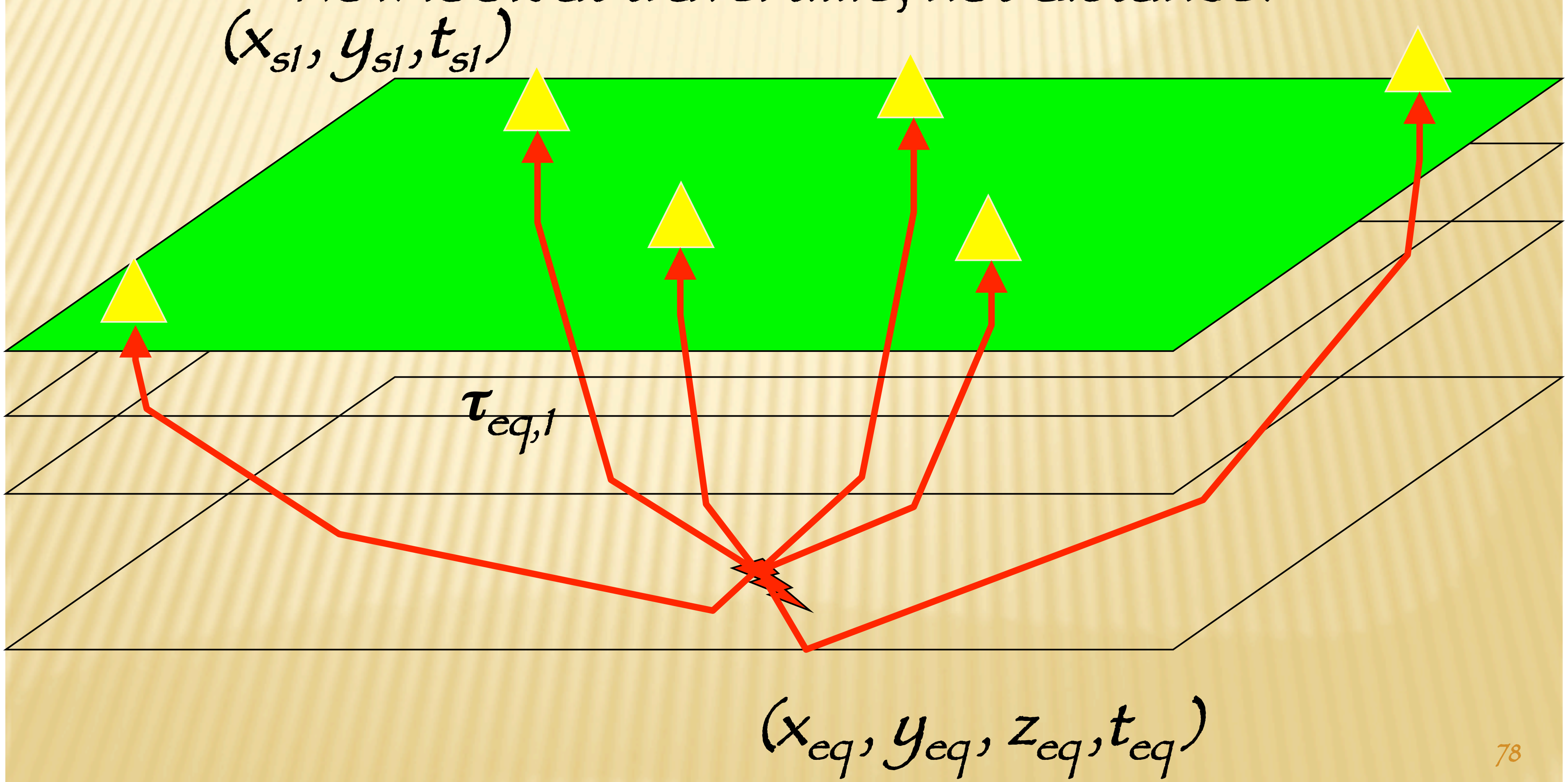
(in a homogeneous half space – raypaths are straight lines, again function of travel time but can also look at distance).



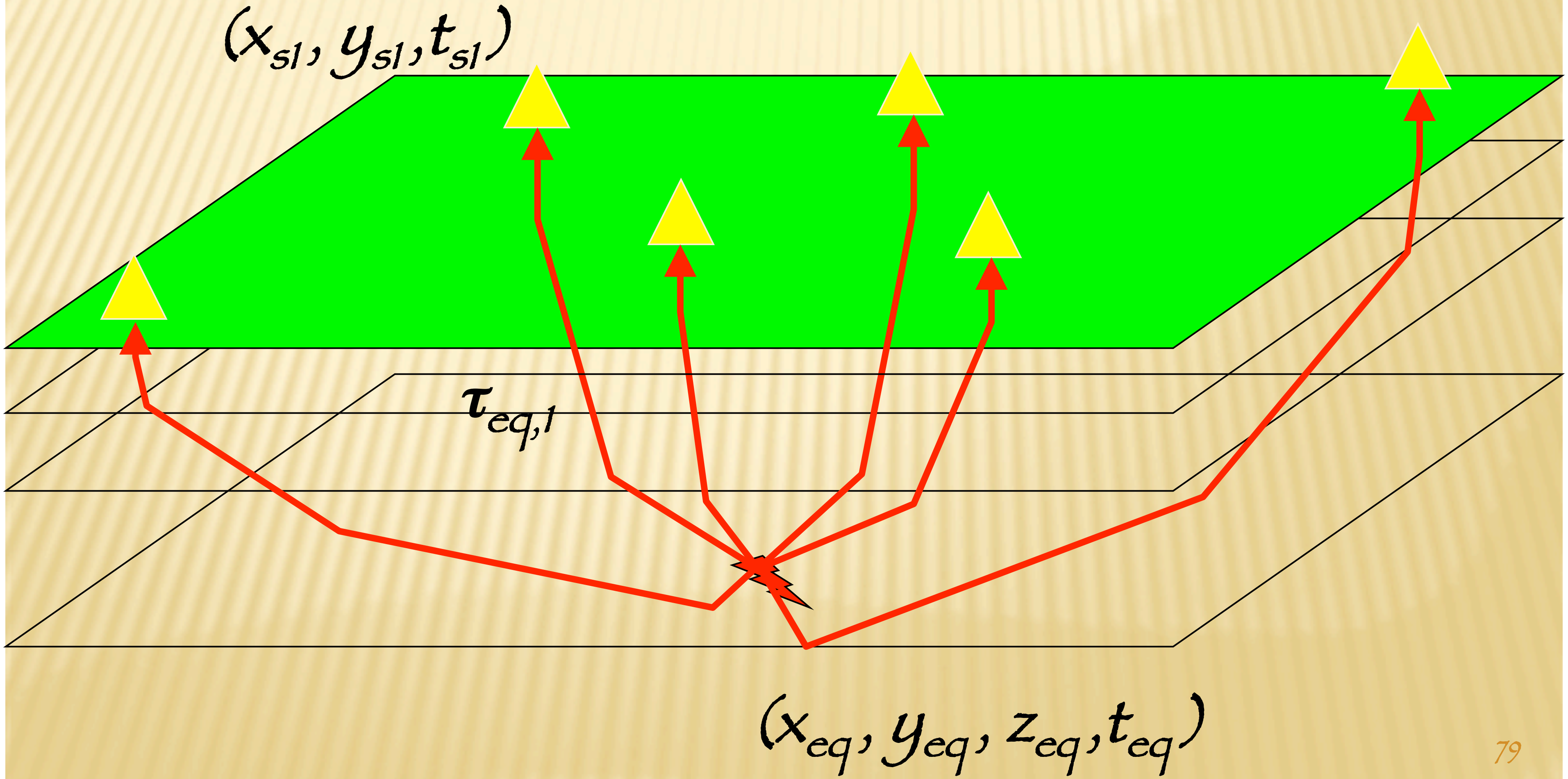
Lets look at more general problem of a layered half space.

Raypaths are now not limited to straight lines (mix of refracted and head waves shown).

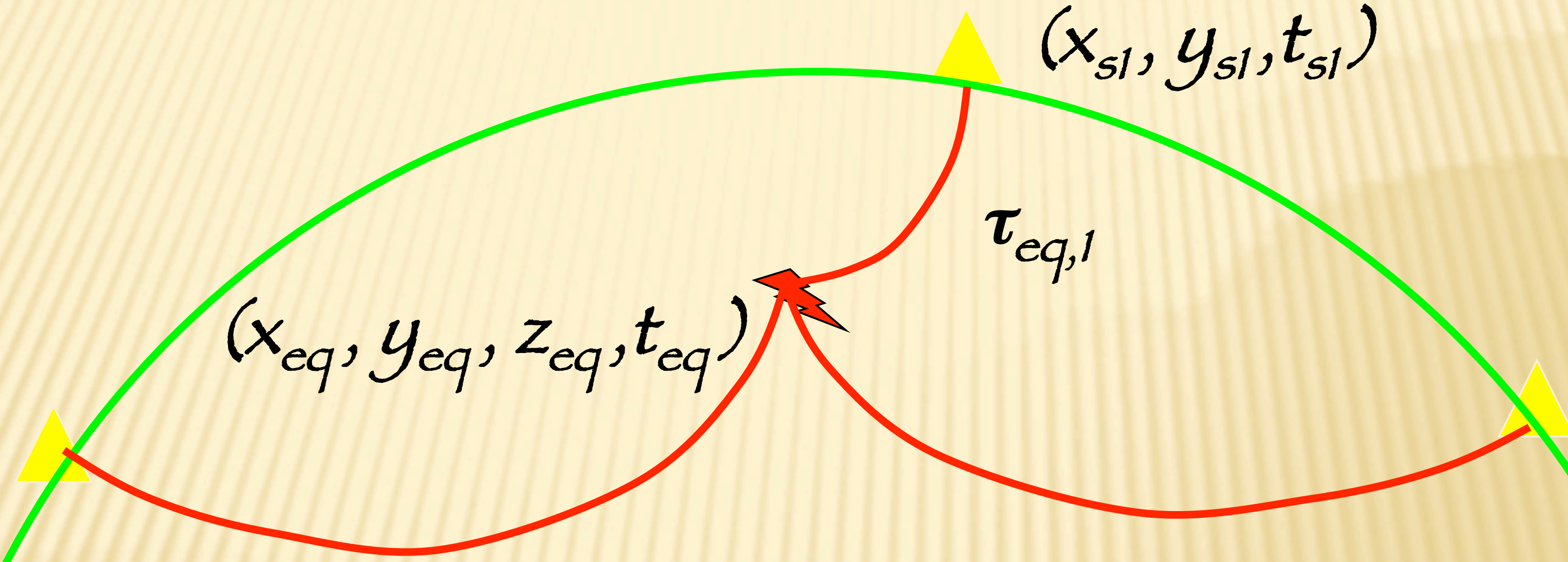
Now look at travel time, not distance.



This view will help us see a number of problems with locating earthquakes (some of which will also apply to GPS).



This development will also work for a radially symmetric earth.



Here again we will look at travel times (τ) rather than distance.

Let χ be a vector in 4-space giving the location of the earthquake

(3 cartesian coodinates plus time)

$$\vec{\chi} = \left(x, y, z, t\right)$$

Let X be a vector in 3-space – location of the station

$$\vec{X}_k = \left(x_k, y_k, z_k\right)$$

Discussion follows Lee and Stewart

What data/information is available to locate an earthquake?

Arrival time of seismic waves at a number of known locations

$$\tau_{k,\text{observed}}\left(x_k, y_k, z_k\right) = \tau_{k,\text{observed}}\left(\vec{X}_k\right)$$

Plus we have a model for how seismic waves travel in the earth.

This allows us to calculate the travel time to station k

$$T_{k,\text{calculated}}\left(\vec{X}_k, \vec{\chi}\right)$$

(does not really depend on *t*, but carry it along)

from an earthquake at (location and time)

$$\vec{\chi} = \left(x, y, z, t\right)$$

So we can do the forward problem.

From the travel time plus the origin time, *t*
(when the earthquake occurred)

we can calculate the arrival time at the kth station

$$\tau_{k,\text{calculated}}\left(\vec{X}_k,\vec{\chi}\right) = T_{k,\text{calculated}}\left(\vec{X}_k,\vec{\chi}\right) + t$$

We want to estimate the 4 parameters of χ

so we will need 4 data (which gives 4 equations) as a minimum

Unless the travel time – distance relationship is linear

(which it is not in general)

we cannot (easily) solve these 4 equations.

So what do we do?

One possibility is to do the forward calculation for a large number of trial solutions (usually on a grid)

and select the trial solution with the smallest difference between the predicted and measured data

This is known as a grid search (inversion!) and is expensive
(but sometimes it is the only way)

Modifications of this method use ways to cut down on the number of trial solutions

monte carlo

steepest descent

simulated annealing

other

Another approach

solve iteratively by

1) Assuming a location

2) Linearizing the travel time equations

3) Use least squares to compute an adjustment to the location, which we will use to produce a new (better) location

4) Go back to step 1 using location from step 3

We do this till some convergence criteria is met

(if we're lucky)

This is basically Newton's method

Least squares "minimizes" the difference between observed and modeled/calculated data.

Assume a location (time included)

$$\vec{\chi}^* = \left(x^*, y^*, z^*, t^*\right)$$

and consider the difference between the calculated and measured values

Least squares minimizes the difference between observed and modeled/calculated data.

for one station we have

$$\tau_{\text{observed}} = \tau_{\text{calculated}} + \nu$$

noise

$$\tau_{\text{observed}} = \tau\left(\vec{X}, \vec{\chi}\right) + \nu$$

Did not write calculated here because I can't calculate this without knowing χ.

First – linearize the expression for the arrival time $\tau(X,\chi)$

$$\tau\left(\vec{X},\vec{\chi}\right) \approx \tau_{\text{calculated}}\left(\vec{X},\vec{\chi}^{*}\right) + \left(x - x^{*}\right)\frac{\partial \tau}{\partial x}\bigg|_{\chi^{*}} + \left(y - y^{*}\right)\frac{\partial \tau}{\partial y}\bigg|_{\chi^{*}} + \left(z - z^{*}\right)\frac{\partial \tau}{\partial z}\bigg|_{\chi^{*}} + \left(t - t^{*}\right)\frac{\partial \tau}{\partial t}\bigg|_{\chi^{*}}$$

$$\tau\left(\vec{X},\vec{\chi}\right) \approx \tau_{\text{calculated}}\left(\vec{X},\vec{\chi}^{*}\right) + \boxed{\frac{\partial \tau}{\partial x}\Delta x + \frac{\partial \tau}{\partial y}\Delta y + \frac{\partial \tau}{\partial z}\Delta z + \frac{\partial \tau}{\partial t}\Delta t}$$

Now can put calculated here because can calculate this using the known (assumed) χ^*, but don't know these.

Now consider the difference between the observed and linearized τ – the residual $\Delta\tau$.

$$\Delta\tau = \tau_{\text{observed}} - \tau_{\text{calculated}}$$

$$\Delta\tau = \tau\left(\vec{X},\vec{\chi}^{*}\right) + \nu - \tau_{\text{calculated}}$$

$$\Delta\tau = \left(\tau_{\text{calculated}}\left(\vec{X},\vec{\chi}\right) + \frac{\partial\tau}{\partial x}\Delta x + \frac{\partial\tau}{\partial y}\Delta y + \frac{\partial\tau}{\partial z}\Delta z + \frac{\partial\tau}{\partial t}\Delta t\right) + \nu$$
$$- \tau_{\text{calculated}}\left(\vec{X},\vec{\chi}^{*}\right)$$

$$\Delta\tau = \frac{\partial\tau}{\partial x}\Delta x + \frac{\partial\tau}{\partial y}\Delta y + \frac{\partial\tau}{\partial z}\Delta z + \frac{\partial\tau}{\partial t}\Delta t + \nu$$

We have the following for one station

$$\Delta\tau = \frac{\partial\tau}{\partial x}\Delta x + \frac{\partial\tau}{\partial y}\Delta y + \frac{\partial\tau}{\partial z}\Delta z + \frac{\partial\tau}{\partial t}\Delta t + \nu$$

Which we can recast in matrix form

$$\Delta\tau = \begin{pmatrix} \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y} & \frac{\partial\tau}{\partial z} & \frac{\partial\tau}{\partial t} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t \end{pmatrix} + \nu$$

For m stations (where $m \geq 4$)

$$\begin{pmatrix} \Delta\tau_1 \\ \Delta\tau_2 \\ \Delta\tau_3 \\ \vdots \\ \Delta\tau_m \end{pmatrix} = \begin{pmatrix} \frac{\partial\tau_1}{\partial x} & \frac{\partial\tau_1}{\partial y} & \frac{\partial\tau_1}{\partial z} & \frac{\partial\tau_1}{\partial t} \\ \frac{\partial\tau_2}{\partial x} & \frac{\partial\tau_2}{\partial y} & \frac{\partial\tau_2}{\partial z} & \frac{\partial\tau_2}{\partial t} \\ \frac{\partial\tau_3}{\partial x} & \frac{\partial\tau_3}{\partial y} & \frac{\partial\tau_3}{\partial z} & \frac{\partial\tau_3}{\partial t} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\tau_m}{\partial x} & \frac{\partial\tau_m}{\partial y} & \frac{\partial\tau_m}{\partial z} & \frac{\partial\tau_m}{\partial t} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_m \end{pmatrix}$$

Jacobian matrix

Which is usually written as

$$b = Ax + \nu$$

Evaluating the time term

$$b = Ax + \nu$$

$$\Delta\vec{\tau} = A\delta\vec{x} + \nu$$

$$A = \begin{pmatrix} \frac{\partial \tau_1}{\partial x} & \frac{\partial \tau_1}{\partial y} & \frac{\partial \tau_1}{\partial z} & 1 \\ \frac{\partial \tau_2}{\partial x} & \frac{\partial \tau_2}{\partial y} & \frac{\partial \tau_2}{\partial z} & 1 \\ \frac{\partial \tau_3}{\partial x} & \frac{\partial \tau_3}{\partial y} & \frac{\partial \tau_3}{\partial z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tau_m}{\partial x} & \frac{\partial \tau_m}{\partial y} & \frac{\partial \tau_m}{\partial z} & 1 \end{pmatrix}$$

$$b = Ax + \nu \qquad \Delta\vec{\tau} = A\delta\vec{x} + \nu$$

Expresses linear relationship between residual observations, $\Delta\tau$, and unknown corrections, δx.

Plus unknown noise terms.

Linearized observation equations

Next use least squares to minimize the sum of the squares of the residuals for all the stations.

$$\Delta\vec{\tau} = A\delta\vec{x} + \nu$$

$$F\left(\chi^*\right) = \sum_{k=1}^{m}\left[\Delta\tau_k\left(\chi^*\right)\right]^2$$

Previous linear least squares discussion gives us

$$A^T\Delta\vec{\tau} = A^T A\delta\vec{x}$$

Design matrix – *A*

Coefficients
Partial derivatives of each observation
With respect to each parameter
Evaluated at provisional parameter values

A has 4 columns (for the 4 parameters)
and
As many rows as satellites (need at least 4)

Can calculate derivatives from the model for the observations

This is called Geiger's method

Published 1910

Not used till 1960!

(when geophysicists first got hold of a computer)

So far

Have not specified type of arrival.

Can do with P only, S only (?!), P and S together, or S-P.

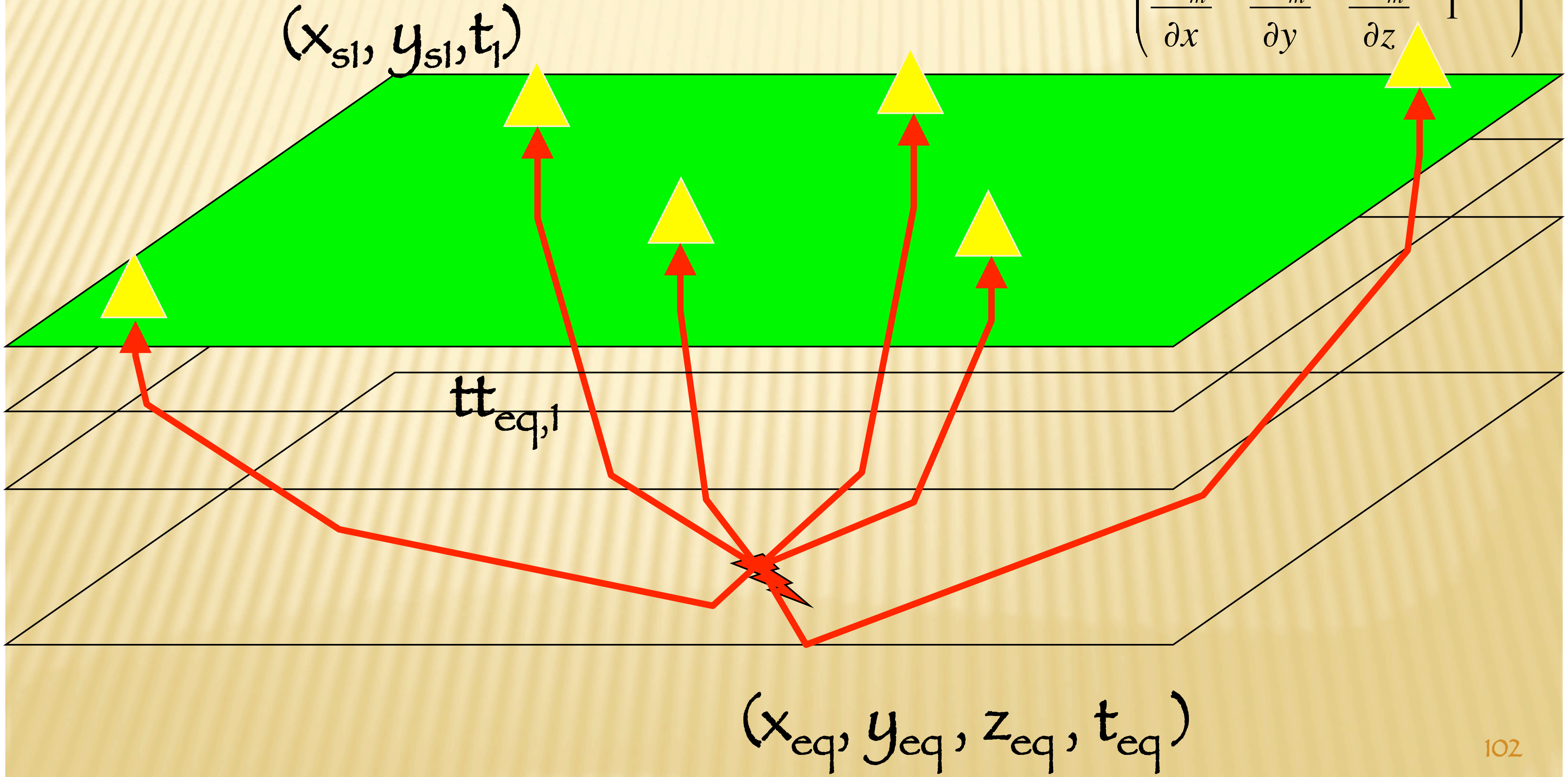
Need velocity model to calculate travel times and travel time derivatives
(so earthquakes are located with respect to the assumed velocity model, not real earth.
Errors are "formal", i.e. with respect to model.)

Velocity models usually laterally homogeneous.

Problems:

Column of 1's – if one of the other columns is constant (or approximately constant) matrix is singular and can't be inverted.

$$A = \begin{pmatrix} \frac{\partial \tau_1}{\partial x} & \frac{\partial \tau_1}{\partial y} & \frac{\partial \tau_1}{\partial z} & 1 \\ \frac{\partial \tau_2}{\partial x} & \frac{\partial \tau_2}{\partial y} & \frac{\partial \tau_2}{\partial z} & 1 \\ \frac{\partial \tau_3}{\partial x} & \frac{\partial \tau_3}{\partial y} & \frac{\partial \tau_3}{\partial z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tau_m}{\partial x} & \frac{\partial \tau_m}{\partial y} & \frac{\partial \tau_m}{\partial z} & 1 \end{pmatrix}$$



How can this happen:
- All first arrivals are head waves from same refractor
- Earthquake outside the network

$$A = \begin{pmatrix} \frac{\partial \tau_1}{\partial x} & \frac{\partial \tau_1}{\partial y} & \frac{\partial \tau_1}{\partial z} & 1 \\ \frac{\partial \tau_2}{\partial x} & \frac{\partial \tau_2}{\partial y} & \frac{\partial \tau_2}{\partial z} & 1 \\ \frac{\partial \tau_3}{\partial x} & \frac{\partial \tau_3}{\partial y} & \frac{\partial \tau_3}{\partial z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tau_m}{\partial x} & \frac{\partial \tau_m}{\partial y} & \frac{\partial \tau_m}{\partial z} & 1 \end{pmatrix}$$