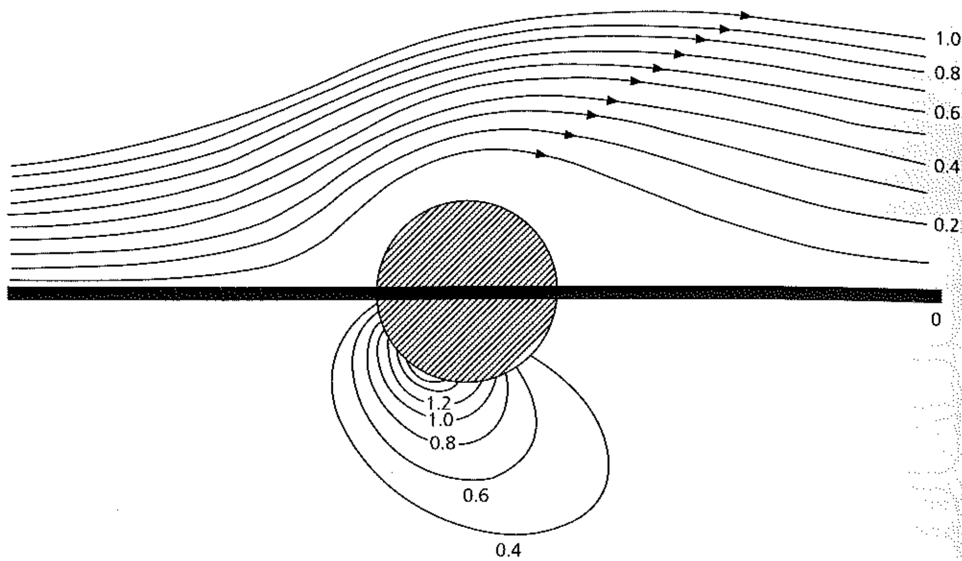

Fig. A.6-5 *Top:* streamlines showing the velocity of fluid flow around an object. Numbers on streamlines show the magnitude of the velocity. *Bottom:* contours of the curl for this velocity field. The curl is greatest near the sphere, where the fluid flow lines are the most curved. (After Batchelor, 1967. Reprinted with the permission of Cambridge University Press.)

an obscure-looking relation that is useful in deriving the existence of *P* and *S* waves.

A.7 Spherical coordinates

The vector operations discussed so far were performed in Cartesian coordinates, in which the unit basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ point in the same direction everywhere. There are, however, situations in which non-Cartesian coordinate systems without these nice properties are useful. In particular, *spherical* coordinates often simplify the solution of problems with a high degree of symmetry about a point.

A.7.1 *The spherical coordinate system*

In a spherical coordinate system, a point defined by a position vector **x** is described by its radial distance from the origin, $r = |\mathbf{x}|$, and two angles. θ is the *colatitude*, or angle between **x** and the x_3 axis, and ϕ, the *longitude*, is measured in the x_1–x_2 plane. Often the *latitude*, $90° - \theta$, is used instead of the colatitude. Spherical coordinates are often used in seismology because the earth is approximately spherically symmetric, varying with depth much more than laterally. Thus properties like velocity and density are often approximated as functions only of r, independent of θ and ϕ.

Figure A.7-1 shows the relations between rectangular and spherical coordinates. If the vector **x** is written as

$$\mathbf{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3, \tag{1}$$

then its components in rectangular coordinates (x_1, x_2, x_3) are described by spherical coordinates as

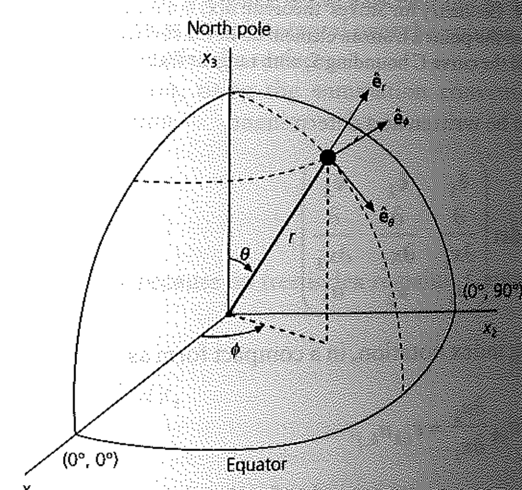

Fig. A.7-1 Relations between spherical (r, θ, ϕ) and Cartesian coordinates (x_1, x_2, x_3). (After Marion, 1970. From *Classical Dynamics of Particles and Systems*, 2nd edn, copyright 1970 by Academic Press, reproduced by permission of the publisher.)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}. \tag{2}$$

Conversely, the spherical coordinates r, θ, and ϕ can be written as

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \quad \theta = \cos^{-1}(x_3/r), \quad \phi = \tan^{-1}(x_2/x_1). \tag{3}$$

In the equatorial $(x_1$–$x_2)$ plane, $\theta = 90°$, $\cos\theta = 0$, $\sin\theta = 1$, so $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, and $x_3 = 0$. This is the same as the polar coordinate system described in Section A.3.1. Along the x_3 axis we have $\theta = 0°$, so $x_1 = x_2 = 0$, and $x_3 = r$. Any of these expressions written in terms of colatitude θ can be converted to latitude $\lambda = 90° - \theta$, using $\cos\theta = \sin\lambda$ and $\sin\theta = \cos\lambda$.

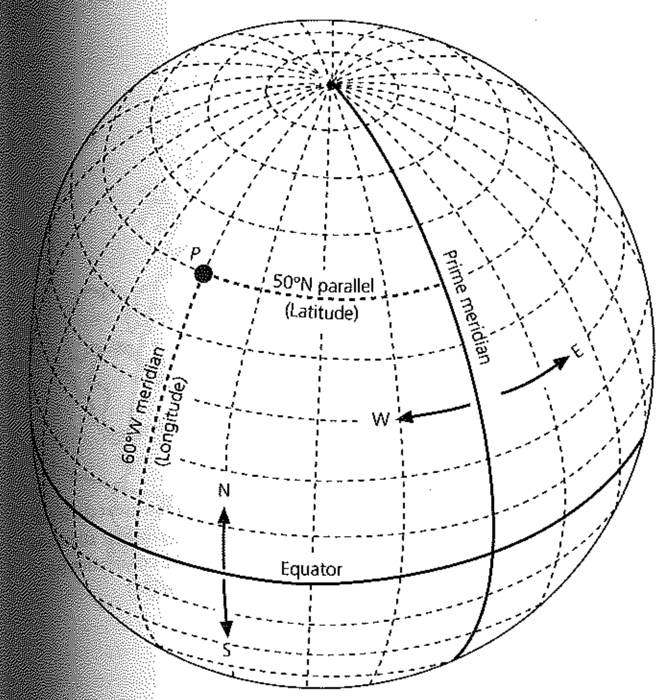

Fig. A.7-2 Geometry of the latitude and longitude system used to locate points on the earth's surface. A point P at 50°N, 60°W ($\theta = 40°$, $\phi = -60°$) is shown. (After Strahler, 1969.)

This coordinate system is the familiar one (Fig. A.7-2) used to locate points within the earth or on its surface, $r = a$. For this purpose, the origin is placed at the center of the earth, and the x_3 axis is defined by a line from the center of the earth through the north pole. The intersections of planes containing the x_3 axis with the earth's surface define *meridians*, lines of constant longitude. The x_1 axis intersects the equator at the *prime meridian*, on which ϕ is defined as zero, which has been chosen to run through Greenwich, England. The intersection of planes perpendicular to the x_3 axis with the earth's surface define *parallels*, lines of constant colatitude or latitude. Meridians are a special case of *great circles*, lines on the surface defined by the intersection of a plane through the origin with the surface of the spherical earth. Parallels are a special case of *small circles*, which are lines on the surface defined by the intersection of the surface of the spherical earth with a plane normal to a radius vector.

These conventions allow the colatitude θ ($0° \le \theta < 180°$) and longitude ϕ ($0° \le \phi < 360°$) to define a unique point on the earth's surface. Often locations are described in terms of latitudes north and south of the equator, and longitudes east and west of Greenwich. North and south latitudes correspond, respectively, to colatitudes less than or greater than 90°. Because ϕ measures longitude east of the prime meridian, west longitudes correspond to values of ϕ less than 0° or greater than 180°. Thus a point at (10°S, 110°W) has $\theta = 90° + 10° = 100°$, and $\phi = -110° = 360° - 110° = 250°$.

At any point, unit spherical basis vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ can be defined in the direction of increasing r, θ, and ϕ. $\hat{e}_r$ points away from the origin, and gives the upward vertical direction. $\hat{e}_\theta$ points south, and $\hat{e}_\phi$ points east. These two are sometimes written in terms of north- and east-pointing unit vectors, $\hat{e}_{NS} = -\hat{e}_\theta$ and $\hat{e}_{EW} = \hat{e}_\phi$.

An important feature of the unit spherical basis vectors is that at different points they are oriented differently with respect to the Cartesian axes. The Cartesian unit basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ point in the same direction everywhere. By contrast, for example, $\hat{e}_r$ points in the $\hat{e}_3$ direction at the north pole, and in the $-\hat{e}_3$ direction at the south pole. This effect is described by the Cartesian $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ components of the unit spherical basis vectors, at a point with colatitude θ and longitude ϕ:

$$\hat{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad \hat{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \hat{e}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{4}$$

The dependence on the colatitude and longitude describes how the orientation with respect to the Cartesian axes changes.

At any point, the spherical basis vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ form an orthonormal set. For problems whose spatial extent is small enough that the curvature of the earth can be ignored, these basis vectors provide a useful local coordinate system.

A.7.2 *Distance and azimuth*

Spherical coordinates are especially useful in describing the geographic relation between two points on the earth's surface. A common application is to find the distance between points and the direction of the great circle arc joining them. A great circle arc is the shortest path between points on a sphere, so if seismic velocity varies only with depth, the fastest path along the surface is the great circle arc, and the fastest paths through the interior are in the plane of the great circle and the center of the earth. Because velocities vary laterally by only a few percent throughout most of the earth (and imperceptibly in the liquid outer core), this is a good approximation for most seismic applications. The source-to-receiver distance is often given in terms of the angle Δ subtended at the center of the earth by the great circle arc between the two points (Fig. A.7-3). If Δ is expressed in radians, then the length s (in km) of the arc along the earth's surface is $R\Delta$, where R is the earth's radius ($\approx$ 6371 km). If Δ is expressed in degrees, $s = R\Delta\pi/180$, so one degree of arc equals 111.2 km.

Consider the great circle arc connecting an earthquake whose epicenter is at (θ_E, ϕ_E) and a seismic station at (θ_S, ϕ_S). Seismic waves that traveled along the great circle arc (or in the plane of this arc and the center of the earth) left the earthquake in a direction given by the *azimuth* angle ζ measured clockwise from the local direction of north at the epicenter to the great

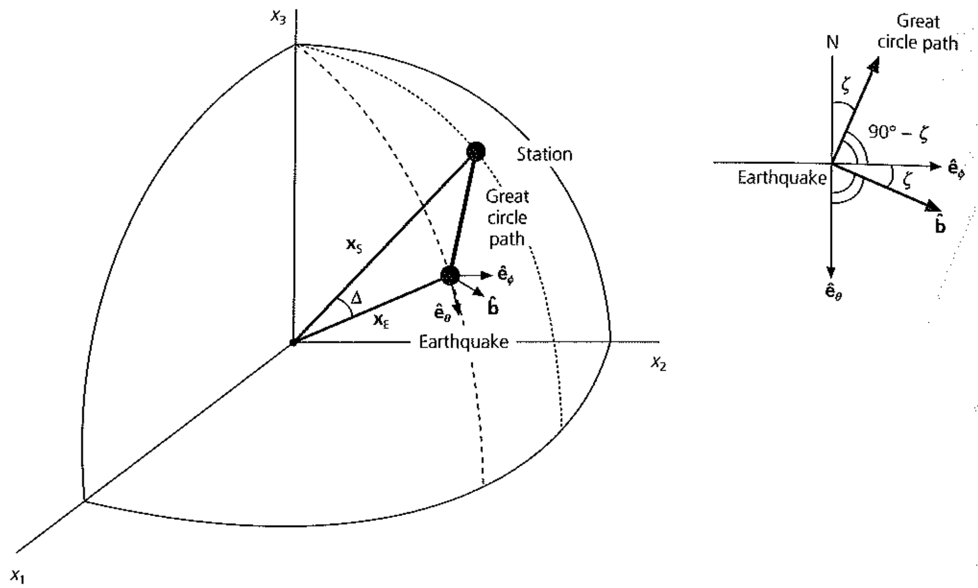

Fig. A.7-3 Geometry of the great circle path between an earthquake epicenter and seismic station (*left*), showing the convention for defining the azimuth, ζ (*right*).

circle arc. These waves arrive at the seismometer from a direction described by the *back azimuth* angle ζ' measured clockwise from the local direction of north at the seismometer to the great circle arc. To find these quantities, the Cartesian components of the position vectors for the earthquake and the station are written, using Eqn 2:

$$\mathbf{x}_E = \begin{pmatrix} R \sin\theta_E \cos\phi_E \\ R \sin\theta_E \sin\phi_E \\ R \cos\phi_E \end{pmatrix} \quad \mathbf{x}_S = \begin{pmatrix} R \sin\theta_S \cos\phi_S \\ R \sin\theta_S \sin\phi_S \\ R \cos\theta_S \end{pmatrix}. \tag{5}$$

The distance Δ, the angle between $\mathbf{x}_S$ and $\mathbf{x}_E$, is given by the scalar product

$$\mathbf{x}_S \cdot \mathbf{x}_E = R^2 \cos\Delta, \tag{6}$$

so

$$\Delta = \cos^{-1}[\cos\theta_E \cos\theta_S + \sin\theta_E \sin\theta_S \cos(\phi_S - \phi_E)]. \tag{7}$$

This formula defines Δ uniquely between 0 and 180°. This shorter portion of the great circle is called the *minor arc* connecting the two points; the longer portion, known as the *major arc*, is (360° – Δ) degrees long.

To compute the azimuth from the earthquake to the station, consider $\hat{\mathbf{b}}$, a unit vector normal to the great circle in the local horizontal plane at $\mathbf{x}_E$, which is written using the vector product of the position vectors

$$\mathbf{x}_S \times \mathbf{x}_E = \hat{\mathbf{b}} R^2 \sin\Delta. \tag{8}$$

Evaluation of the vector product gives

$$\hat{\mathbf{b}} = \frac{1}{\sin\Delta} \begin{pmatrix} \sin\theta_S \cos\theta_E \sin\phi_S - \sin\theta_E \cos\theta_S \sin\phi_E \\ \cos\theta_S \sin\theta_E \cos\phi_E - \cos\theta_E \sin\theta_S \cos\phi_S \\ \sin\theta_S \sin\theta_E \sin(\phi_E - \phi_S) \end{pmatrix}. \tag{9}$$

The azimuth angle ζ, measured clockwise from north, is then given (Fig. A.7-3) by

$$\cos\zeta = \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\phi = \frac{1}{\sin\Delta}(\cos\theta_S \sin\theta_E - \sin\theta_S \cos\theta_E \cos(\phi_S - \phi_E)) \tag{10}$$

and

$$\sin\zeta = \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\theta = \frac{1}{\sin\Delta} \sin\theta_S \sin(\phi_S - \phi_E). \tag{11}$$

Use of both sin ζ and cos ζ makes the angle ζ unambiguous (0° ≤ ζ < 360°). The azimuth from an earthquake to a receiver is useful, because earthquakes radiate more energy in some directions than in others (Chapter 4), so measurements at different azimuths yield information about the source.

The back azimuth ζ', obtained by reversing the indices E and S in Eqns 10 and 11, shows the direction from which seismic energy arrives at a seismometer. Seismometers typically record the north–south and east–west components of horizontal ground motion. Using the back azimuth, these observations can be converted into *radial* (along the great circle path) and *transverse* (perpendicular to the great circle path) components by a vector transformation (Eqn A.5.9). This distinction is made because waves appearing on these components propagated differently (Section 2.4). The azimuth and back azimuth

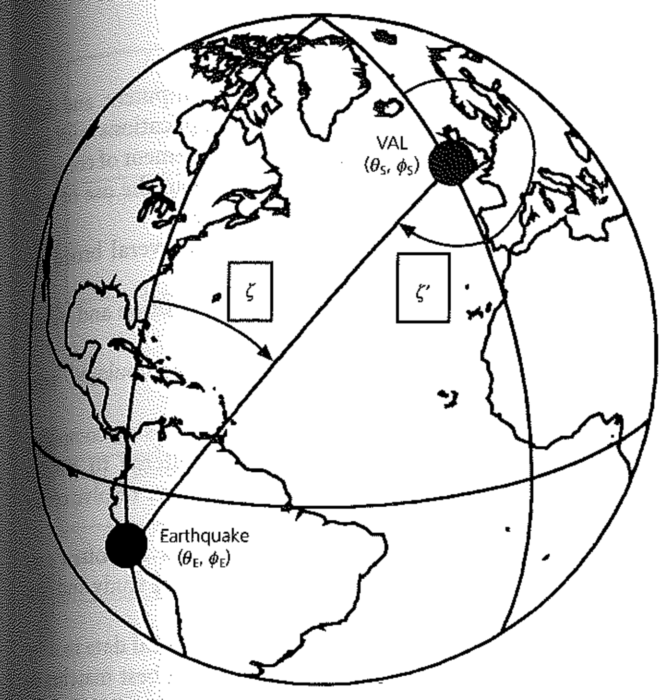

Fig. A.7-4 Geometry of the great circle path for an earthquake in the Peru trench recorded at station VAL (Valentia, Ireland). The azimuth, ζ, and back azimuth, ζ', are not simply related, due to the sphericity of the earth.

angles are measured clockwise from north, a geographic convention which contrasts with the mathematical one of measuring angles counterclockwise from the x_1 direction. Figure A.7-4 illustrates this geometry for an earthquake in the Peru trench ($\theta_E = 102°$, $\phi_E = -78°$) recorded at station VAL (Valentia, Ireland; $\theta_S = 38°$, $\phi_S = -10.25°$). The resulting distances and azimuths are $\Delta = 86°$, $\zeta = 35°$, $\zeta' = 245°$.[1]

This analysis assumes that the earth is perfectly spherical. In fact, the earth is flattened by its rotation into a shape close to an oblate ellipsoid, so the radius varies with colatitude approximately as

$$r(\theta) = R_e(1 - f\cos^2\theta), \tag{12}$$

where R_e is the equatorial radius, 6378 km. The flattening factor f is approximately 3.35×10^{-3}, or about 1/298, so the polar radius R_p is 6357 km. An average radius can be defined as the radius of a sphere with the same volume as the earth, if it were a perfect ellipsoid. Because the volume of an ellipsoidal earth would be $(4/3)\pi R_e^2 R_p$, and a sphere of radius R has volume $(4/3)\pi R^3$, the average radius is 6371 km. For certain applications the ellipticity is included in precise distance calculations.

[1] These distance–azimuth equations also have nonseismological applications because ships and aircraft follow the shortest (great circle) paths between two points when possible.

A.7.3 *Choice of axes*

Spherical coordinates are also used with axes different from the geographic ones. Because the physics of a problem does not depend on the choice of coordinates, a set of coordinates that simplifies the relevant expressions is used. For example, in earthquake source studies, the x_3 axis can be chosen to go from the center of the earth to the location of the earthquake. The prime meridian, and hence x_1, axis can be selected so that the fault is oriented in the direction $\phi = 0$. These axes simplify the description of the seismic waves radiated by an earthquake, because the distance Δ from the source is now the colatitude. Moreover, the radiation pattern generally has a high degree of symmetry about the fault, so simple functions of ϕ appear. By contrast, the radiation pattern need have no symmetry about the North pole and Greenwich meridian, so a description in those coordinates would usually be more complicated.

Fortunately, a coordinate system referred to the earthquake location does not make describing the propagation of waves from the source any more difficult. Because earth structure varies primarily with depth, the spherical symmetry about the center of the earth is independent of the axis orientation chosen. The geographical convention in which the earth rotates about the x_3 axis is helpful for navigation. In most seismological applications, however, the north direction has no particular significance because the propagation of seismic waves is essentially unaffected by the earth's rotation. The choice of a prime meridian is arbitrary; in the early nineteenth century some American maps had it through Washington DC, and some French maps had it through Paris.

A.7.4 *Vector operators in spherical coordinates*

Because at a point on the sphere the unit spherical basis vectors are oriented up, south, and east, the basis vectors at different locations are generally not parallel. This makes the vector differential operators more complicated, because these operators involve taking spatial derivatives of vectors. In Cartesian coordinates the unit basis vectors are not affected by this differentiation because they do not change orientation, so only derivatives of the components need be taken. In spherical coordinates, because a vector **u** is

$$\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_\phi\hat{\mathbf{e}}_\phi, \tag{13}$$

differential operators acting on **u** must incorporate the derivatives of the basis vectors. Thus, in spherical coordinates, for a scalar field ψ and a vector field **u**:

$$\text{grad}\,\psi = \hat{\mathbf{e}}_r \frac{\partial\psi}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\mathbf{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{14}$$

$$\text{div}\,\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} \tag{15}$$