4. The Theorems of Euler and Chasles

4.1. Spherical Displacements

Euler's Theorem

We have seen that a spherical displacement or a pure rotation is described by a 3×3 rotation matrix. According to Euler's theorem,
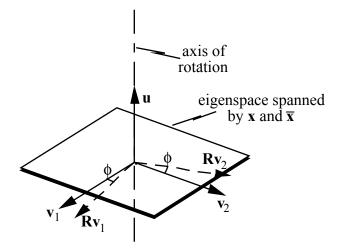
"Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O."
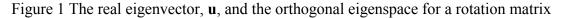
We will start with a general spherical displacement and show Euler's theorem is valid. Consider a body fixed frame that is displaced from $\{F\}$ to $\{M\}$. The displacement of any point on the rigid body can be written as:

$$\mathbf{P} = \mathbf{Rp}$$

where $\mathbf{p}$ and $\mathbf{P}$ are the position vectors before and after displacement respectively, and we are deleting the superscripts and subscripts associated with $^F\mathbf{R}_M$. Let us find the eigenvalues and eigenvectors of $\mathbf{R}$ by writing:

$$\mathbf{Rp} = \lambda\mathbf{p}$$

Figure 1 The real eigenvector, $\mathbf{u}$, and the orthogonal eigenspace for a rotation matrix

The characteristic equation is:

$$|\mathbf{R} - \lambda\mathbf{I}| = 0$$

If the elements of $\mathbf{R}$ are denoted by R_{ij}, the above equation can be written as:

$$-\lambda^3 + \lambda^2(R_{11} + R_{22} + R_{33})$$
$$-\lambda[(R_{22}R_{33} - R_{32}R_{23}) + (R_{11}R_{33} - R_{31}R_{13}) + (R_{11}R_{22} - R_{12}R_{21})] + |R| = 0$$

Because $\mathbf{R}$ is a rotation matrix, $|\mathbf{R}| = 1$. Also, because $\mathbf{R}^{-1} = \mathbf{R}^T$,

$$R_{11} = R_{22}R_{33} - R_{23}R_{32}$$
$$R_{22} = R_{11}R_{33} - R_{13}R_{31}$$
$$R_{33} = R_{11}R_{22} - R_{21}R_{12}$$

This simplifies the characteristic equation:

$$-\lambda^3 + \lambda^2(R_{11} + R_{22} + R_{33}) - \lambda(R_{11} + R_{22} + R_{33}) + 1 = 0$$

Factoring the left hand side, we get:

$$(\lambda - 1)\left(\lambda^2 - \lambda(R_{11} + R_{22} + R_{33}) + 1\right) = 0$$

We see that $\lambda = 1$ is an eigenvalue of $\mathbf{R}$. In other words, there exists a real vector, $\mathbf{u}$, such that all points on the line, $\mathbf{p} = \alpha\mathbf{u}$ remain fixed (invariant) under the transformation $\mathbf{R}$. Thus $\mathbf{R}$ is a rotation that leaves this line or axis fixed.

Define ϕ so that

$$\cos\phi = \tfrac{1}{2}\left((R_{11} + R_{22} + R_{33}) - 1\right) \tag{1}$$

Then the remaining eigenvalues are:

$$\lambda = e^{i\phi} = \cos\phi + i\sin\phi, \quad \lambda = e^{-i\phi} = \cos\phi - i\sin\phi$$

Thus the eigenvalues and eigenvectors are:

$$\lambda_1 = e^{i\phi}, \ \mathbf{p}_1 = \mathbf{x}, \ \text{say}$$

$$\lambda_2 = e^{-i\phi}, \ \mathbf{p}_2 = \overline{\mathbf{x}} \tag{2}$$

$$\lambda_3 = 1, \ \mathbf{p}_3 = \mathbf{u}.$$

where $\bar{\mathbf{x}}$ is the complex conjugate of $\mathbf{x}$.

Since $\mathbf{R}$ is orthogonal, $\mathbf{x}$ and $\bar{\mathbf{x}}$ are perpendicular to $\mathbf{u}$ and span the plane perpendicular to the axis of rotation. A real basis for this plane can be constructed using the vectors:

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{x}+\bar{\mathbf{x}}), \qquad\qquad \mathbf{v}_2 = \frac{i}{2}(\mathbf{x}-\bar{\mathbf{x}}) \tag{3}$$

$$
\begin{aligned}
\mathbf{R}\mathbf{v}_1 &= \frac{1}{2}(\lambda_1\mathbf{x}+\lambda_2\bar{\mathbf{x}}) & \mathbf{R}\mathbf{v}_2 &= \frac{i}{2}(\lambda_1\mathbf{x}-\lambda_2\bar{\mathbf{x}}) \\
&= \frac{1}{2}\left(e^{i\phi}\mathbf{x}+e^{-i\phi}\bar{\mathbf{x}}\right) & &= \frac{i}{2}\left(e^{i\phi}\mathbf{x}-e^{-i\phi}\bar{\mathbf{x}}\right) \\
&= \mathbf{v}_1\cos\phi + \mathbf{v}_2\sin\phi & &= -\mathbf{v}_1\sin\phi + \mathbf{v}_2\cos\phi
\end{aligned}
$$

As shown in Figure 1, the effect of the transformation, $\mathbf{R}$, is to rotate vectors in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ through an angle ϕ about $\mathbf{u}$, while vectors along $\mathbf{u}$ are invariant. Thus the transformation $\mathbf{R}$ rotates the rigid body about $\mathbf{u}$ through an angle ϕ. This proves Euler's theorem.

There is a *canonical representation* of any rotation matrix which allows us to view it as a rotation through an angle ϕ about the z axis[1]. If we consider the triad of orthonormal vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}\}$, we can easily show that $\mathbf{R}$ can be written in the form:

$$\mathbf{R} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T \tag{4}$$

where,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is essentially a similarity transformation. If we view the rotation from a new frame, $\{F'\}$, whose orientation is given by the coordinate transformation, ${}^{F}\mathbf{R}_{F'} = \mathbf{Q}$, it is clear that in this new frame, the displacement is a rotation about the z axis through an angle ϕ.

---

[1] Note that we could also have chosen to construct our canonical representation of a rotation by viewing it as a rotation about the x or the y axis.

4.2.   Representations for finite rotations

We have seen that any rotation or spherical displacement can be described by a rotation matrix. However, because the rotation matrix is orthogonal, its nine elements are not independent of each other. Specifically, they satisfy six independent equations:

$$R_{11}^2 + R_{12}^2 + R_{13}^2 = 1$$
$$R_{21}^2 + R_{22}^2 + R_{23}^2 = 1$$
$$R_{31}^2 + R_{32}^2 + R_{33}^2 = 1$$
$$R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32} = 0$$
$$R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33} = 0$$
$$R_{13}R_{12} + R_{23}R_{22} + R_{33}R_{32} = 0$$

Therefore there are only three independent parameters that completely describe a given rotation and the rotation matrix is a very redundant representation. In order to develop more compact representations, we pursue two types of representations. The first relies on decomposing a given rotation into three finite successive rotations and these three rotation angles, called Euler angles, completely describe the given rotation. The second is derived from the results of the previous section and explicitly identifies the axis of rotation and the angle of rotation. The Euler angles are defined in standard kinematics and dynamics texts and are not discussed here. Instead, we discuss axis-angle representations in some detail. In this section we define Euler parameters which provide another compact representation of the axis and angle of rotation. We also develop a simple result that allow us to obtain the rotation matrix from the axis and angle of rotation.

The axis-angle representation

Consider the rotation of a rigid body about **u** through an angle ϕ as shown in Figure 2. Consider the triangle QMN in the figure. We can write:

$$\mathbf{P} - \mathbf{p} = \mathbf{a}_1 + \mathbf{a}_2$$

or,

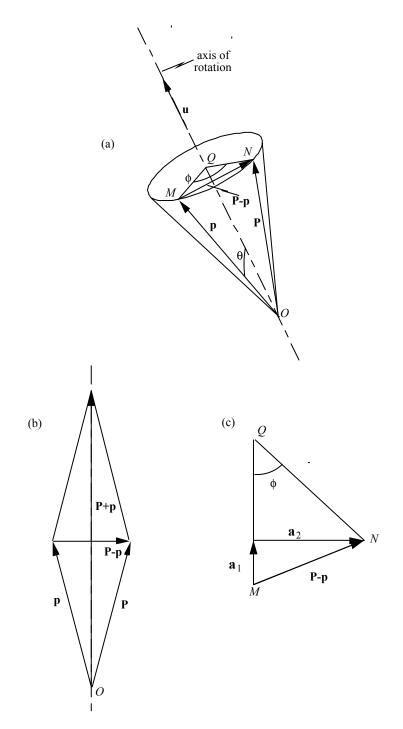$$\mathbf{Rp} = \mathbf{p} + \mathbf{a}_1 + \mathbf{a}_2$$

Figure 2 Rotation of **p** about **u** through ϕ. The vector **P** is related to **p** by the rotation matrix, **P=R p**. Since the tail of **P** and **p** are coincident at O, we can draw a cone whose apex is O such that both vectors are generators of the cone emanating from O.

From the geometry,

$$QN = QM = |\mathbf{p}| \sin\theta = |\mathbf{u} \times \mathbf{p}|$$
$$|\mathbf{a}_2| = QN \sin\phi = |\mathbf{u} \times \mathbf{p}| \sin\phi$$

Since $\mathbf{a}_2$ is perpendicular to $\mathbf{u}$ and $\mathbf{p}$,

$$\mathbf{a}_2 = (\mathbf{u} \times \mathbf{p}) \sin\phi.$$

The length QM can also be expressed as $QM = |\mathbf{p} - (\mathbf{p}.\mathbf{u})\mathbf{u}|$. Therefore,

$$\mathbf{a}_1 = [\mathbf{p}\text{-}(\mathbf{p}.\mathbf{u})\ \mathbf{u}]\ (1\text{-}\cos\phi).$$

Substituting $\mathbf{a}_1$ and $\mathbf{a}_2$ yields

$$\mathbf{R}\ \mathbf{p} \quad = \mathbf{p} + \mathbf{a}_1 + \mathbf{a}_2$$

$$= \mathbf{p} + [\mathbf{p}\text{-}(\mathbf{p}.\mathbf{u})\ \mathbf{u}]\ (1\text{-}\cos\phi) + (\mathbf{u} \times \mathbf{p}) \sin\phi..$$

If we let $\hat{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$, this equation yields:

$$\mathbf{R} = \mathbf{I}\cos\phi + \mathbf{u}\mathbf{u}^T(1-\cos\phi) + \hat{\mathbf{u}}\sin\phi \tag{5}$$

This is called the *Rodrigues' formula*, a result that should actually be attributed to Euler, Lexell, and Rodrigues. It describes the rotation matrix in terms of the angle and axis of the rotation.

To obtain the angle, ϕ, and axis of rotation, $\mathbf{u}$, from the rotation matrix, we can easily derive the following formulas:

$$\cos\phi = 0.5 \times (\text{Trace}(\mathbf{R}) - 1) \tag{6}$$

$$\hat{\mathbf{u}} = \frac{1}{2\sin\phi}\left(\mathbf{R} - \mathbf{R}^T\right) \tag{7}$$

There are many solutions for the angle of rotation because the inverse cosine function is multivalued. If we find ϕ from (6) restricting the range of the inverse cosine function to the interval $[0,\pi]$, we can find the axis of rotation from (7) provided ϕ is not either 0 or π. If $\mathbf{R} = \mathbf{I}$,

$\phi=0$ and the axis of rotation is not well defined. If the trace of the rotation matrix is -1, $\phi=\pi$, and from the Euler-Lexell formula:

$$\mathbf{R} = 2\mathbf{u}\mathbf{u}^T - \mathbf{I}$$

from which $\mathbf{u}$ can be solved.

For every rotation of angle ϕ about the axis $\mathbf{u}$, we can also obtain an equivalent axis-angle representation with a rotation $-\phi$ about the axis $-\mathbf{u}$. Also for every solution $(\mathbf{u}, \phi)$, we have other solutions $(\mathbf{u}, \phi\pm2k\pi)$ for all integer values of k.

Euler parameters

We now turn to a representation based on half the rotation angle and the axis of rotation. Define the four *Euler parameters*:

$$c_0 = \cos\frac{\phi}{2}, \quad c_1 = u_x \sin\frac{\phi}{2}, \quad c_2 = u_y \sin\frac{\phi}{2}, \quad c_3 = u_z \sin\frac{\phi}{2} \tag{8}$$

Note $c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1$. This is the equation of a unit sphere (or hypersphere) in R^4, also called the projective three sphere. Any point on this sphere corresponds to a unique axis and angle of rotation and therefore a unique rotation matrix. However, there are two points (diametrically opposite) on the sphere that give the same axis and angle. $\mathbf{c}$ and $-\mathbf{c}$ denote the same rotation. For example, a rotation of 90° degrees about the x-axis yields

$$\mathbf{c} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)^T$$

and a rotation of $-270°$ about the same axis yields

$$\mathbf{c} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0\right)^T$$

although they are really the same rotation.

If $\mathbf{c}$ is partitioned as follows:

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ \mathbf{d} \end{bmatrix}$$

it can be easily seen that Rodrigues formula can be written as:

$$\mathbf{R} = \mathbf{I}\left(2c_0^2 - 1\right) + 2\mathbf{d}\mathbf{d}^T + 2c_0\hat{\mathbf{d}} \tag{9}$$

This is the rotation matrix expressed in terms of Euler parameters.

4.3.   Spatial displacements and transformations

Chasles' Theorem

One of the most fundamental results in spatial kinematics is a theorem that is usually attributed to Chasles (1830), although Mozzi and Cauchy are credited with earlier results that are similar. The theorem states:

"The most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line."

Because this displacement is reminiscent of the displacement of a screw, it is called a screw displacement, and the line or axis is called the screw axis. We prove this theorem for the planar case and then for the spatial case.

For a planar displacement, one can find a fixed point of the displacement. This is a point that is left unchanged by the displacement. Consider a general 3×3 homogeneous transfer matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_{2\times 2} & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_x \\ d_y \end{bmatrix}.$$

If $\mathbf{R}$ is not the identity matrix, there is one fixed point on the rigid body for any displacement (in contrast to spatial displacements) called the *pole* (or the instantaneous center) of the displacement. If $\mathbf{c}$ is the position vector of the pole,

$$\mathbf{c} = \mathbf{R}\mathbf{c} + \mathbf{d}$$

which allows us to find the pole:

$$\mathbf{c} = (\mathbf{I}\text{-}\mathbf{R})^{-1}\,\mathbf{d} \tag{10}$$

This point corresponds to the eigenvector of the matrix **A** for a unit eigenvalue.

Thus any planar displacement can be considered as either a pure rotation about a point called the pole (if (**R-I**) is not singular) or a pure translation in the direction given by **d** (if **R=I**). Since pure rotations and pure translations are special cases of a screw motion, Chasles theorem is proved for the planar case.

If we consider a general 4×4 homogeneous transfer matrix, **A**, it has four eigenvalues two of which are equal to 1. (The other two are complex conjugate with a magnitude of 1). However there are no real eigenvectors corresponding to $\lambda=1$. This implies that a general displacement has no fixed points.

To prove Chasles' theorem for spatial displacements, consider the following similarity transformation of the matrix **A**:

$$\Lambda = \begin{bmatrix} \mathbf{Q}^T & -\mathbf{Q}^T\mathbf{c} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{c} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{Q}^T\mathbf{R}\mathbf{Q} & \mathbf{Q}^T\mathbf{R}\mathbf{c} - \mathbf{Q}^T\mathbf{c} + \mathbf{Q}^T\mathbf{d} \\ 0 & 1 \end{bmatrix}$$

Now choose **Q** as we did for the canonical representation of the rotation matrix in Equation (4) so that it consists of the eigenvectors of **R**:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u} \end{bmatrix} \tag{11}$$

The 3×3 orthogonal matrix part of **Λ** reduces to the rotation matrix corresponding to a rotation about the z axis:

$$\mathbf{Q}\mathbf{R}^T\mathbf{Q} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we define $\mathbf{d}'=\mathbf{Q}^T\mathbf{d}$, the 3×1 translation vector in **Λ** can be simplified as follows:

$$\mathbf{Q}^T \mathbf{Rc} - \mathbf{Q}^T \mathbf{c} + \mathbf{Q}^T \mathbf{d} = (\mathbf{Q}^T \mathbf{RQ} - \mathbf{I})\mathbf{Q}^T \mathbf{c} + \mathbf{Q}^T \mathbf{d}$$

$$= \begin{bmatrix} \cos\phi - 1 & -\sin\phi & 0 \\ \sin\phi & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c'_x \\ c'_y \\ c'_z \end{bmatrix} + \begin{bmatrix} d'_x \\ d'_y \\ d'_z \end{bmatrix}$$

where $\mathbf{c'} = \mathbf{Q}^T \mathbf{c}$ and $\mathbf{d'} = \mathbf{Q}^T \mathbf{d}$. If the top 2×2 submatrix of $(\mathbf{Q}^T \mathbf{RQ} - \mathbf{I})$ is nonsingular, we can always find $\mathbf{c'}$ such that

$$\begin{bmatrix} \cos\phi - 1 & -\sin\phi \\ \sin\phi & \cos\phi - 1 \end{bmatrix} \begin{bmatrix} c'_x \\ c'_y \end{bmatrix} = \begin{bmatrix} d'_x \\ d'_y \end{bmatrix} \tag{12}$$

In other words, we can always solve the first two equations of

$$(\mathbf{Q}^T \mathbf{RQ} - \mathbf{I})\ \mathbf{c'} = -\mathbf{d'}$$

for the first two components of $\mathbf{c'}$ as we did in Equation (10) and let the third component be zero. In this case, $\mathbf{\Lambda}$ has the form:

$$\mathbf{\Lambda} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

where k is given by the third component of the vector $\mathbf{Q}^T \mathbf{d}$. In this case, the displacement can be described by a rotation about the z axis through an angle ϕ and a concurrent translation along the z axis through a distance k. This is a screw displacement and proves Chasles' theorem for a general displacement.

If the top 2×2 submatrix of $(\mathbf{Q}^T \mathbf{RQ} - \mathbf{I})$ is singular, then $\mathbf{Q}^T \mathbf{RQ} = \mathbf{I}$. In this special case, $\mathbf{\Lambda}$ is a pure translation given by the vector $\mathbf{d}$. This is a special case of the screw displacement. Thus we have proved Chasles' theorem for this special case as well.

The geometric interpretation is shown in Figure 3. We are interested in the displacement that takes frame $\{G_1\}$ into frame $\{G_2\}$. The matrix

$$^{G_1}\mathbf{A}_{E_1} = \begin{bmatrix} \mathbf{Q} & \mathbf{c} \\ 0 & 1 \end{bmatrix}$$

represents the coordinate transformation from $\{E_1\}$ to $\{G_1\}$. In the reference frame $\{E_1\}$, the displacement is a pure rotation about the z axis through an angle ϕ, acommpanied by a concurrent translation along the same axis of $k=h\phi$. Thus,

$$^{E_1}\mathbf{A}_{E_2} = \mathbf{\Lambda} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward similarity transformation confirms:

$$\begin{aligned} ^{G_1}\mathbf{A}_{G_2} &= {}^{G_1}\mathbf{A}_{E_1} \, {}^{E_1}\mathbf{A}_{E_2} \, {}^{E_2}\mathbf{A}_{G_2} \\ &= {}^{G_1}\mathbf{A}_{E_1} \, \mathbf{\Lambda} \left[{}^{G_1}\mathbf{A}_{E_1} \right]^{-1} \end{aligned} \tag{14}$$

Determination of the screw axis from the homogeneous transformation matrix

Given the rotation matrix, $\mathbf{R}$, we can find $\mathbf{u}$, a unit vector along the screw axis, and ϕ, the angle of rotation about the screw axis from Equations (6-7). In order to completely specify the screw displacement, we want to find at the position vector of at least one point on the screw axis and the translation of a point on the rigid body along the screw axis.

We proceed according to the development in the previous subsection. First, find an appropriate proper orthogonal matrix $\mathbf{Q}$ such that its third column is $\mathbf{u}$. One may choose $\mathbf{Q}$ to be the matrix defined in Equation (11) but this requires the computation of the eigenvectors of $\mathbf{R}$ and is unnecessary. If we let $\mathbf{d}_p$ denote the projection of the vector $\mathbf{d}$ on a plane perpendicular to $\mathbf{u}$,

$$\mathbf{d_p} = \mathbf{d} - (\mathbf{d} \cdot \mathbf{u}) \, \mathbf{u} \tag{15}$$

then we can choose

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{bmatrix} \tag{16}$$
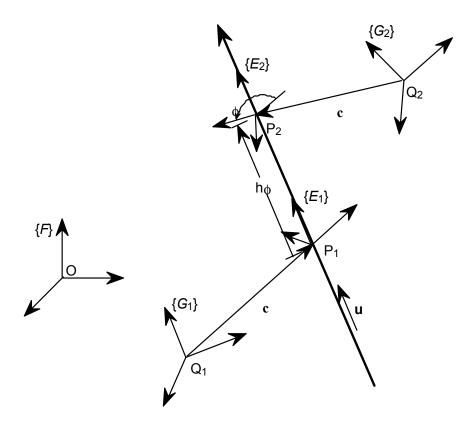
where,

Figure 3 The displacement of a body fixed frame from $\{G_1\}$ to $\{G_2\}$, the canonical representation of this displacement from $\{E_1\}$ to $\{E_2\}$, and the corresponding screw axis.

$$\mathbf{v} = \frac{\mathbf{d}_p}{d_p}, \qquad \mathbf{w} = \mathbf{u} \times \mathbf{v}$$

and d_p is the magnitude of $\mathbf{d}_p$ or the norm $\|\mathbf{d}_p\|$. Now Equation (12) simplifies to:

$$\begin{bmatrix} c_x' \\ c_y' \end{bmatrix} = -\begin{bmatrix} \cos\phi - 1 & -\sin\phi \\ \sin\phi & \cos\phi - 1 \end{bmatrix}^{-1}$$

$$= -\begin{bmatrix} \cos\phi - 1 & -\sin\phi \\ \sin\phi & \cos\phi - 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{v} \cdot \mathbf{d} \\ \mathbf{w} \cdot \mathbf{d} \end{bmatrix}$$

$$= \frac{d_p}{2(1 - \cos\phi)} \begin{bmatrix} (1 - \cos\phi) \\ \sin\phi \end{bmatrix}$$

Note that we have used the fact $\mathbf{d.w} = 0$. Thus, we get the position vector of a point on the screw axis:

$$\mathbf{c}' = \frac{d_p}{2} \begin{bmatrix} 1 \\ \frac{\sin\phi}{(1-\cos\phi)} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \mathbf{Q}\mathbf{c}' = \frac{d_p}{2}\left(\mathbf{v} + \frac{\sin\phi}{(1-\cos\phi)}\mathbf{w}\right) \tag{17}$$

The displacement of a point on the axis is given by the third component of **d'**:

$$k = \mathbf{u}.\mathbf{d} \tag{18}$$

and the pitch of the screw, h, is given by

$$h = \frac{k}{\phi} \tag{19}$$

4. References

[1] Bottema, O. and Roth, B., *Theoretical Kinematics*. Dover Publications, 1990.

[2] Brand, L., *Vector and Tensor Analysis*, John Wiley, 1947.

[3] Hunt, K.H., *Kinematic Geometry of Mechanisms*, Clarendon Press, Oxford, 1978.

[4] McCarthy. J.M., *Introduction to Theoretical Kinematics*, M.I.T. Press, 1990.

[5] Paul, R., *Robot Manipulators, Mathematics, Programming and Control*, The MIT Press, Cambridge, 1981.