Earth Science Applications of Space Based Geodesy

DES-7355

Tu-Th 9:40-11:05

Seminar Room in 3892 Central Ave. (Long building)

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http://www.ceri.memphis.edu/people/smalley/ESCI7355/ESCI_7355_Applications_of_Space_Based_Geodesy.html

Class 8

Covariance and Cofactor matrix in GPS

If observations had no errors and the model was perfect then the estimations from

$$\hat{x} = \left(A^T A\right)^{-1} A^T \vec{b}$$

Would be perfect

Errors, v, in the original observations b will map into errors v_x in the estimates of x and this mapping will take the same form as the estimation

$$\left(\hat{x} + \nu_x\right) = \left(A^T A\right)^{-1} A^T \left(\vec{b} + \vec{v}\right)$$

$$\hat{x} = \left(A^T A\right)^{-1} A^T \vec{b}$$

$$\nu_x = \left(A^T A\right)^{-1} A^T \vec{v}$$

If we have an expected (a priori) value for the error in the data, σ, we can compute the expected error in the parameters

Consider the covariance matrix

and for this discussion suppose that the observations are uncorrelated (covariance matrix is therefore diagonal)

$$C_{ii} = E\left(\nu_i^2\right) = \sigma_i^2$$

$$C = E\left(\nu\nu^T\right)$$

Assume further that we can characterize the error in the observations by a single number, σ.

$$C_{ii} = \sigma^2 I$$

then

$$C_x = E\left(v_x v_x^T\right) = E\left(\left(\left(A^T A\right)^{-1} A^T \vec{v}\right)\left(\left(A^T A\right)^{-1} A^T \vec{v}\right)^T\right)$$

$$C_x = E\left(\left(A^T A\right)^{-1} A^T \vec{v} v^T A \left(A^T A\right)^{-1}\right)$$

$$C_x = \left(A^T A\right)^{-1} A^T E\left(\vec{v} v^T\right) A \left(A^T A\right)^{-1}$$

$$C_x = \left(A^T A\right)^{-1} A^T \sigma^2 I A \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \left(A^T A\right)^{-1} \left(A^T A\right) \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

Expected covariance is σ^2 (a number) times cofactor matrix, compare to

$$\vec{v}_x = \left(A^T A\right)^{-1} A^T \vec{v}$$

$$\hat{x} = \left(A^T A\right)^{-1} A^T \vec{b}$$

Covariance or cofactor matrix

$$\left(A^T A\right)^{-1}$$

Interpretation of covariance

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

Variance of measurements

Measurement errors may be independent (our assumption – why we could factor out constant σ^2)

We saw before that A is dependent on the "direction" from antenna to satellite – so it is a function of the geometry

But total effect, after Least Squares, can be non-diagonal.

Since A is function of geometry only, the cofactor matrix is also a function of geometry only.

$$\left(A^{T}A\right)^{-1}$$

Can use cofactor matrix to quantify the relative strength of the geometry.

Also relates measurement errors to expected errors in position estimations

In the old days,

before the full constellation of satellites was flying,

one had to plan – design – the GPS surveying sessions based on the (changing) geometry.

A is therefore called the “design” matrix

Don’t have to worry about this anymore

(most of the time).

Look at full covariance matrix

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

$$C_x = \sigma^2 \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{x\tau} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} & \sigma_{y\tau} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 & \sigma_{z\tau} \\ \sigma_{\tau x} & \sigma_{\tau y} & \sigma_{\tau z} & \sigma_\tau^2 \end{pmatrix}$$

$$\sigma_{ij} = \sigma_{ji}$$

Off diagonal elements indicate degree of correlation between parameters.

Correlation coefficient

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}$$

Depends only on cofactor matrix

Independent of observation variance (the σ^2' s cancel out)

+1 perfect correlation – what does it mean – the two parameters behave practically identically (and are not independent?!).

0 – no correlation, independent

-1 perfect anti-correlation – practically opposites (and not independent?!).

So far all well and good –

but Cartesian coordinates are not the most useful.

We usually need estimates of horizontal and vertical positions and velocities on earth (ellipsoid?).

We also need error estimates on the position and velocity.

Since the errors are a function of the geometry only, one might expect that the vertical errors are larger than the horizontal errors.

How do we find the covariance / cofactor matrices in the local (north, east, up) coordinate system?

Have to transform the matrix

from its representation in one coordinate system

to its representation in another

using the rules of error propagation.

First how do we transform a small relative vector in Cartesian coordinates (u,v,w) to local topocentric coordinates (n,e,u)?

$$\Delta \vec{L} = G\Delta \vec{X}$$

$$\begin{pmatrix} \Delta n \\ \Delta e \\ \Delta u \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Where ϕ and λ are the lat and long of the location (usually on the surface of the earth) respectively

Errors (small magnitude vectors) transform the same way

$$\vec{v}_L = G\vec{v}_x$$

Why?

Using linearity

$$\Delta\vec{L} = G\Delta\vec{X}$$

$$G(\Delta\vec{X} + \vec{v}_x) = G\Delta\vec{X} + G\vec{v}_x = \Delta\vec{L} + G\vec{v}_x = \Delta\vec{L} + \vec{v}_L$$

Errors (small magnitude vectors) transform the same way

$$\vec{v}_{L} = G\vec{v}_x$$

Now – how does the covariance

$$C = E\left(\vec{v}\vec{v}^T\right)$$

Transform?

Plug in – get

"law of propagation of errors"

law of propagation of errors

$$C_L = E\left(\vec{v}_L \vec{v}_L^T\right)$$

$$C_L = E\left(G\vec{v}_x \left(G\vec{v}_x\right)^T\right)$$

$$C_L = E\left(G\vec{v}_x \vec{v}_x^T G^T\right)$$

$$C_L = GE\left(\vec{v}_x \vec{v}_x^T\right)G^T$$

$$\boxed{C_L = GC_xG^T}$$

(does this look familiar?)

law of propagation of errors

$$C_L = E\left(\vec{v}_L \vec{v}_L^T\right)$$

$$C_L = E\left(G\vec{v}_x \left(G\vec{v}_x\right)^T\right)$$

$$C_L = E\left(G\vec{v}_x \vec{v}_x^T G^T\right)$$

$$C_L = GE\left(\vec{v}_x \vec{v}_x^T\right)G^T$$

$$\boxed{C_L = GC_x G^T}$$

(does this look familiar?)

(it better -- transforming tensors!!)

This is a general result for affine transformations (multiplication of a column vector by any rectangular matrix)

An affine transformation is any transformation that preserves

collinearity
(i.e., all points lying on a line initially still lie on a line after transformation)

and ratios of distances
(e.g., the midpoint of a line segment remains the midpoint after transformation).

http://mathworld.wolfram.com/AffineTransformation.html

Geometric contraction, expansion, dilation, reflection, rotation, shear, similarity transformations, spiral similarities, and translation

are all affine transformations, as are their combinations.

In general, an affine transformation is a composition of

rotations, translations, dilations, and shears.

While an affine transformation preserves *proportions* on lines, it does not necessarily preserve angles or lengths.

http://mathworld.wolfram.com/AffineTransformation.html

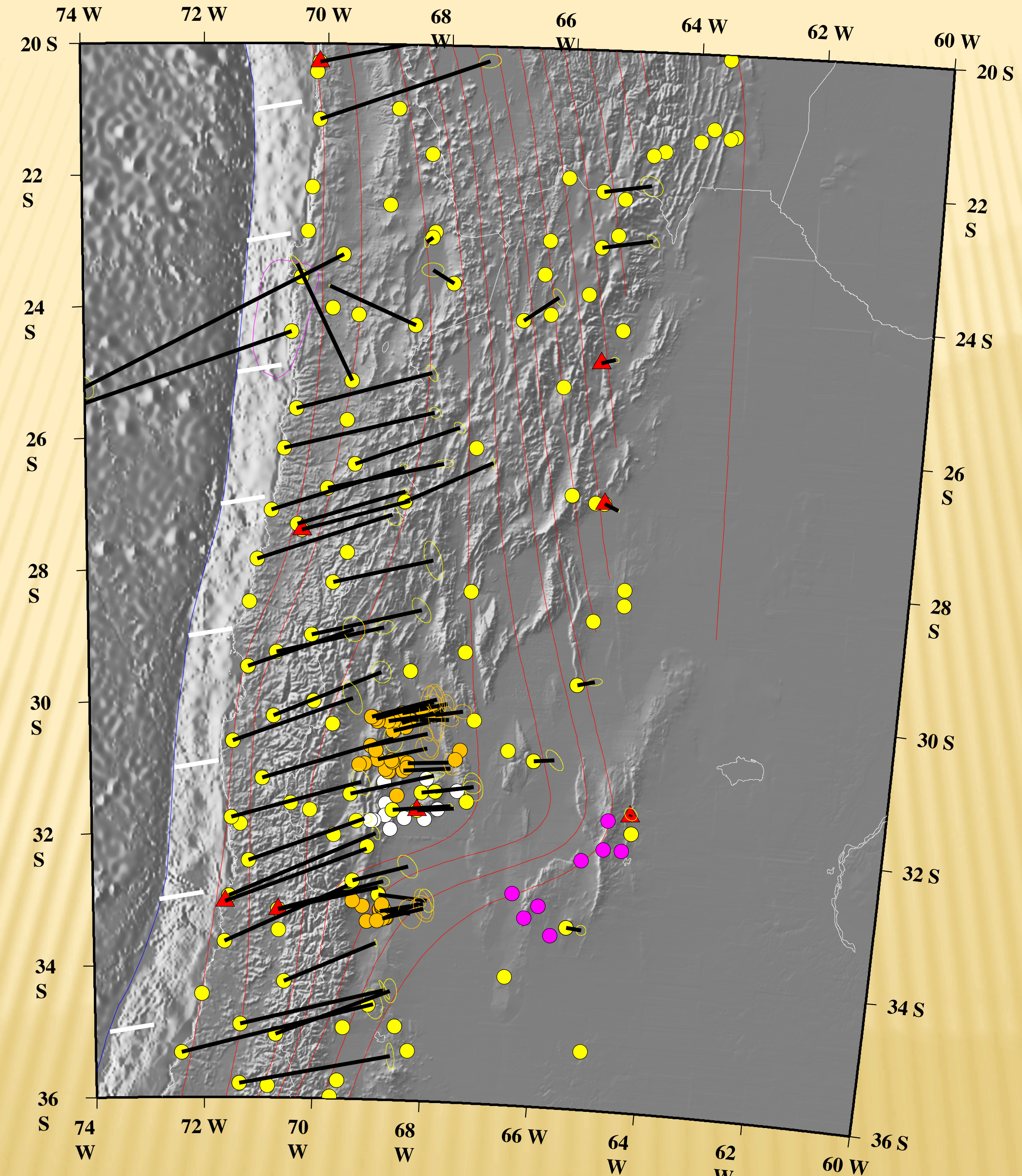
Look at full covariance matrix

(actually only the spatial part)

$$C_x = \sigma^2 \left(A^T A\right)^{-1}$$

$$C_L = \sigma^2 \begin{pmatrix} \sigma_n^2 & \sigma_{ne} & \sigma_{nh} \\ \sigma_{en} & \sigma_e^2 & \sigma_{eh} \\ \sigma_{hn} & \sigma_{he} & \sigma_h^2 \end{pmatrix}$$

Can use this to plot error ellipses on a map (horizontal plane).


74 W
72 W
70 W
68 W
66 W
64 W
62 W
60 W
20 S
22 S
24 S
26 S
28 S
30 S
32 S
34 S
36 S

Error estimators –

Remember the expression for the RMS in 2-D from before

$$DRMS = \left(\sigma_x{}^2 + \sigma_y{}^2\right)^{\frac{1}{2}}$$

We can now apply this to the covariance matrix

Error estimates called "dilution of Precision" – DOP – are defined in terms of the diagonal elements of the covariance matrix

$$GDOP = \left(\sigma_n{}^2 + \sigma_e{}^2 + \sigma_h{}^2 + \sigma_\tau{}^2\right)^{\frac{1}{2}}$$

$$PDOP = \left(\sigma_n{}^2 + \sigma_e{}^2 + \sigma_h{}^2\right)^{\frac{1}{2}}$$

$$HDOP = \left(\sigma_n{}^2 + \sigma_e{}^2\right)^{\frac{1}{2}}$$

$$VDOP = \sigma_h$$

$$TDOP = \sigma_\tau$$

G – geometric, P – position, H – horizontal, V – vertical,

T - time

The DOPs map the errors of the observations (represented/quantified statistically by the standard deviations)

Into the parameter estimate errors

$$\sigma\, GDOP = \sigma\left(\sigma_n^{\ 2} + \sigma_e^{\ 2} + \sigma_h^{\ 2} + \sigma_\tau^{\ 2}\right)^{\frac{1}{2}}$$

$$\sigma\, PDOP = \sigma\left(\sigma_n^{\ 2} + \sigma_e^{\ 2} + \sigma_h^{\ 2}\right)^{\frac{1}{2}}$$

$$\sigma\, HDOP = \sigma\left(\sigma_n^{\ 2} + \sigma_e^{\ 2}\right)^{\frac{1}{2}}$$

$$\sigma\, VDOP = \sigma\, \sigma_h$$

$$\sigma\, TDOP = \sigma\, \sigma_\tau$$

So for a σ of 1 m and an χ–DOP of 5, for example,

errors in the position χ

(where χ is one of G, P, H, V, T)

Would be *5σ=5* m

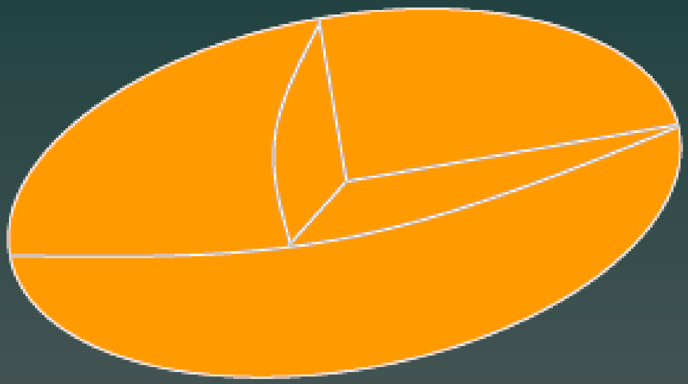
"Good" geometry gives 'small' DOP

"Bad" geometry gives 'large' DOP

(it is relative, but *PDOP>5* is considered poor)

3-D Accuracy Terms

- Error Ellipsoid



  – 19.9% of positions will lie inside this "error ellipsoid"

www.eng.auburn.edu/department/an/Teaching/BSEN_6220/GPS/Lecture%20Notes/Carrier_Phase_GPS.pdf

In 2-D

There is a 40% chance of being inside the 1-σ error ellipse (compared to 68% in 1-D)

Normally show 95% confidence ellipses, is 2.54 σ in 2-D

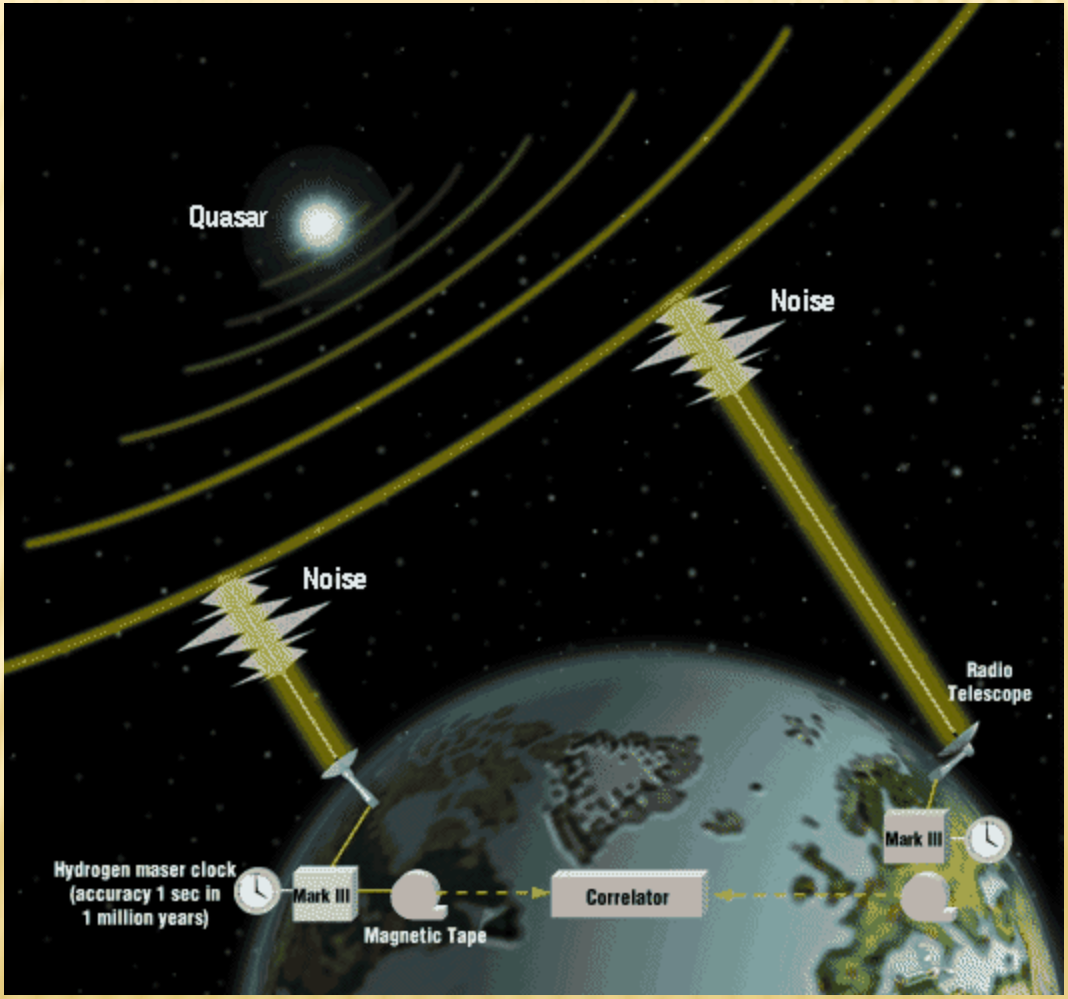
(is only 2σ in 1-D)
Can extend to 3-D

Another method of estimating location

Phase comparison/Interferometer

- VLBI
- GPS-Carrier Phase Observable

VLBI
Uses techniques/physics similar to GPS but with natural sources
(in same frequency band and suffers from similar errors)



http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

Correlate signal at <u>two (or more) sites</u> to find time shift

Need more than 1 receiver.

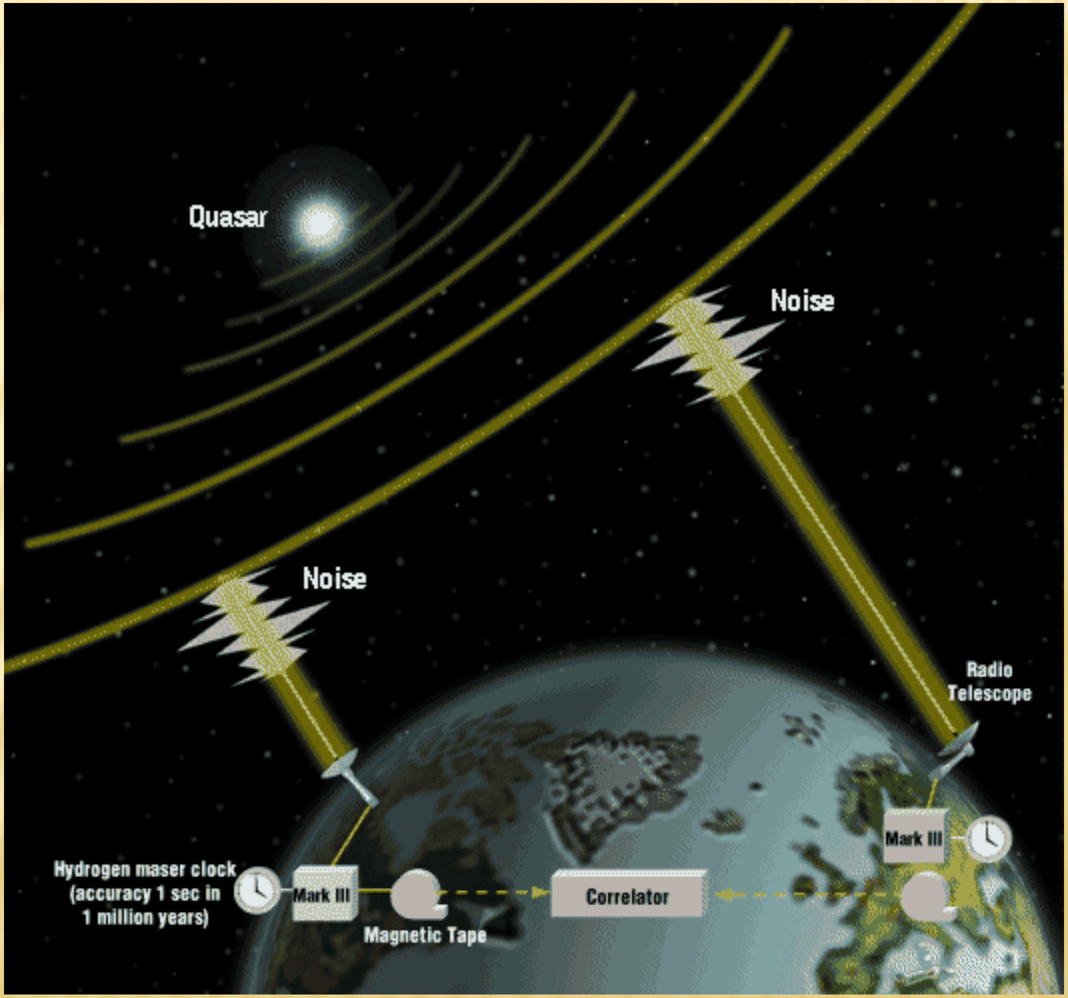
Differential (difference) method

(similar to PRN correlation with GPS codes

or

Aligning two seismograms that are almost same but have time shift)

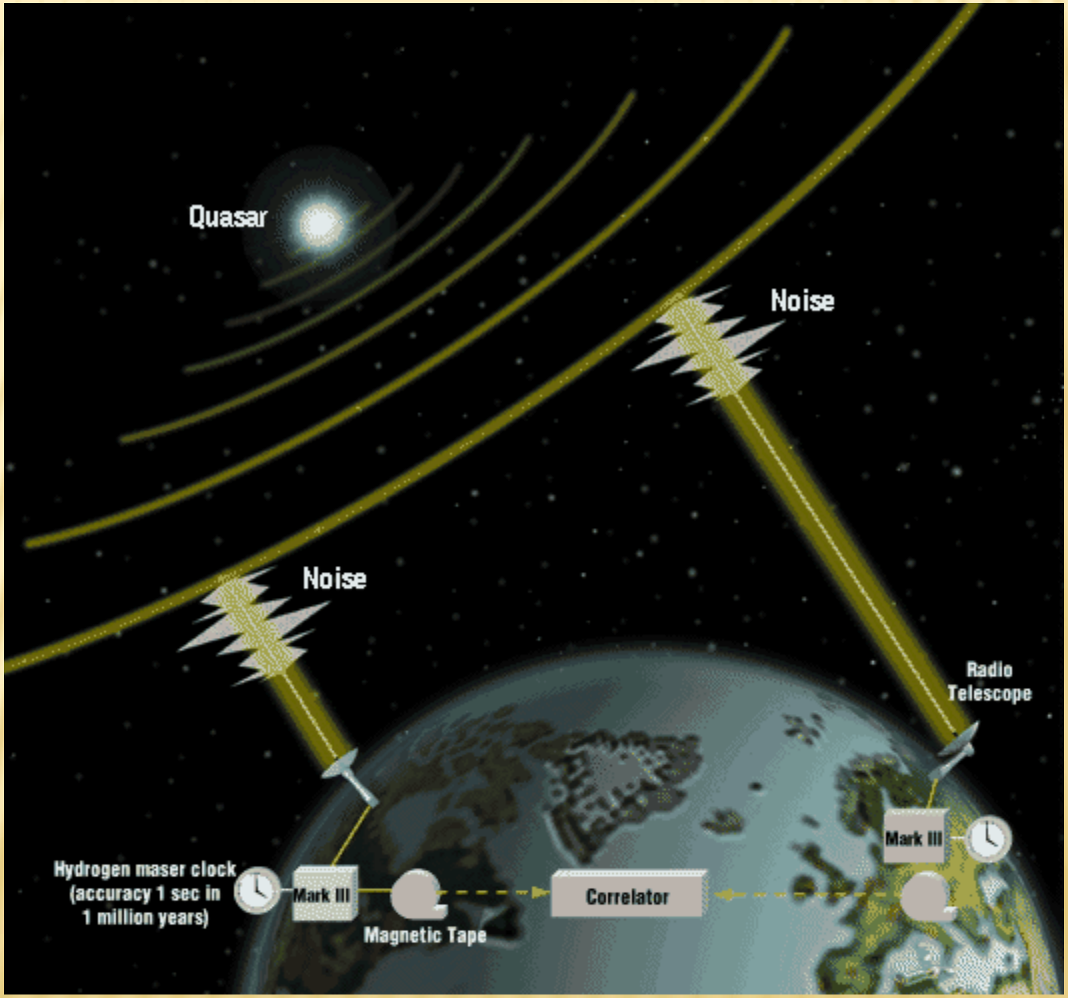
Assume you are receiving a plane wave from a distant quasar



http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

Two radio antennas observe signal from quasar <u>simultaneously</u>.

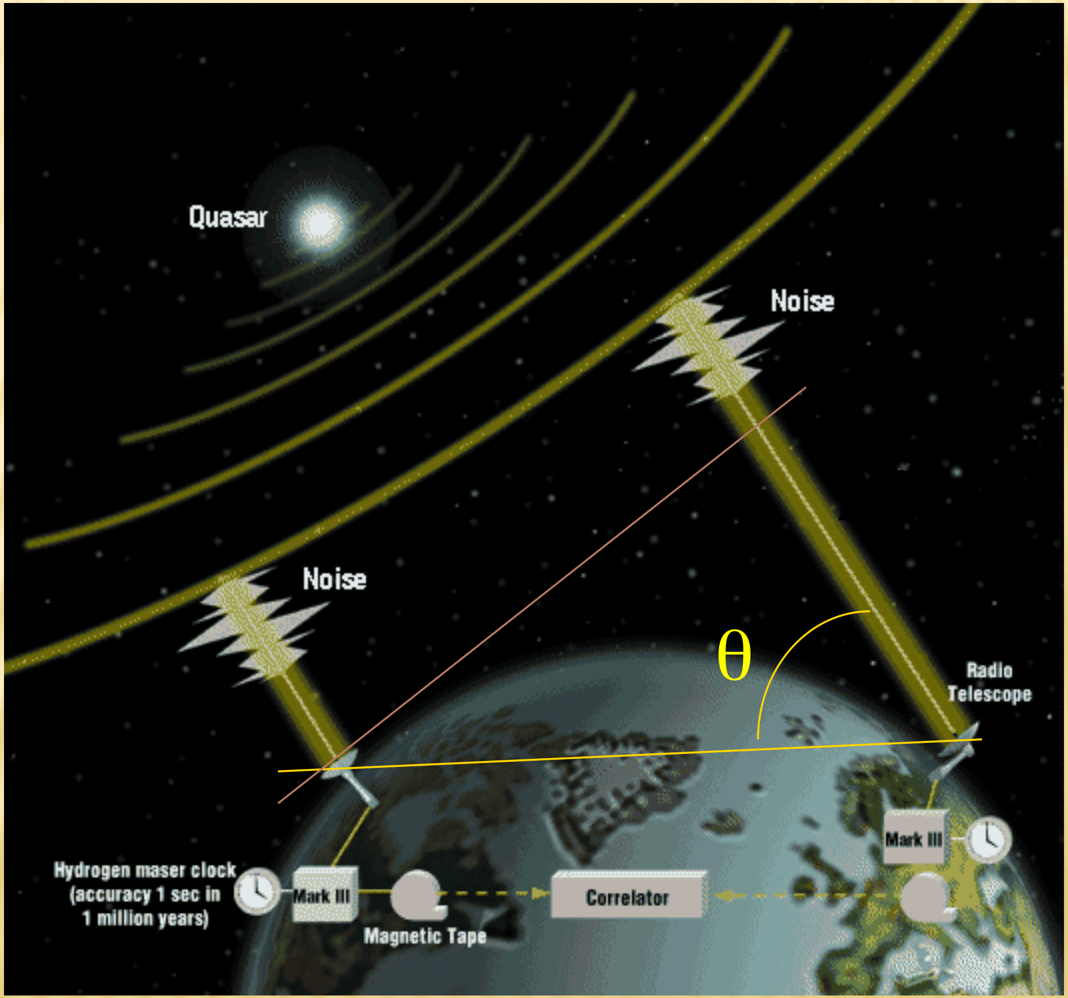
The signal arrives at the two antennae at different times



The distance or baseline length b between the two antennas can be defined as:

$$b * \cos(\theta) = c * \Delta T$$

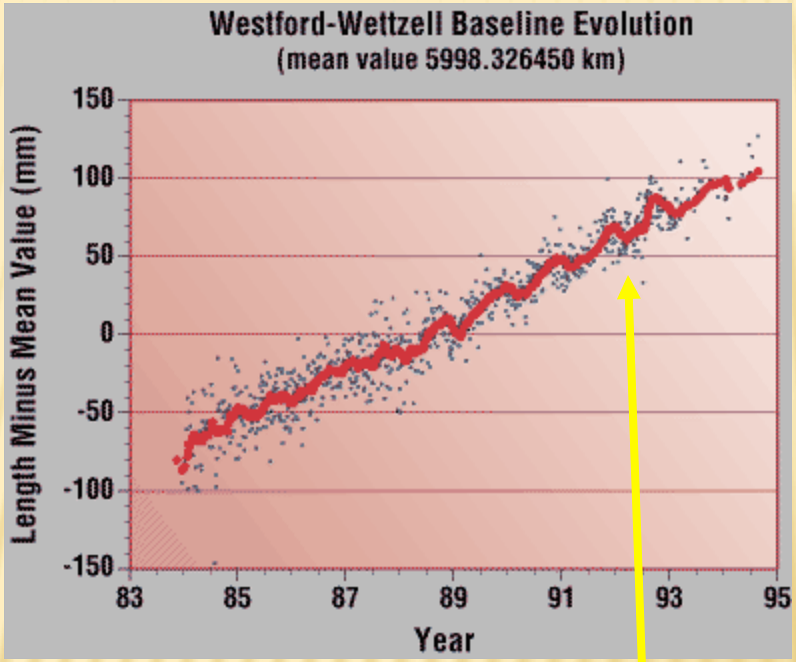
where θ is the angle between the baseline and the quasar



http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

Baseline length

Massachusetts to Germany

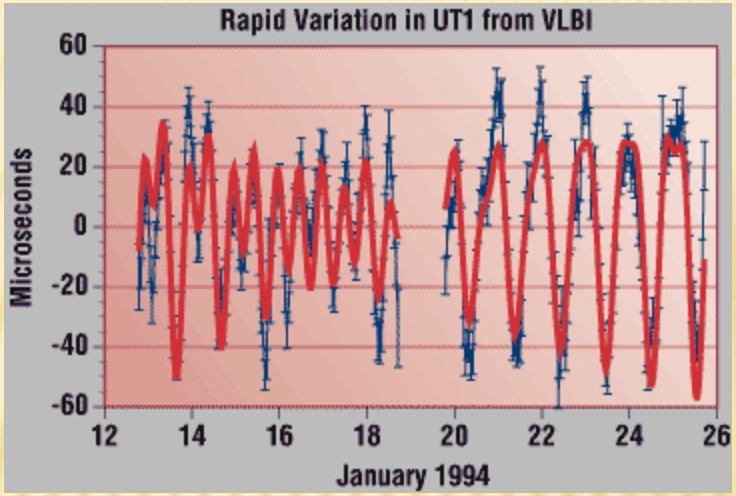


What is this variation?

Seasonal variation, geophysical phenomena, modeling problems?

http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

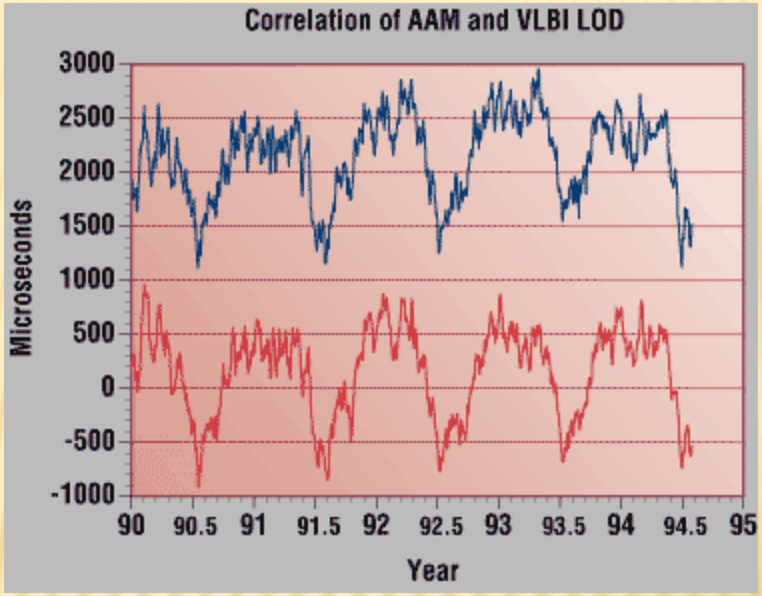
Short period (hours/days) variations in LOD



Mostly from ocean tides and currents

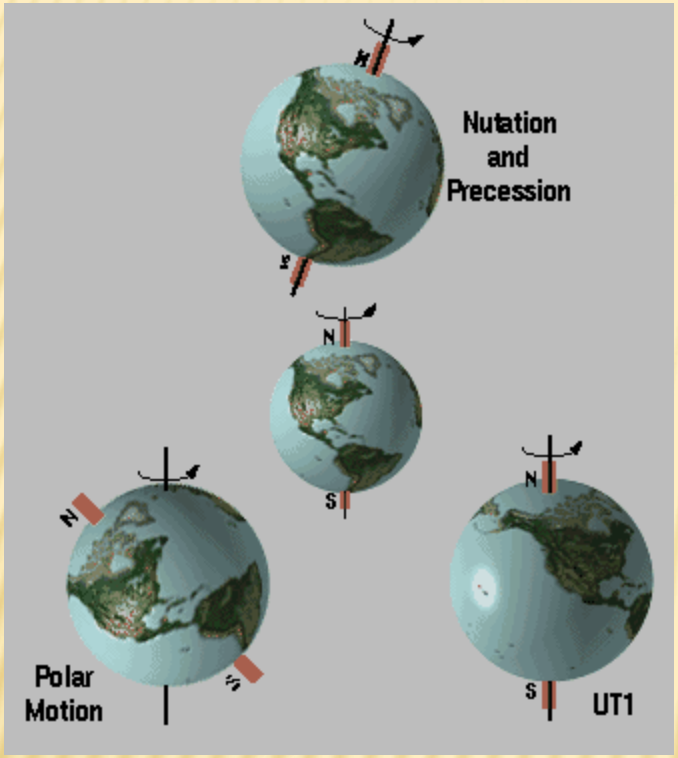
http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

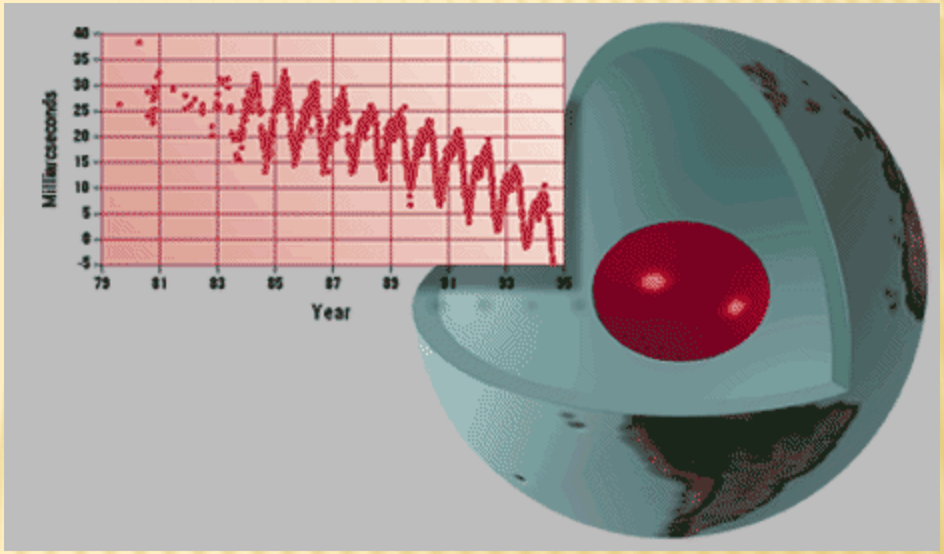
Correlation of Atmospheric Angular Momentum with (longer period – weeks/months) variations in LOD.



http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

(longer term – months/years/…) changes in LOD and EOP

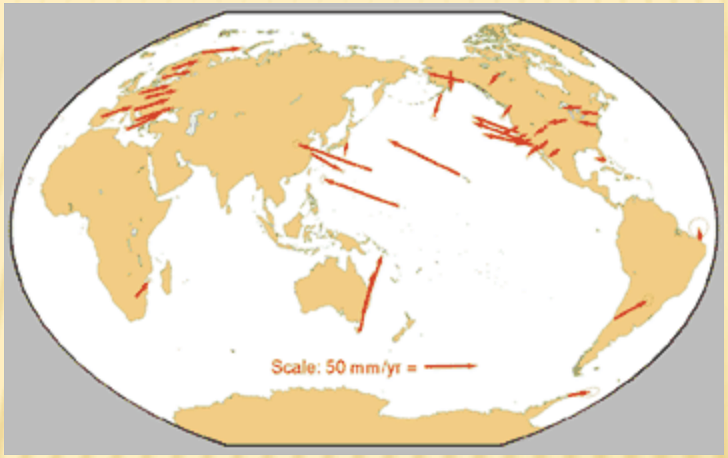




Exchange angular momentum between large earth structures (eg. core!) and Moon, Sun.

http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

Plate velocities



http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

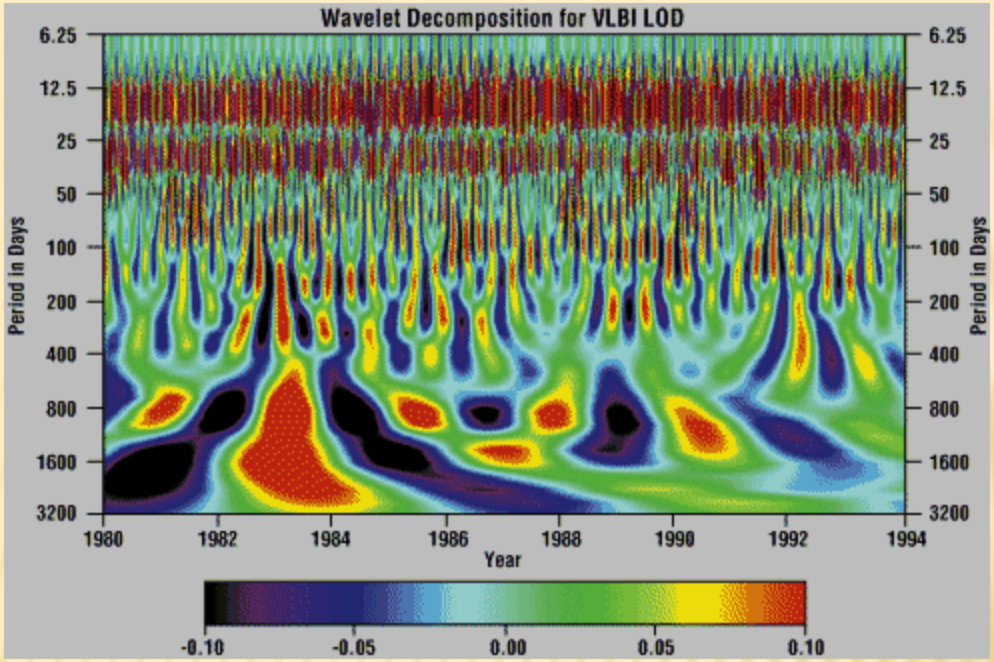
VLBI

Not the most portable or inexpensive system –

But best definition of inertial reference frame external to earth.

Use to measure changes in LOD and EOP due to gravitational forces and redistribution of angular momentum.

http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html



vertical cut - spectral decomposition LOD at that instant.

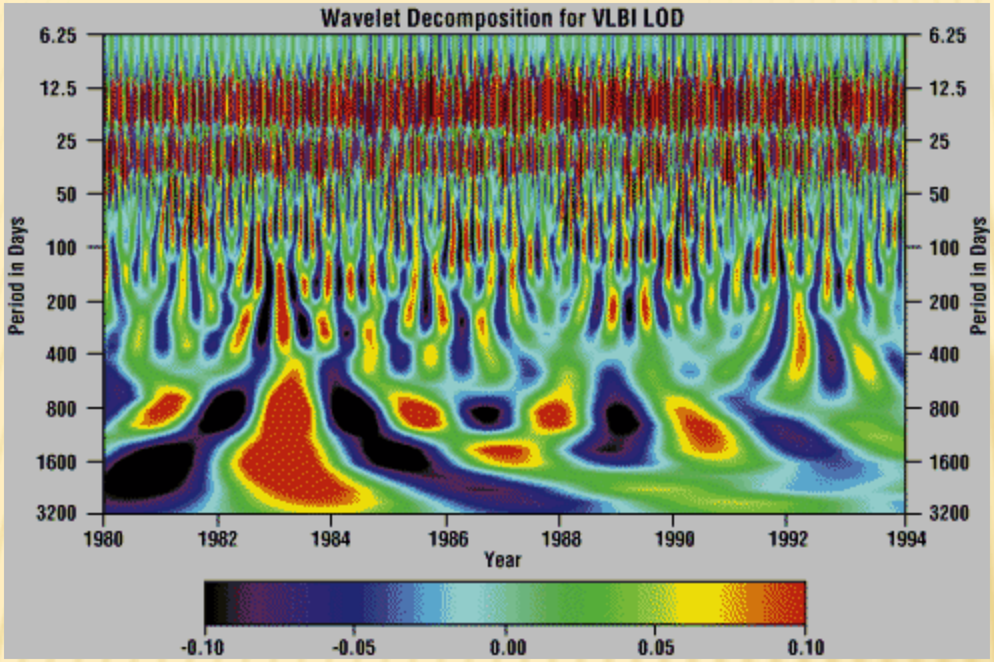
horizontal cut - how strength of component varies with time.

Combining both – 2-D view dynamic nature of LOD.

http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

dark red – peaks

dark blue - troughs.

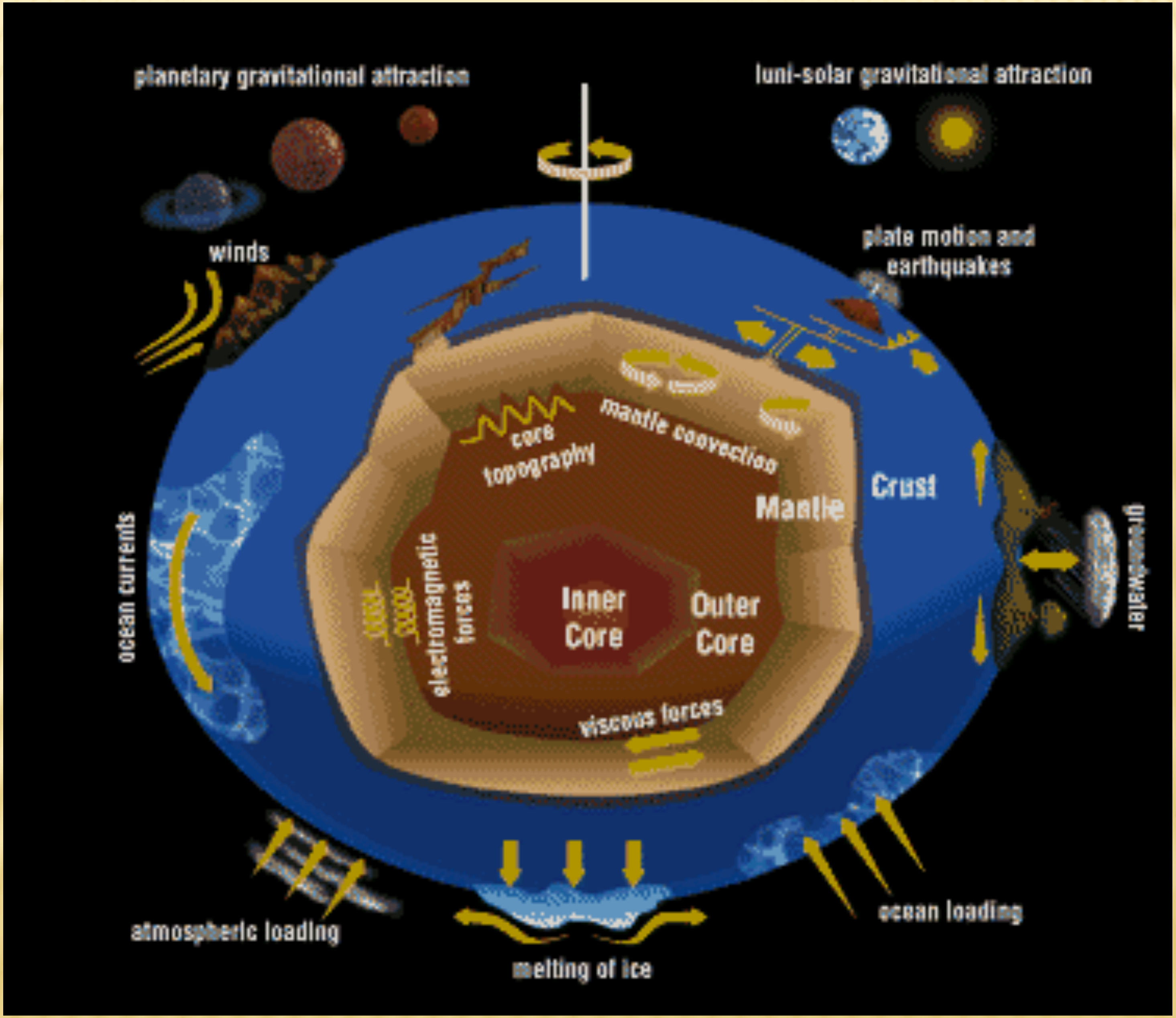


dominant features

monthly and half-monthly lunar tides, the ~800-day quasi-biennial oscillation, and the ~1600-day El Nino (dark red structure in 1983). Yearly and half-yearly seasonal excitations caused by meteorological variations have been removed for clarity.

http://www.colorado.edu/engineering/ASEN/asen5090/asen5090.html

Factors affecting EOP



Great book –

Longitude, by Dava Sobel,

describes one of the first great scientific competitions-- to provide ship captains with their position at sea.

This was after the loss of two thousand men in 1707, when British warships ran aground entering the English Channel.

The competition was between the Astronomers Royal (fat cats), using distance to the moon and its angle with the stars, and a man named John Harrison (unknown commoner), who made clocks.

Strang, http://www.siam.org/siamnews/general/gps.htm

The accuracy demanded in the 18th century was a modest 1/2 degree in longitude.

The earth rotates that much in two minutes.

For a six-week voyage this allows a clock error of three seconds per day.

Newton recommended the moon, and a German named Mayer won 3000 English pounds for his lunar tables. Even Euler got 300 for providing the right equations.

But lunar angles had to be measured, on a rolling ship at sea, within 1.5 minutes of arc.

Strang, http://www.siam.org/siamnews/general/gps.htm

The big prize was practically in sight for the lunar method, when Harrison came from nowhere and built clocks that could do better.

(You can see the clocks at Greenwich

Competing in the long trip to Jamaica, Harrison' s clock lost only five seconds and eventually (the fat cats fought it) won the prize.

Strang, http://www.siam.org/siamnews/general/gps.htm

The modern version of this same competition was between VLBI and GPS.

Very Long Baseline Interferometry uses "God's satellites," the distant quasars.

The clock at the receiver has to be very accurate and expensive.

The equipment can be moved (on a flatbed truck), but it is certainly not handheld.

Strang, http://www.siam.org/siamnews/general/gps.htm

There are valuable applications of VLBI, but it is GPS that ~~will~~ appears everywhere.

GPS is perhaps the second most important military contribution to civilian science, after the Internet.

The key is the atomic clock in the satellite, designed by university physicists to confirm Einstein's prediction of the gravitational red shift.

Strang, http://www.siam.org/siamnews/general/gps.htm

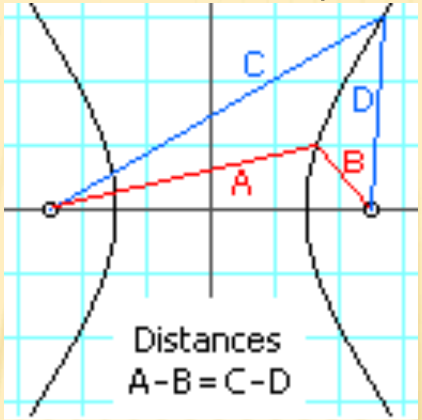
Using pseudo-range, the receiver solves a nonlinear problem in geometry.

What it knows is the the distance d_{ij} between itself and the satellites.

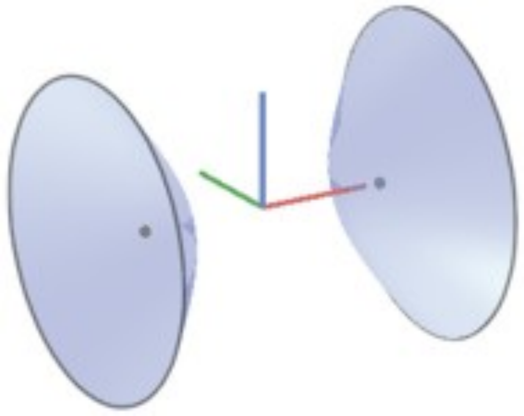
What if we know the difference in arrival times of the same signal at two or more receivers.

Strang, http://www.siam.org/siamnews/general/gps.htm

In a plane (2-D), when we know the difference d_{12} between the distances to two points, the receiver is located on a hyperbola.



In space (3-D) this becomes a hyperboloid.



Strang, http://www.siam.org/siamnews/general/gps.htm

Then the receiver lies at the intersection of three hyperboloids, determined by d_{12}, d_{13}, and d_{14}.

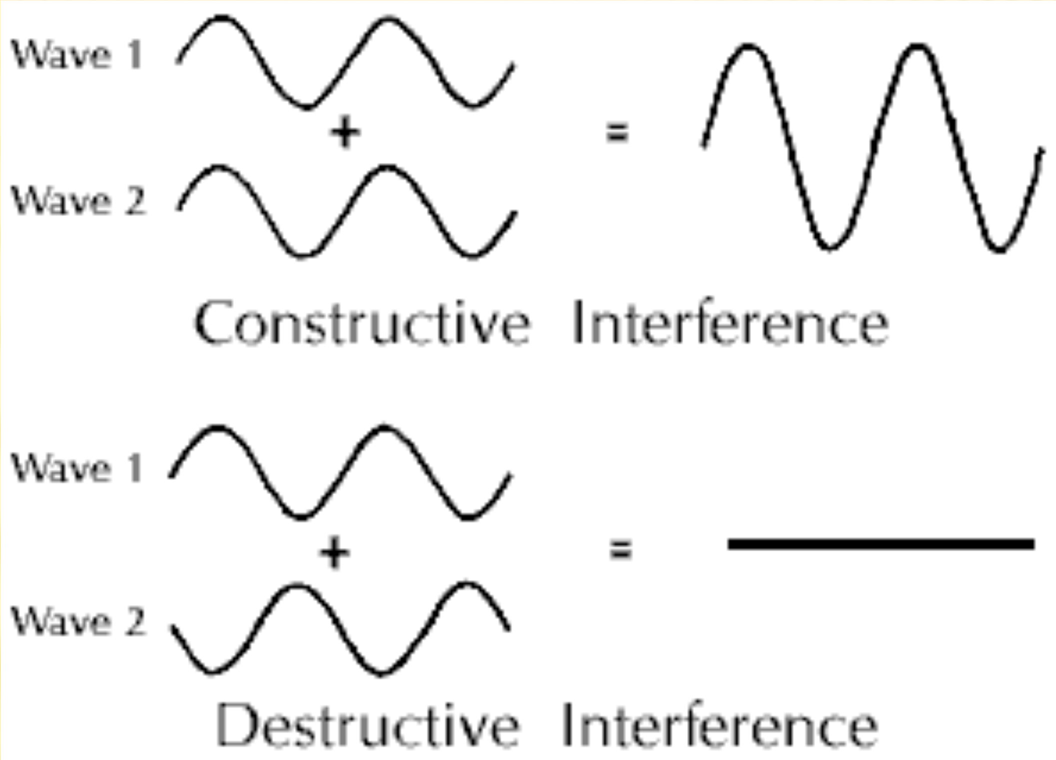
Two hyperboloids are likely to intersect in a simple closed curve.

The third probably cuts that curve at two points. But again, one point is near the earth and the other is far away.

Strang, http://www.siam.org/siamnews/general/gps.htm

Interferometer

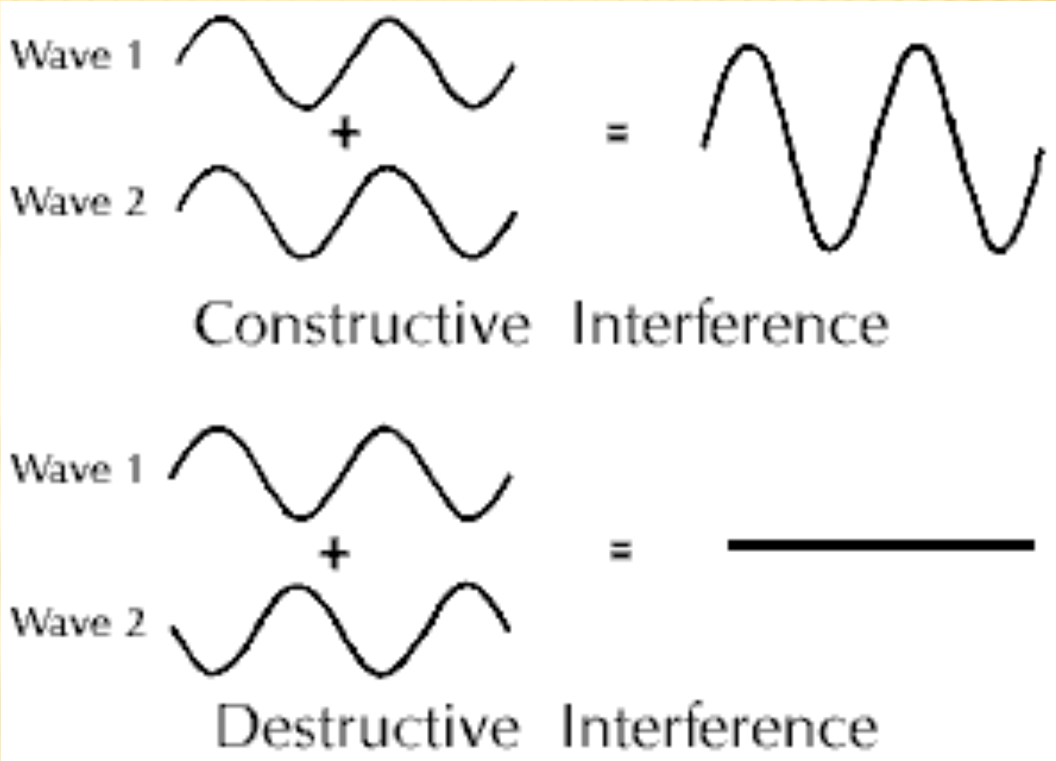
Based on interference of waves



http://www.space.com/scienceastronomy/astronomy/interferometry_101.html

How to make principle of interference useful?

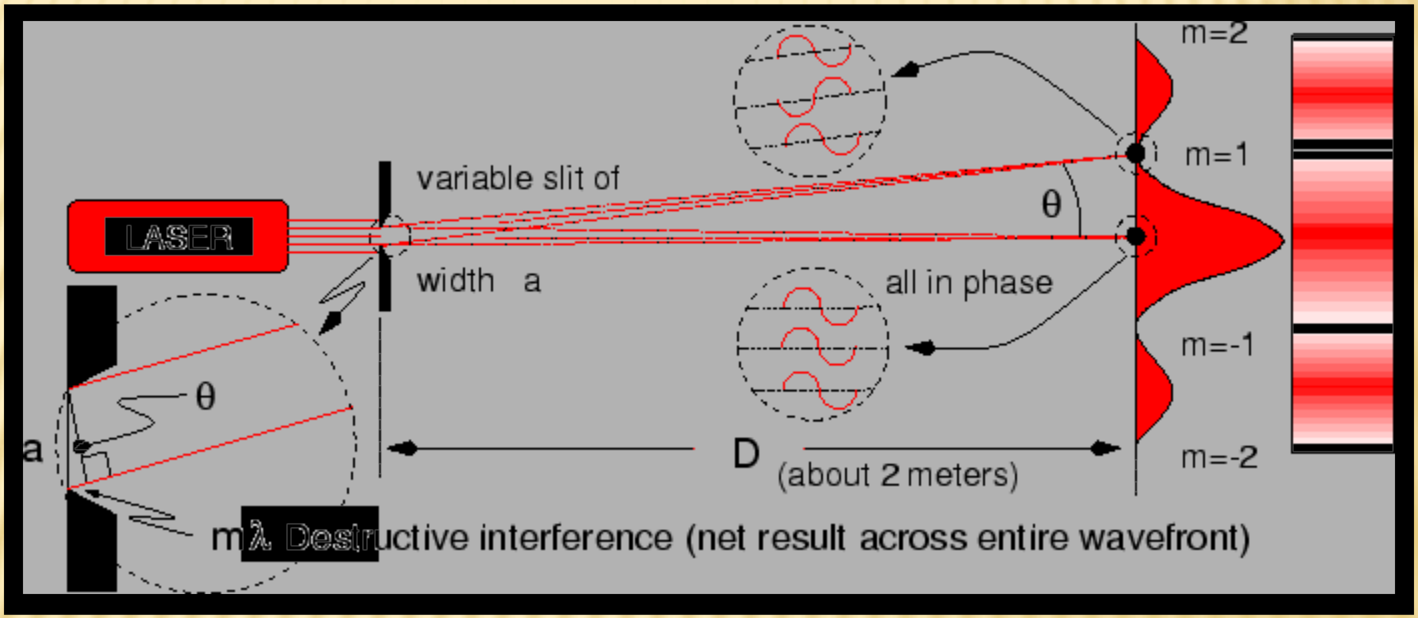
i.e. how does one get relative phase difference to vary, so the interference varies?



http://www.space.com/scienceastronomy/astronomy/interferometry_101.html

Interference from single slit

As move across screen get phase difference from different lengths of paths through slits

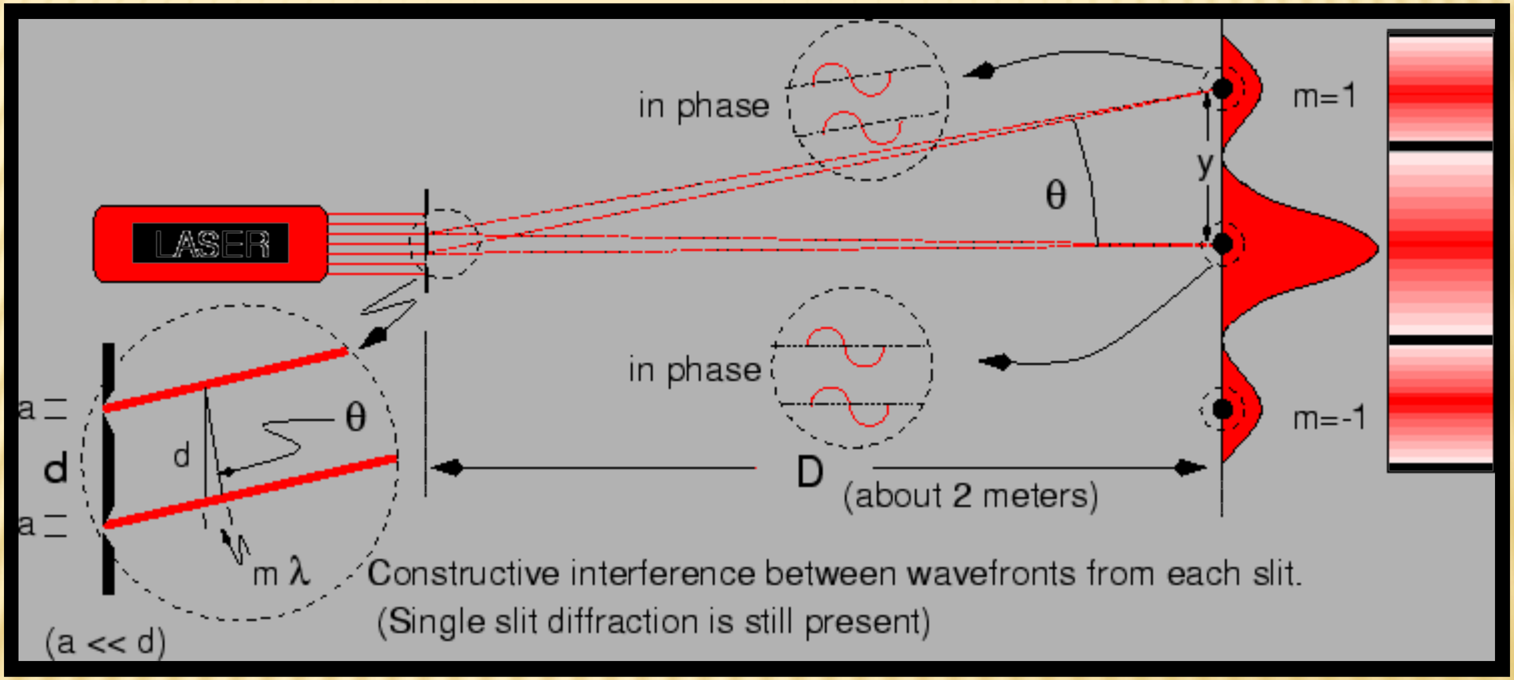


Makes "fringes"

As phase goes through change of 2π

Interference from double (multiple) slit

Similar for multi-slits, but now interference is between the waves leaving each slit



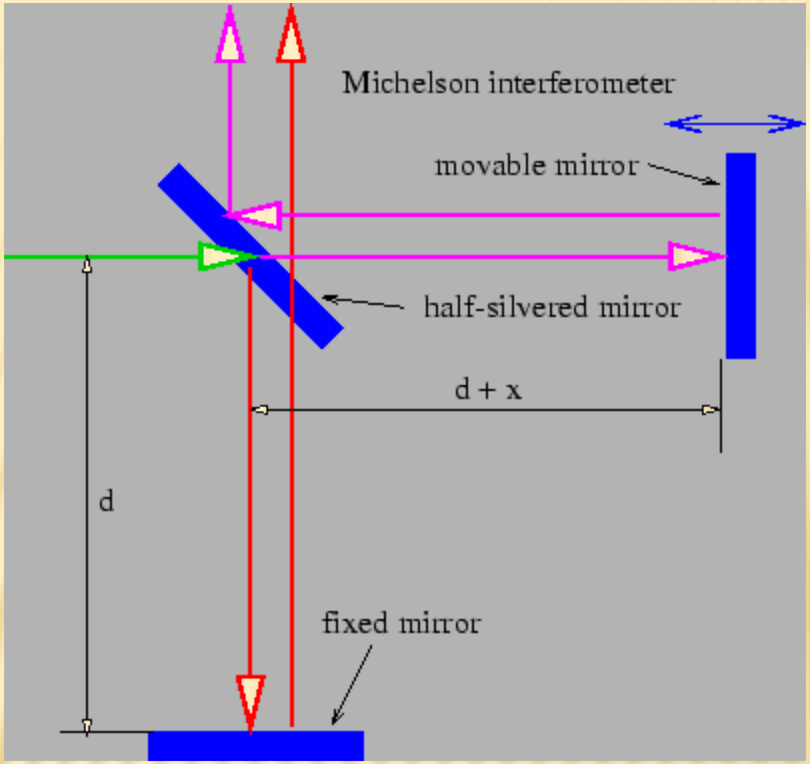
Light going through slits has to be "coherent" (does not work with "white" light)

http://badger.physics.wisc.edu/lab/manual2/node17.html

The phase change comes from the change in geometric length between the two "rays"

(change in length of 1/2 wavelength causes π change in phase – and destructive interference)

Michelson Interferometer
Make two paths from same source
(for coherence, can't do with white light)



Can change geometric path length with movable mirror.
Get interference fringes when recombine.

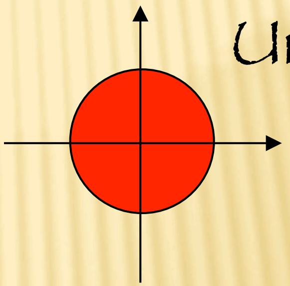
http://www.physics.nmt.edu/~raymond/classes/ph13xbook/node13.html

Definition of vector norms

Vector Norms

L^2 (Euclidean) norm :

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

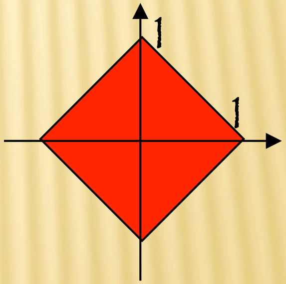


Unit circle

$$\|\mathbf{x}\|_2 < 1$$

L^1 norm :

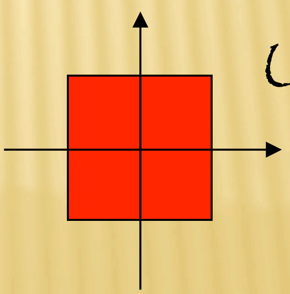
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$



$$\|\mathbf{x}\|_1 < 1$$

∞ or "max" norm :

$$\|\mathbf{x}\|_\infty = \max_i |x_i|$$



Unit square

$$\|\mathbf{x}\|_\infty < 1$$

From lec3.gps

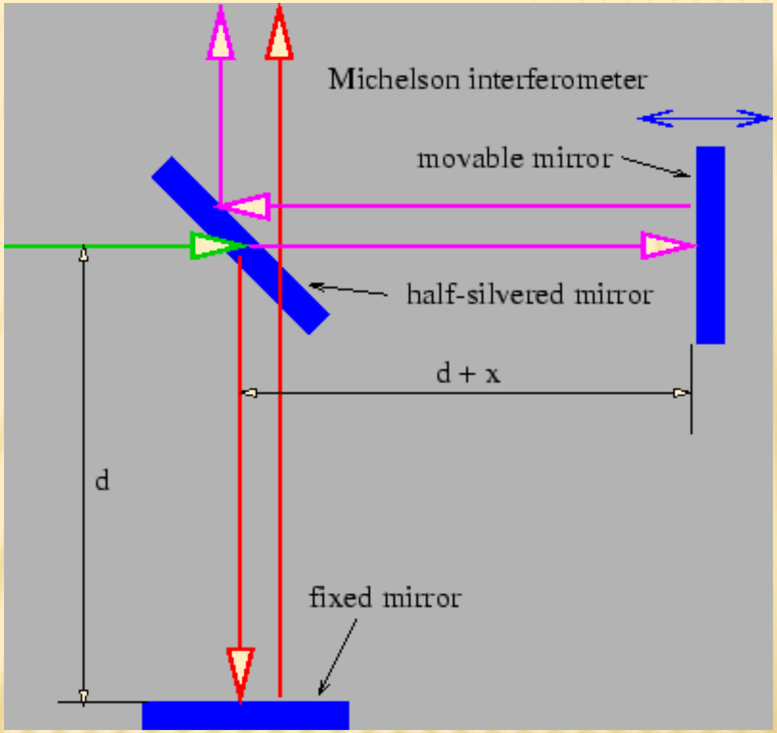
Interferemetry

The phase change comes from the change in distance (#wavelengths) between the two "rays"

(at constant velocity a) change geometric distance traveled

(change in length of 1/2 wavelength causes π change in phase – and destructive interference)

Michelson Interferometer
Make two paths from same source
(for coherence, can do with white light!)



Can change geometric path length with movable mirror (eg mount on speaker).
Get interference "fringes" when recombine.

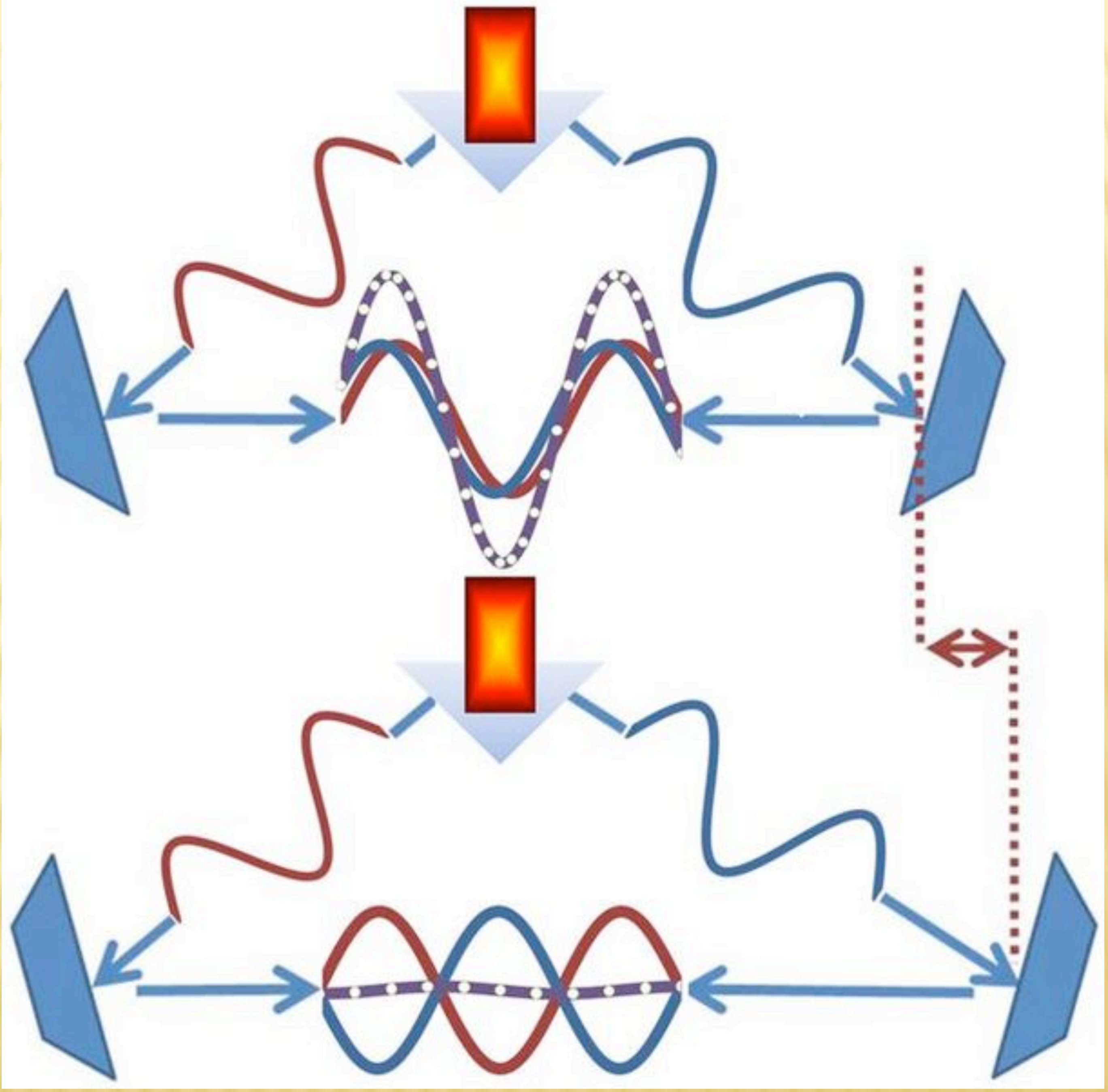
http://www.physics.nmt.edu/~raymond/classes/ph13xbook/node13.html

http://www.physics.uq.edu.au/people/mcintyre/applets/michelson/michelson.html

Note from animation

Can "integrate" (count continuously) the fringes and see how they change,

but there is a certain ambiguity
(each set of fringes looks same as others)
[no "reference" fringe]



Another way to get phase change

Change the "optical path length" (e.g. by changing velocity)

What counts is number of "cycles" (wavelengths), not geometric distance.

Change optical path length by changing index of refraction along path

(this is what happens to GPS in ionosphere and troposphere

– error for crustal motion,

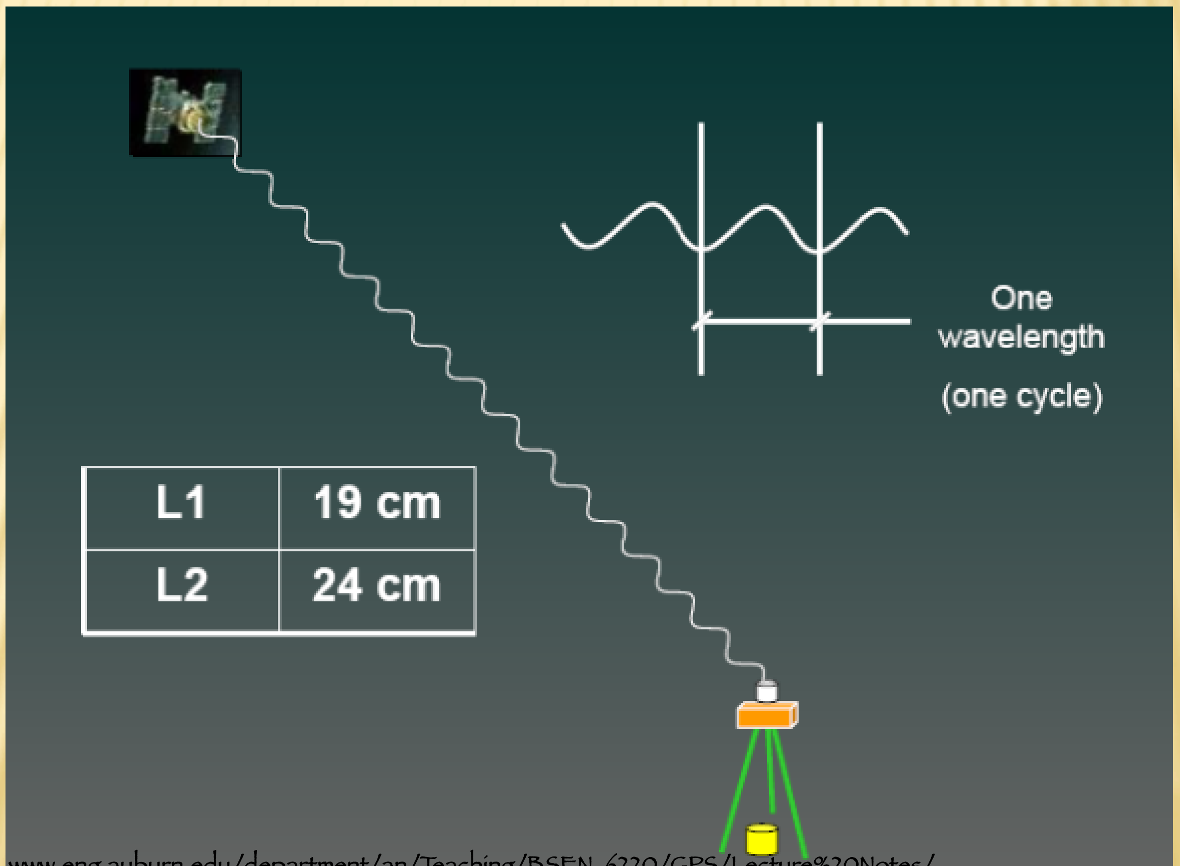
signal for ionospheric physics, weather, etc.)

GPS Carrier (beat) phase observable

(The word “beat” is usually not included in the “carrier phase observable” name, which can cause some [major] confusion)

The key is to count radio wavelengths between satellites and receiver.

This number (the phase) is an integer plus a fraction.



www.eng.auburn.edu/department/an/Teaching/BSEN_6220/GPS/Lecture%20Notes/Carrier_Phase_GPS.pdf

Strang, http://www.siam.org/siamnews/general/gps.htm

Phase measurements

One can convert phase to distance by multiplying the phase by the wavelength

(so phase measurements are another way to measure the distance from the satellite to the receiver – another "pseudo" distance measurement)

The wavelengths of the carrier waves are very short –

Approximately

19cm for L1 and

24cm for L2 –

compared to the C/A (length of one "chip" at ~1MHz is ~300m) and P code chip lengths.

http://www.gmat.unsw.edu.au/snap/gps/gps_survey/chap3/323.htm

Phase measurements

Phase can be measured to about 1% of λ (3.6°)

This gives a precision of

~2 mm for L1

~2.4 mm for L2

Phase measurements

this means that carrier phase can be measured to

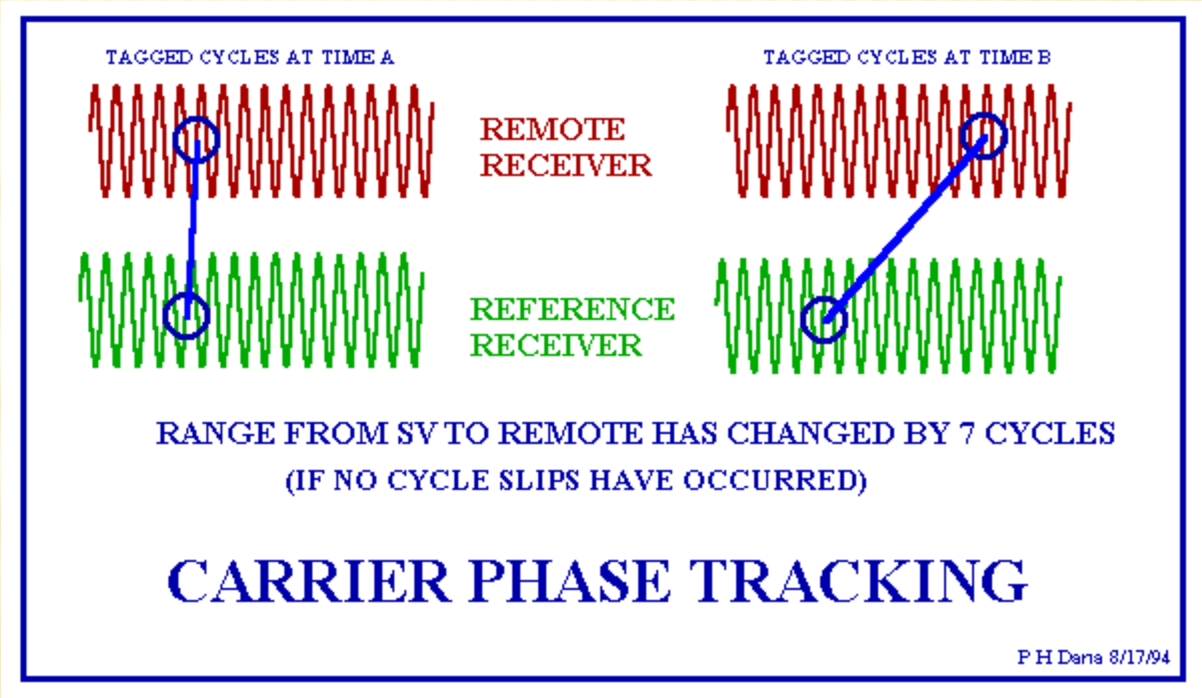
millimeter precision

compared with a few meters for C/A code measurements (but to get this you really need WAAS), and several decimeters for P code measurements.

http://www.gmat.unsw.edu.au/snap/gps/gps_survey/chap3/323.htm

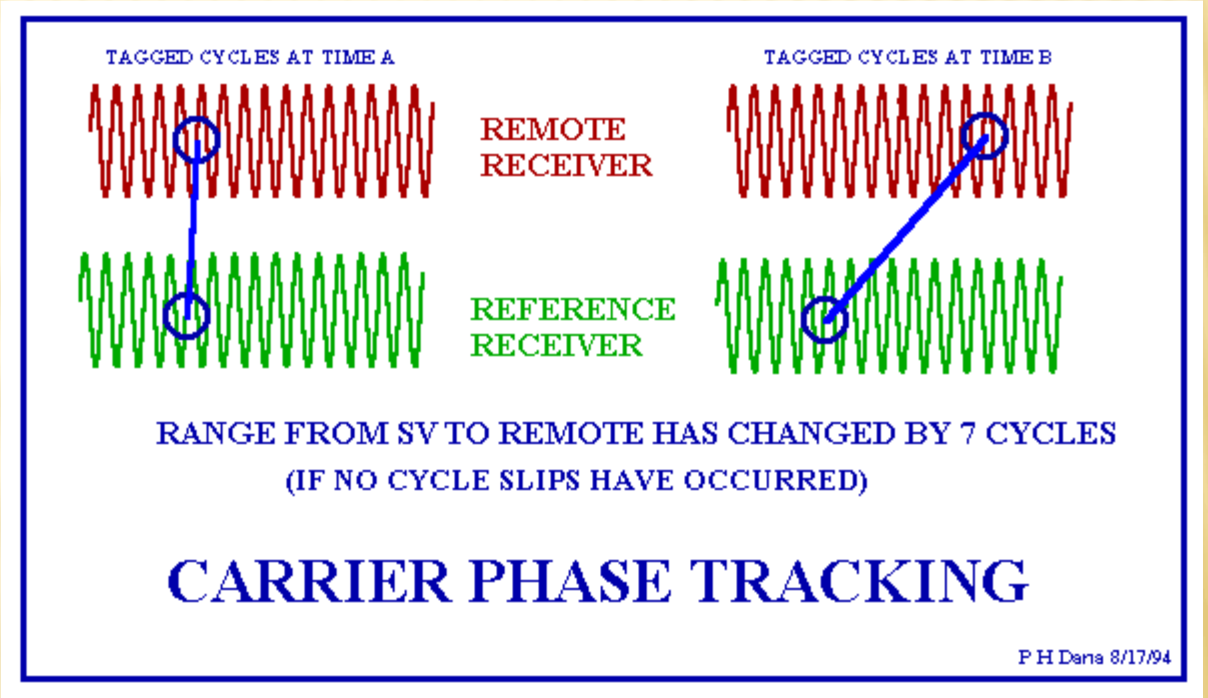
Tracking carrier phase signals, however, provides no time of transmission information.

The carrier signals, while modulated with time tagged binary codes, carry no time-tags that distinguish one cycle from another.



Dana, http://www.colorado.edu/geography/gcraft/notes/gps/gps_f.html

The measurements used in carrier phase tracking are differences in carrier phase cycles and fractions of cycles over time.

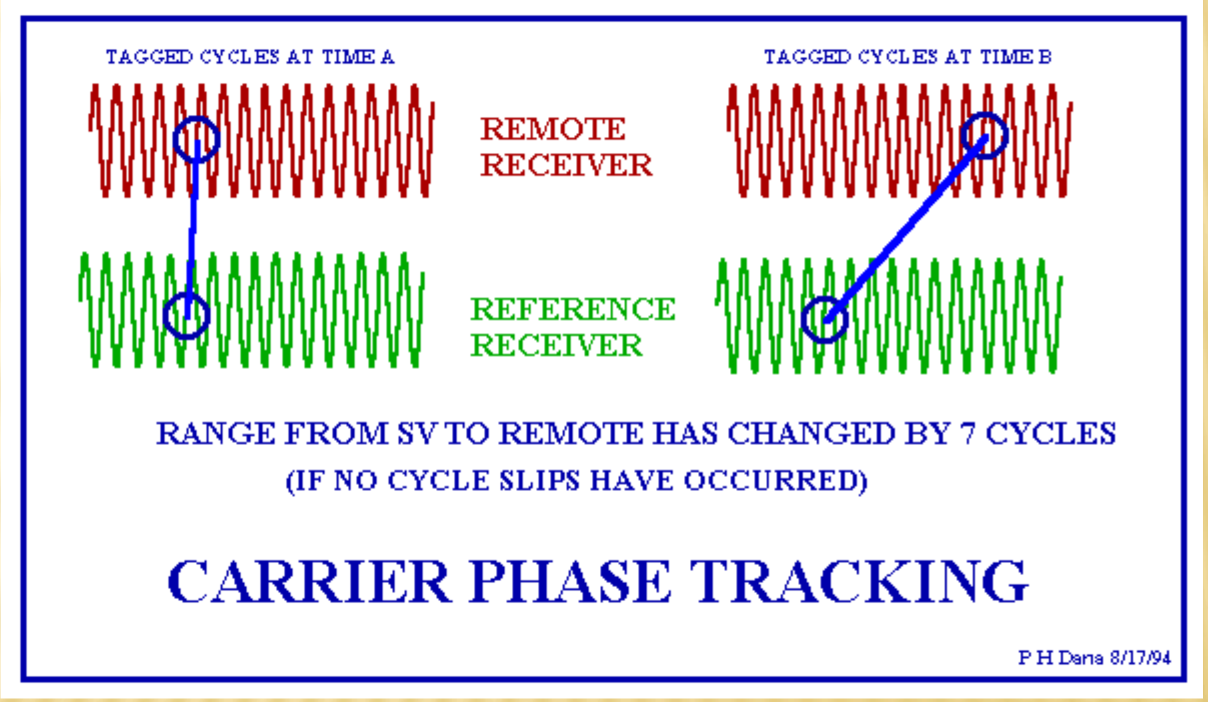


Dana, http://www.colorado.edu/geography/gcraft/notes/gps/gps_f.html

Unfortunately

phase measurement is "ambiguous" as it cannot discriminate one (either L1 or L2) <u>cycle</u> from another

(they all “look” the same).



http://www.gmat.unsw.edu.au/snap/gps/gps_survey/chap3/323.htm

In other words, time-of-transmission information for the signal cannot be imprinted onto the carrier wave as is done using PRN codes

(this would be possible only if the PRN code frequency was the same as the carrier wave,

rather than 154 or 120 times lower – and longer - in the case of the P code,

and 1540 or 1200 times lower – and longer - for the C/A code).

http://www.gmat.unsw.edu.au/snap/gps/gps_survey/chap3/323.htm

The basic phase measurement is therefore in the range

0° to 360°

(or 0 to 2π)

http://www.gmat.unsw.edu.au/snap/gps/gps_survey/chap3/323.htm

Phase measurements review:

Phase measurement PRECISE

But AMBIGUOUS

Another complication -

Phase measurements have to be corrected for

propagation effects

(several to 10' s of meters) to benefit from the increased precision

The key is to count radio wavelengths between satellites and receiver.

This number (the phase) is an integer plus a fraction.

The integer part (called the ambiguity) is the tricky problem.

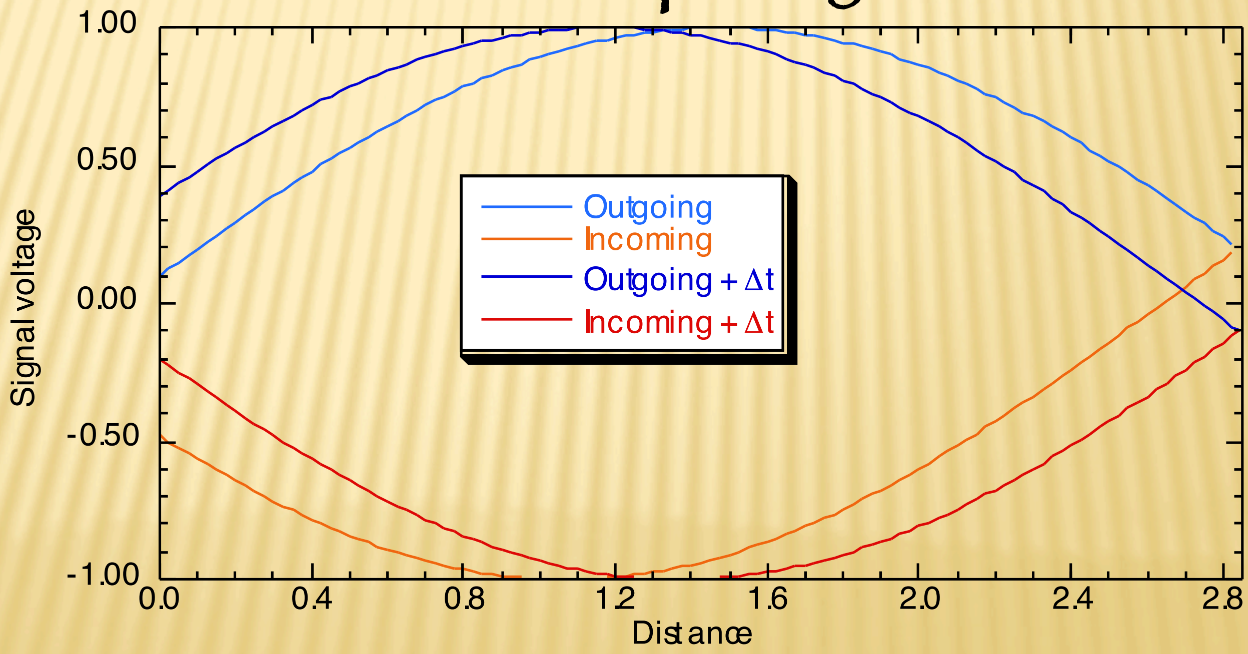
It has to be right, because one missing wavelength means an error of 19 cm or 24 cm (the satellite transmits on two frequencies).

Strang, http://www.siam.org/siamnews/general/gps.htm

Difference of phase measurement at two points less than one wave lenth apart.

(stays constant with time and depends on distance [for stationary source])

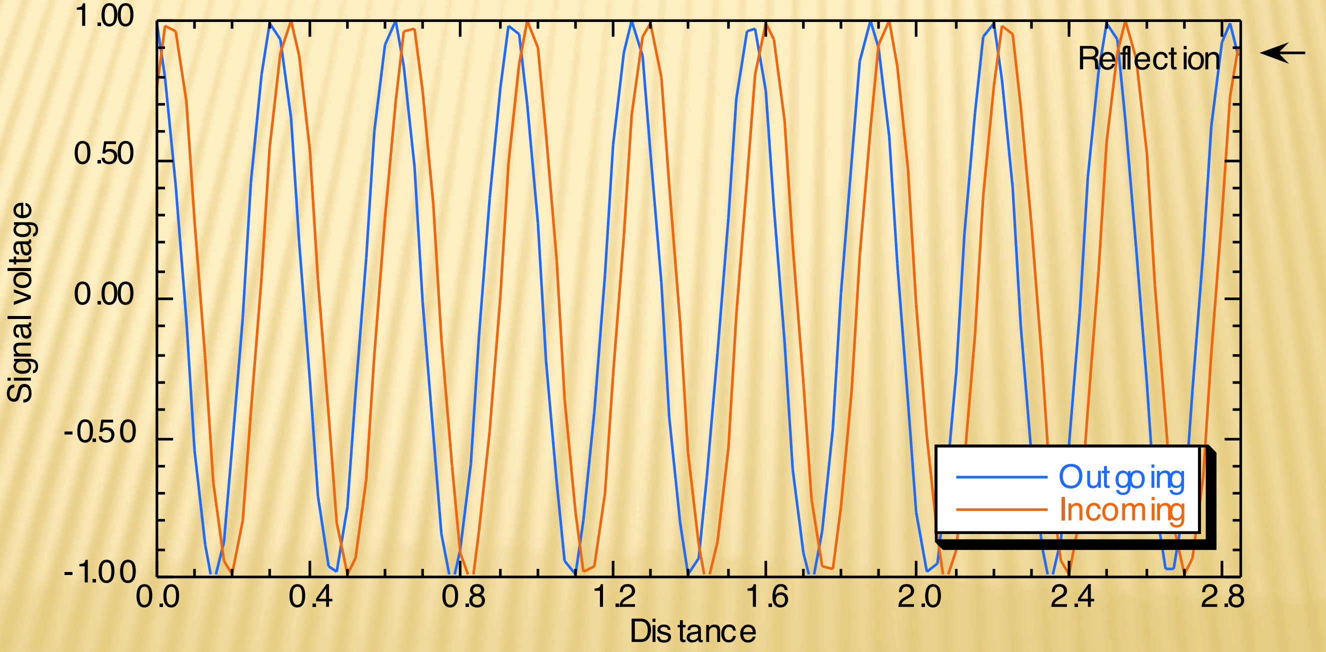
Low frequency



Higher frequency.

Phase difference still says something about distance but have to know number of cycles?

How to do this?



Note that the phase is not constant for fixed positions of the transmitter and receiver.

The rate of phase change, and therefore the frequency, is constant in this case.
(frequency is rate of change of phase [when the rate of change of phase is not constant - instantaneous frequency is the instantaneous rate of change of phase])

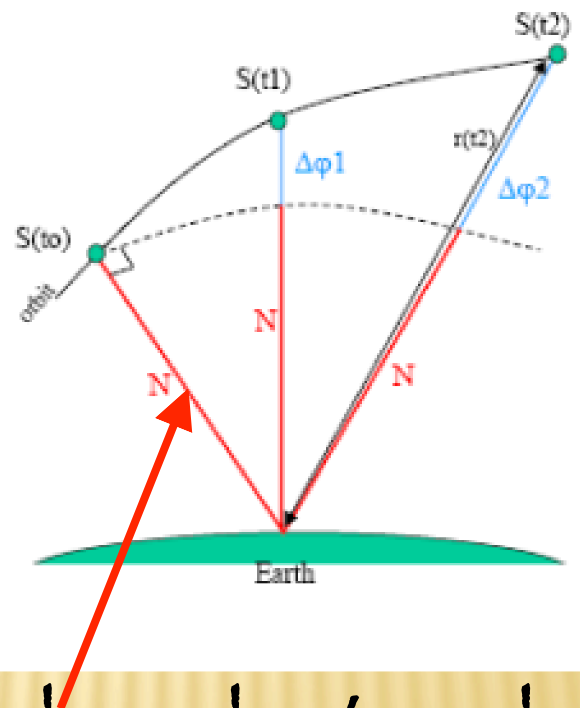
Moving transmitters and receivers cause the rate of phase change to vary, and therefore the frequency to vary --- a Doppler shift.

How to use the phase.

We can keep track of phase once we lock onto it.

Phase measurements

- When a satellite is locked (at t_o), the GPS receiver starts tracking the incoming phase
- It counts the (real) number of phases as a function of time = $\Delta\varphi\ (t)$
- But the initial number of phases N at t_o is unknown…
- However, if no loss of lock, N is constant over an orbit arc



But can't tell how many whole cycles/wavelengths there are between satellite and receiver – called the (integer) ambiguity.

From E. Calais

Determining this integer is like swimming laps in a pool

after an hour, the fractional part is obvious, but it is easy to forget the number of laps completed.

Strang, http://www.siam.org/siamnews/general/gps.htm

You could estimate it by dividing total swim time by approximate lap time.

For a short swim, the integer is probably reliable.

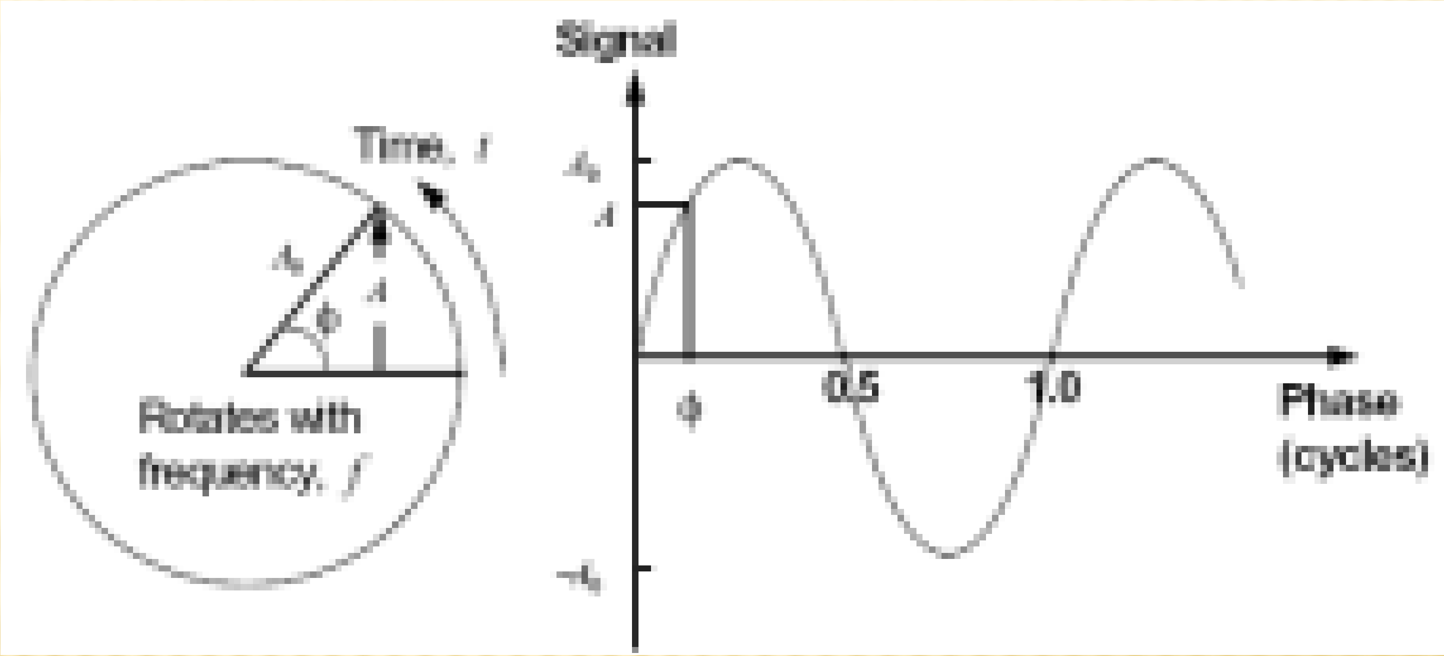
But the longer you swim, the greater the variance in the ratio.

In GPS, the longer the baseline between receivers, the harder it is to find this whole number.

Strang, http://www.siam.org/siamnews/general/gps.htm

Phase, frequency and Clock time

Phase is angle of rotation



Unit is cycles

Note is ambiguous by whole "rotations"

Concept of time

(or at least keeping track of it)

based on periodic "motion"

Day – rotation of earth on own axis
Year – rotation of earth around sun
Quartz crystal (or atomic clock) – oscillations
Etc.

Phase is "%" of period.
But does not count whole periods.

Need way to convert phase to time units.

write

$$T(t) = k\left(\phi(t) - \phi_0\right)$$

Where

T(t) is time according to our clock at (some "absolute" time) t

$\phi_0 = \phi(t{=}0)$ is the time origin (our clock reads *0* at ϕ_0)

k is the calibration constant converting cycles to seconds

Frequency

Expressed as cycles-per-second
(SI unit is Hertz)

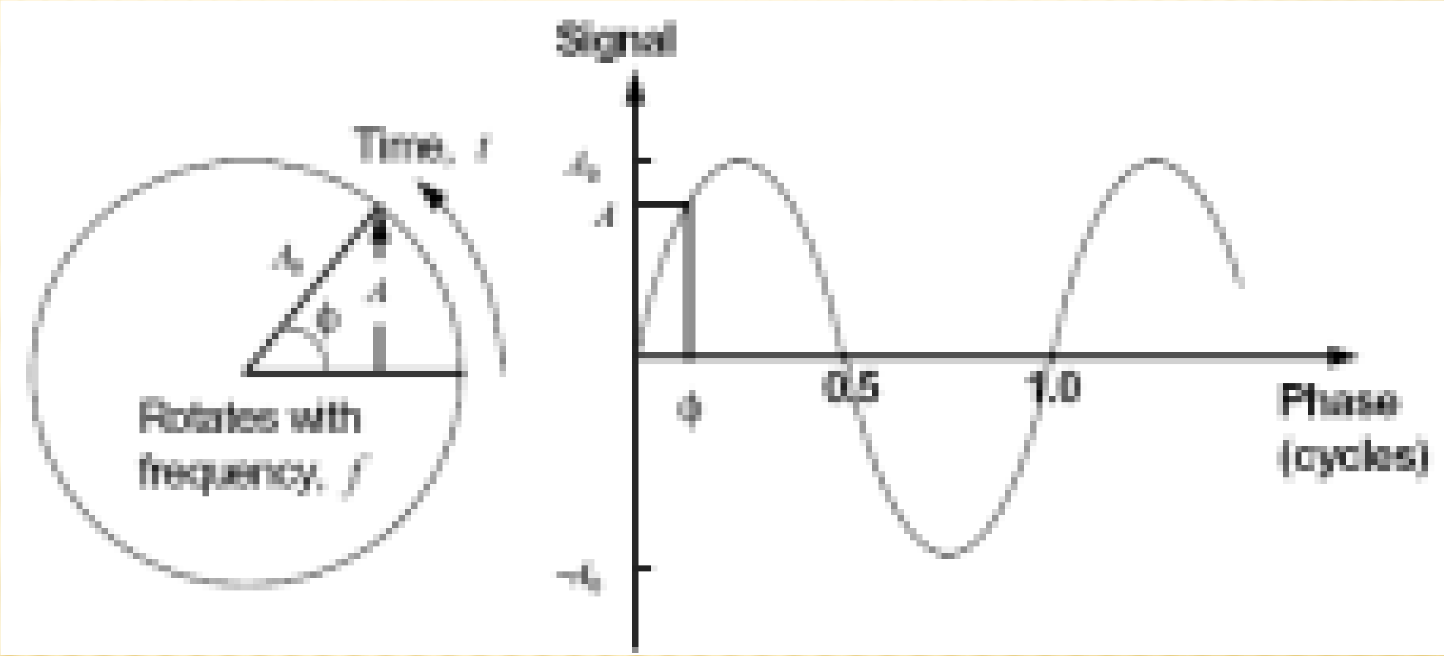
Assumes rotation rate is constant

Better definition – rate of change of phase with respect to time (this also covers instantaneous phase)

$$f = \frac{d\phi(t)}{dt}$$

$$f = \frac{d\phi(t)}{dt} = \text{constant}$$

pure sine/cosine



Phase changes linearly with time

Blewitt, Basics of GPS in "Geodetic Applications of GPS"

We will treat

-- Phase as the fundamental quantity

-- Frequency as the derived quantity or dependent variable

Basis for "ideal" clock

Constant frequency

$$\phi_{ideal} = f_0 t + \phi_0$$

$$T_{ideal} = k\, f_0 t$$

$$\phi_{ideal} = f_0 t + \phi_0$$

$$T_{ideal} = k\, f_0 t$$

This suggests that

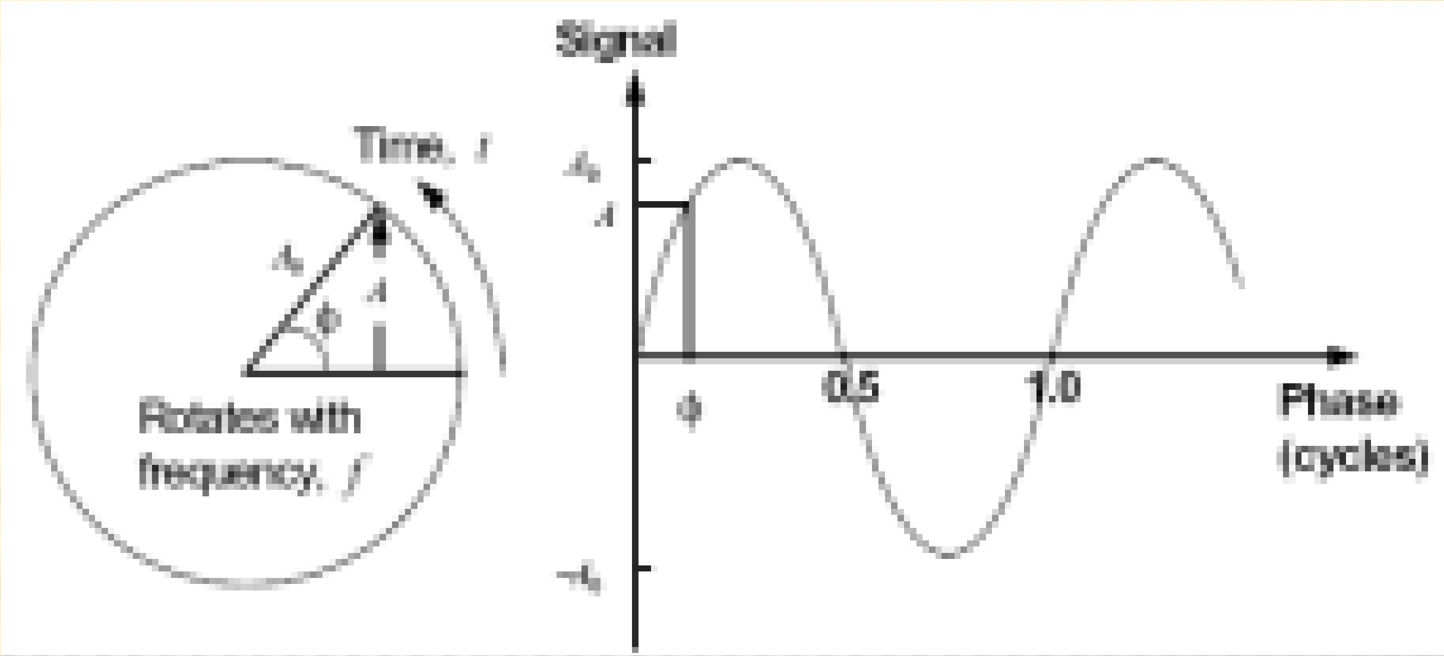
$k=1/f_0$

so

$$T(t) = \frac{\left(\phi(t) - \phi_0\right)}{f_0}$$

So we can describe the signal below as

$$A(t) = A_0 \sin\left(2\pi\phi(t)\right)$$



If one measures $A(t)$ one can determine $\phi(t)$

Signal for ideal clock

$$A_{ideal}(t) = A_0 \sin\left(2\pi\phi_{ideal}(t)\right)$$

$$A_{ideal}(t) = A_0 \sin\left(2\pi\left(f_0 t + \phi_0\right)\right)$$

$$A_{ideal}(t) = A_0 \cos\left(2\pi\phi_0\right)\sin\left(2\pi f_0 t\right) + A_0 \sin\left(2\pi\phi_0\right)\cos\left(2\pi f_0 t\right)$$

$$A_{ideal}(t) = A_0^S \sin\left(\omega_0 t\right) + A_0^C \cos\left(\omega_0 t\right)$$

Signal for real clock

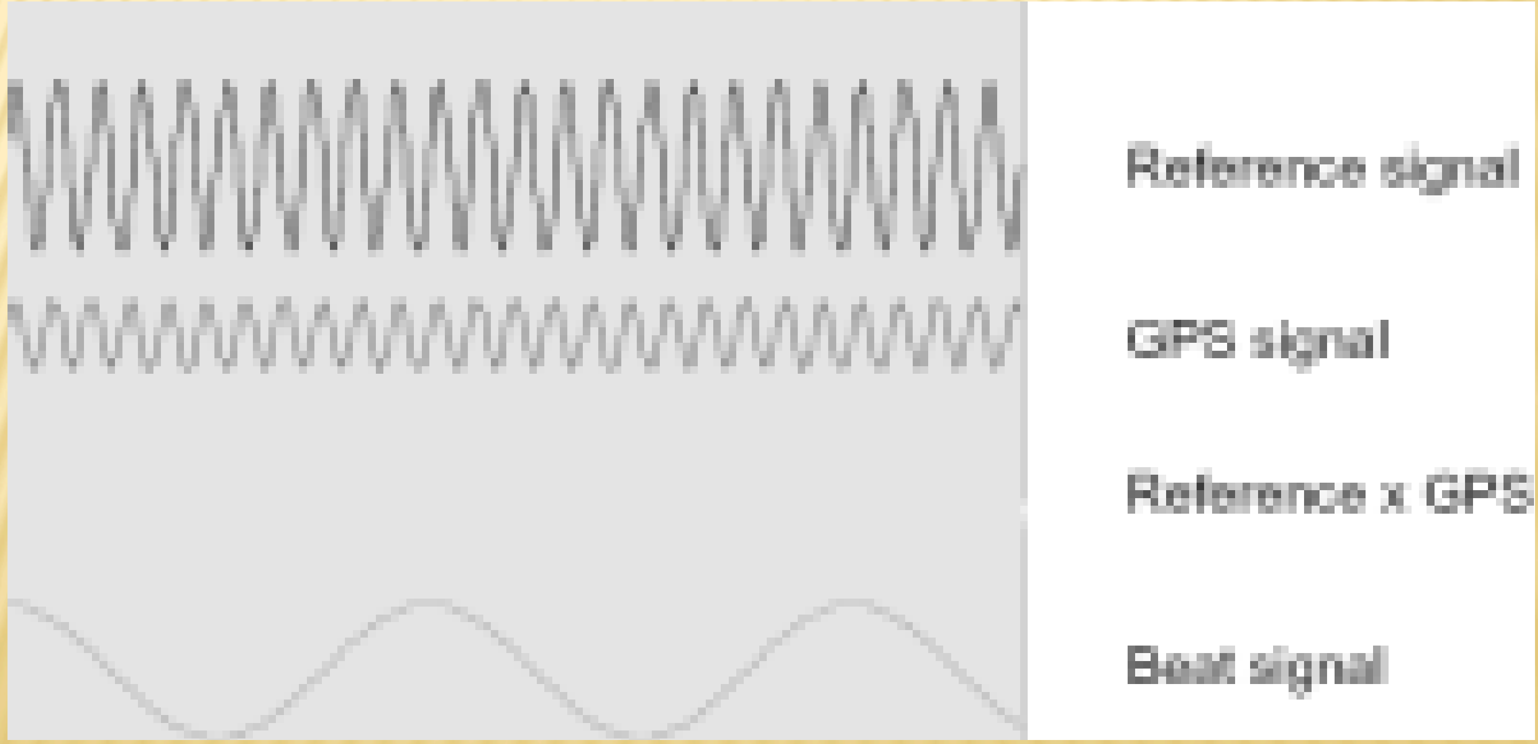
$$A_{real}(T) = A_0^S \sin\left(\omega_0 T\right) + A_0^C \cos\left(\omega_0 T\right)$$

GPS signal of this form PLUS “modulation” by + or – 1.

To "receive" a GPS signal the

received signal (whose frequency has been shifted by the Doppler effect – more later) is <u>mixed</u> with a receiver generated copy of the signal

producing a beat due to the difference in frequency

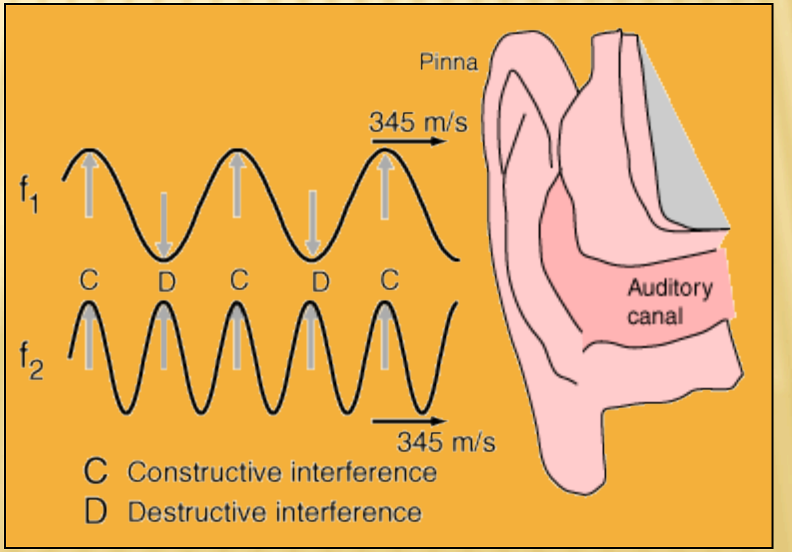


Blewitt, Basics of GPS in "Geodetic Applications of GPS"

When two sound waves of different frequency approach your ear, the alternating constructive and destructive interference causes the sound to be alternatively soft and loud

-a phenomenon which is called "beating" or producing beats.

-The beat frequency is equal to the absolute value of the difference in frequency of the two waves.



http://hyperphysics.phy-astr.gsu.edu/hbase/sound/beat.html

Beat Frequencies in Sound

The sound of a beat frequency or beat wave is a fluctuating volume caused when you add two sound waves of slightly different frequencies together.

If the frequencies of the sound waves are close enough together, you can hear a relatively slow variation in the volume of the sound.

A good example of this can be heard using two tuning forks that are a few Hz apart. (or in a twin engine airplane or boat when the engines are not "synched" = you hear a "wa-wa-wa-wa-… noise)

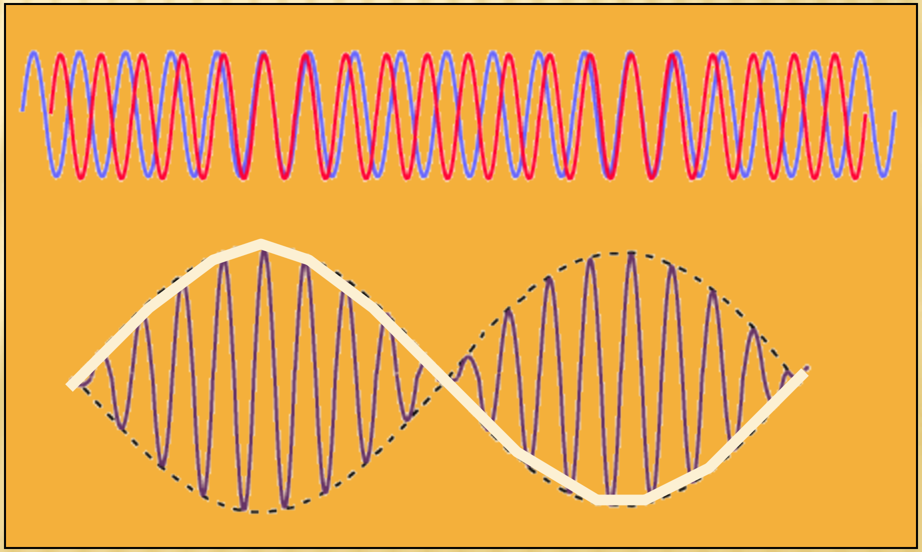
http://www.school-for-champions.com/science/sound_beat.htm

Beats are caused by the interference of two waves at the same point in space.

$$\cos(2\pi f_1) + \cos(2\pi f_2) = 2A\cos\left(2\pi\frac{f_1 - f_2}{2}\right)\cos\left(2\pi\frac{f_1 + f_2}{2}\right)$$

$$f_{beat} = \left|\frac{f_1 - f_2}{2}\right|$$

Beat -- Frequency of minimia, which happens twice per cycle.



http://hyperphysics.phy-astr.gsu.edu/hbase/sound/beat.html

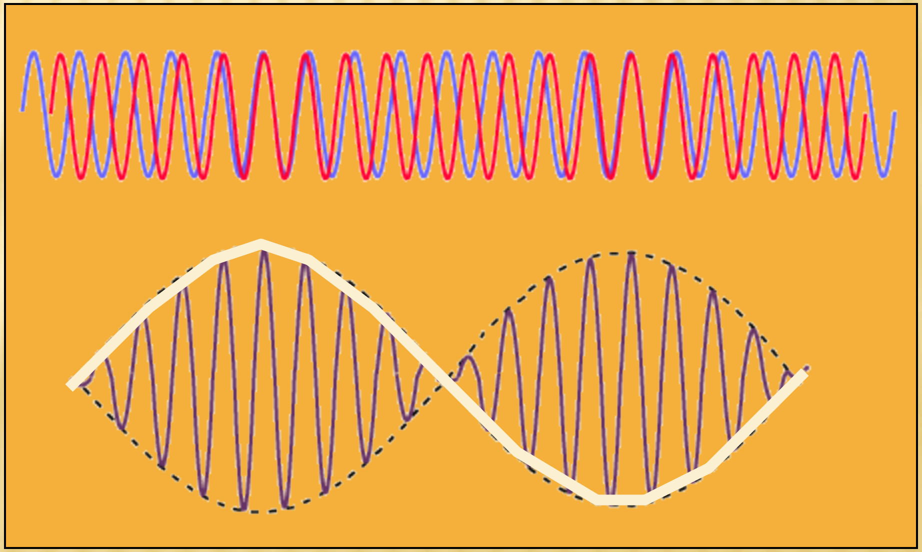
Note the frequencies are

half the difference and

the average of the original frequencies.

$$\cos(2\pi f_1) + \cos(2\pi f_2) = 2A\cos\left(2\pi\frac{f_1 - f_2}{2}\right)\cos\left(2\pi\frac{f_1 + f_2}{2}\right)$$

Different than multiplying (mixing) the two frequencies.



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