Plane Wave Reflection and Transmission Coefficients for the Fluid-Solid Interface

Consider the following geometry:

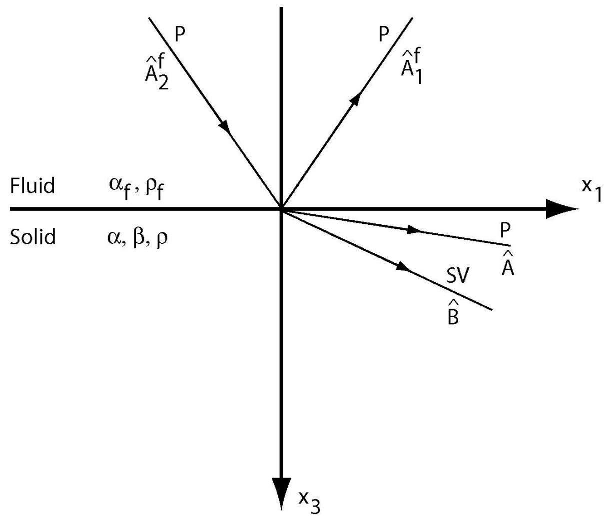

Figure 1: Incident P wave from upper left interacting with a fluid/solid boundary.

The Cartesian displacements are given by

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}$$
$$u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \qquad (1)$$

Fourier transforming

$$\hat{u}_1 = \frac{\partial \hat{\phi}}{\partial x_1} - \frac{\partial \hat{\psi}}{\partial x_3}$$
$$\hat{u}_3 = \frac{\partial \hat{\phi}}{\partial x_3} + \frac{\partial \hat{\psi}}{\partial x_1} \qquad (2)$$

The plane wave P and SV potentials are given by

(in the fluid)

$$\hat{\phi}^f(\omega) = \hat{A}_1^f e^{-i\omega\left(px_1 - \eta_{\alpha_f} x_3\right)} + \hat{A}_2^f e^{-i\omega\left(px_1 + \eta_{\alpha_f} x_3\right)} \qquad (3)$$

(in the solid halfspace)

$$\hat{\phi}(\omega) = \hat{A}e^{-i\omega(px_1+\eta_\alpha x_3)}$$
$$\hat{\psi}(\omega) = \hat{B}e^{-i\omega(px_1+\eta_\beta x_3)} \tag{4}$$

where p is the ray parameter (1/c) and the vertical slownesses are give by

$$\eta_v = \left(\frac{1}{v^2} - p^2\right)^{1/2} \tag{5}$$

To relate the fluid and solid wave potentials, consider the boundary conditions at $x_3 = 0$.

$$\hat{u}_3^f = \hat{u}_3$$
$$\hat{\tau}_{33}^f = \hat{\tau}_{33}$$
$$\hat{\tau}_{31} = 0 \tag{6}$$

Note that three relations are needed at the boundary because there are three unknown wave potential coefficients that must be found in terms of the amplitude of the incident acoustic wave. Or in other words, three equations must be used to determine three unknowns.

The stress components are found in terms of the transformed potentials.

$$\hat{\tau}_{ij} = \lambda\hat{\varepsilon}_{kk}\delta_{ij} + 2\mu\hat{\varepsilon}_{ij} \tag{7}$$

$$\hat{\tau}_{33} = \lambda\nabla^2\hat{\phi} + 2\mu\left(\frac{\partial^2\hat{\phi}}{\partial x_3^2} + \frac{\partial^2\hat{\psi}}{\partial x_1 \partial x_3}\right) \tag{8}$$

$$\hat{\tau}_{31} = \mu\left(2\frac{\partial^2\hat{\phi}}{\partial x_1 \partial x_3} + \frac{\partial^2\hat{\psi}}{\partial x_1^2} - \frac{\partial^2\hat{\psi}}{\partial x_3^2}\right) \tag{9}$$

Evaluating the vertical displacements, normal and shear stress components (equations 2, 8, 9) at $x_3 = 0$ (equations 6) yields a set of three equations for three unknowns in the coefficients of the potentials. These three relations can be written in matrix form as

$$\underset{\sim}{D}\begin{pmatrix}\hat{A}_1^f\\ \hat{A}\\ \hat{B}\end{pmatrix}=\begin{pmatrix}\eta_{\alpha_f}\\ \rho_f\\ 0\end{pmatrix}\hat{A}_2^f \tag{10}$$

where,

$$\underset{\sim}{D}=\begin{pmatrix}\eta_{\alpha_f} & \eta_\alpha & p\\ \rho_f & -\left(\rho-2\mu p^2\right) & -2\mu p\eta_\beta\\ 0 & 2p\eta_\alpha & \left(p^2-\eta_\beta{}^2\right)\end{pmatrix} \tag{11}$$

The solution to equation (10) is

$$\begin{pmatrix}\hat{A}_1^f\\ \hat{A}\\ \hat{B}\end{pmatrix}=\underset{\sim}{D}^{-1}\begin{pmatrix}\eta_{\alpha_f}\\ \rho_f\\ 0\end{pmatrix}\hat{A}_2^f \tag{12}$$

where $\underset{\sim}{D}^{-1}$ can be found from

$$\underset{\sim}{D}^{-1}=\frac{1}{\det\underset{\sim}{D}}\begin{pmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{pmatrix} \tag{13}$$

and the cofactors of the matrix $\underset{\sim}{D}$ are found from the minors M_{ik} through

$$C_{ik}=\left(-1\right)^{i+k}M_{ik} \tag{14}$$

Evaluating the determinent yields

$$\det\underset{\sim}{D}=\eta_{\alpha_f}\left[-\left(p^2-\eta_\beta{}^2\right)\left(\rho-2\mu p^2\right)+4\mu p^2\eta_\alpha\eta_\beta\right]+\frac{\rho_f\eta_\alpha}{\beta^2} \tag{15}$$

and the cofactors

$$\underset{\sim}{D}^{-1} = \frac{1}{\det \underset{\sim}{D}} \begin{pmatrix} -\left(p^2-\eta_\beta{}^2\right)\left(\rho-2\mu p^2\right)+4\mu p^2\eta_\alpha\eta_\beta & \dfrac{\eta_\alpha}{\beta^2} & -2\mu p\eta_\alpha\eta_\beta+\left(\rho-2\mu p^2\right)p \\ -\rho_f\left(p^2-\eta_\beta{}^2\right) & \eta_{\alpha_f}\left(p^2-\eta_\beta{}^2\right) & 2\mu p\eta_{\alpha_f}\eta_\beta+\rho_f p \\ 2\rho_f p\eta_\alpha & -2p\eta_\alpha\eta_{\alpha_f} & -\left(\rho-2\mu p^2\right)\eta_{\alpha_f}-\rho_f\eta_\alpha \end{pmatrix} \tag{16}$$

We can identify the reflection and transmission coefficients for the potentials as

$$R_{PP} = \frac{\hat{A}_1^f}{\hat{A}_2^f}$$
$$T_{PP} = \frac{\hat{A}}{\hat{A}_2^f}$$
$$T_{PS} = \frac{\hat{B}}{\hat{A}_2^f} \tag{17}$$

Using equations (12), (15), and (16), these are

$$R_{PP} = \frac{\eta_{\alpha_f}\left[-\left(p^2-\eta_\beta{}^2\right)\left(\rho-2\mu p^2\right)+4\mu p^2\eta_\alpha\eta_\beta\right]-\dfrac{\rho_f\eta_\alpha}{\beta^2}}{\det \underset{\sim}{D}} \tag{18}$$

$$T_{PP} = \frac{-2\rho_f\eta_{\alpha_f}\left(p^2-\eta_\beta^2\right)}{\det \underset{\sim}{D}} \tag{19}$$

$$T_{PS} = \frac{4\rho_f p\eta_\alpha\eta_{\alpha_f}}{\det \underset{\sim}{D}} \tag{20}$$

Note, for vertical incidence (p=0)

$$R_{PP} = \frac{\rho\alpha - \rho_f\alpha_f}{\rho\alpha + \rho_f\alpha_f}$$
$$T_{PP} = \frac{2\rho_f\alpha}{\rho\alpha + \rho_f\alpha_f}$$
$$T_{PS} = 0 \tag{21}$$

which are the reflection and transmission coefficients for two fluids in contact.

Finally, use equations (2) to evaluate the spectral displacements at x_3=0 in the solid medium.

$$\hat{u}_1 = (-i\omega)e^{-i\omega p x_1}\left[pT_{PP} - \eta_\beta T_{PS}\right]\hat{A}_2^f$$
$$\hat{u}_3 = (-i\omega)e^{-i\omega p x_1}\left[\eta_\alpha T_{PP} + pT_{PS}\right]\hat{A}_2^f \tag{22}$$

The pressure on the solid halfspace from the acoustic field is given by

$$\begin{aligned}\hat{P}(\omega) &= -\hat{\tau}_{33}^f(x_3 = 0)\\ &= -(-i\omega)^2 e^{-i\omega p x_1}\rho_f\left[R_{PP} + 1\right]\hat{A}_2^f\end{aligned} \tag{23}$$

Thus, we see that the ratio of the displacement to pressure is, for example,

$$\frac{\hat{u}_3(\omega)}{\hat{P}(\omega)} = \frac{1}{i\omega}\frac{\left[\eta_\alpha T_{PP} + pT_{PS}\right]}{\rho_f\left[R_{PP} + 1\right]} \tag{24}$$

This means that the displacement field in the solid is proportional to the time integral of the pressure field in the fluid. An impulse in pressure, for example, would give rise to a step in displacement.

An acoustic wave incident on an elastic halfspace displays several unusual wave propagation effects because of the low acoustic velocity in the atmosphere compared to velocities in the elastic medium. Take an earth model of an atmosphere over unconsolidated sediments.

α=1.8 km/s
β=0.6 km/s
ρ=2.0 gm/cc
α_f=0.33 km/s
ρ_f=0.00129 gm/cc

The very large acoustic impedance contrast ensures that the reflected pressure wave be close to unity. However, the transmitted P and converted SV waves show a variety of effects as various critical angles occur for the incident pressure wave. Figure 2 shows the behavior the reflection and transmission coefficients where the real and imaginary parts are plotted as a function of incidence angle of the incident pressure.

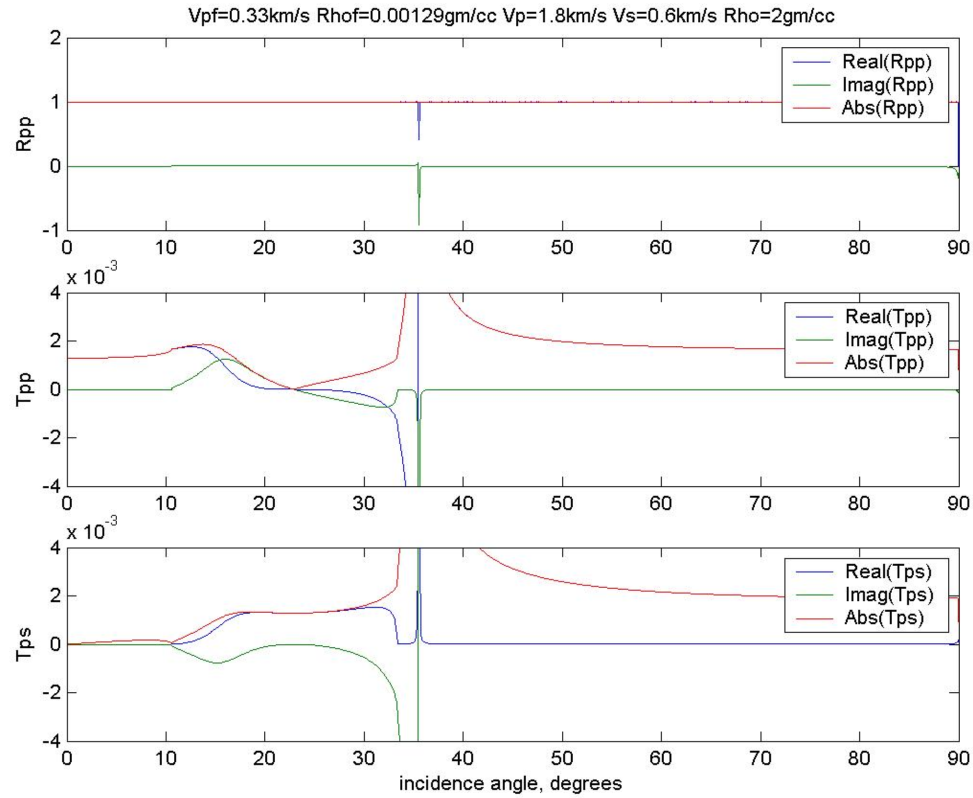

Figure 2: Real and imaginary parts of RPP, TPP, and TPS as a function of incidence angle in the incident pressure.

Critical angles for the incident pressure field occur when

$$\frac{\sin i_{c_\alpha}}{0.33} = \frac{1}{1.8}$$
$$i_{c_\alpha} = 10.56^o \qquad (25)$$

for a P head wave in the solid medium and when

$$\frac{\sin i_{c_\beta}}{0.33} = \frac{1}{0.6}$$
$$i_{c_\beta} = 33.37^o \qquad (26)$$

for an SV head wave in the solid medium. At these angles the vertical slownesses for P and SV, η_α and η_β, respectively, become zero then purely complex as ray parameter increases. To keep our solutions analytic, we choose the negative square root or

$$\eta_v = -i\left(p^2 - \frac{1}{v^2}\right)^{1/2} \quad for\ p > \frac{1}{v} \tag{27}$$

Recall that a complex reflection or transmission coefficient in the frequency domain implies a constant phase shift for the time history in the time domain.

Another interesting effect occurs for the reflection and transmissions when $p \sim 0.94\beta$ or $i \sim 35.5^o$ for this model. The denominators of the reflection and transmission coefficients undergo minima that cause large amplitude responses. This can be better seen in Figure 3 where amplitude and phase are plotted at full scale. Figure 4 shows the behavior of the denominator and Figure 5 shows detail around the minimum. At this value of ray parameter the reflection and transmission coefficients have a maximum which would be the Rayleigh pole if there were no atmosphere and free surface boundary conditions prevailed. It can be easily shown that the imaginary part of $\det \underset{\sim}{D}$ is proportional to the fluid density which, for Earth's atmosphere, is three orders of magnitude smaller than sediment density.

Technically, the transmission coefficients do not have a true Rayleigh pole since the denominators do not have a zero. Because of the upper fluid halfspace, the surface wave is internal to the structure and should rightly be called a Stonely wave. Nevertheless, we interpret this result as showing that an incident pressure pulse will excite a Rayleigh wave in the solid medium at this (real) angle of incidence and that ground motions (Figures 6 and 7) can be three orders of magnitude greater at this angle of incidence compared to all other angles of incidence. This is an example of an acoustically coupled Rayleigh wave at the atmosphere-earth surface interface.

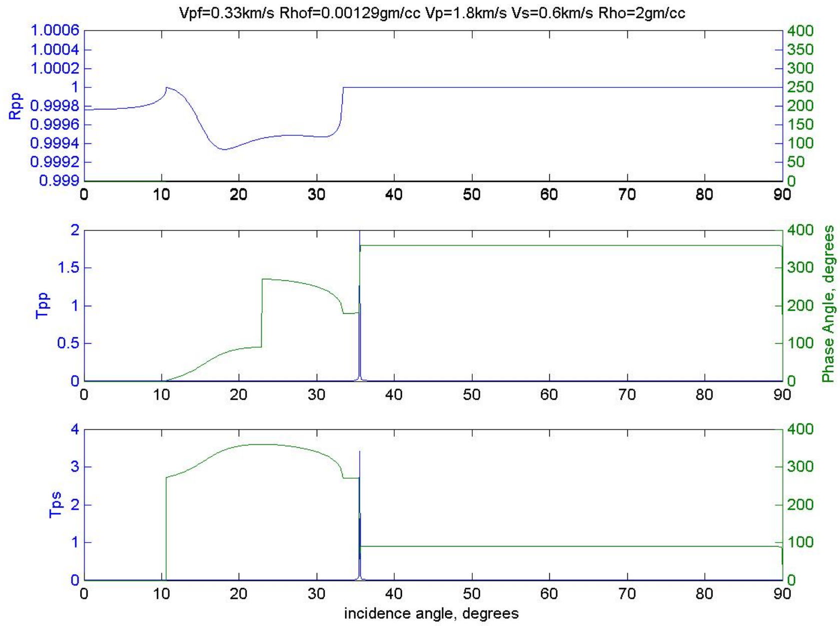

Figure 3: Amplitude and phase of the three reflection and transmission coefficients plotted at full scale.

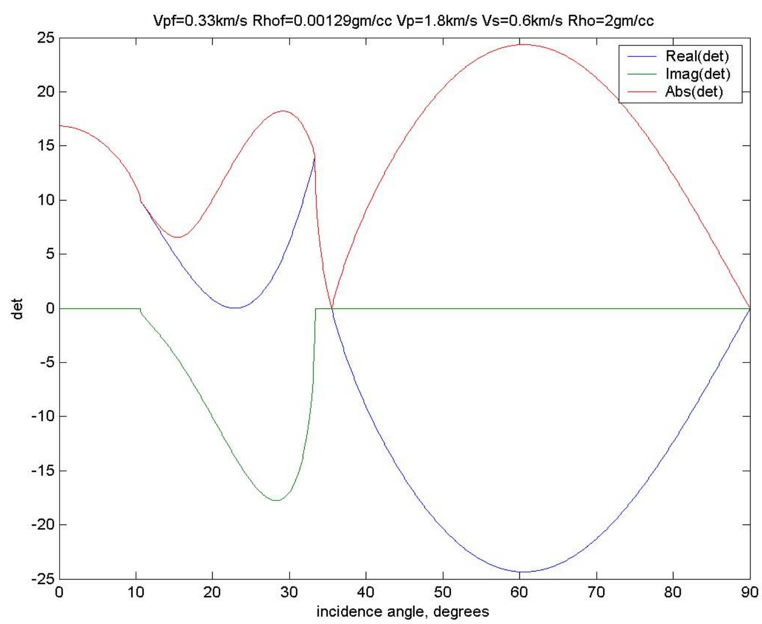

Figure 4: Behavior of $\det \underset{\sim}{D}$ (equation 15) with incidence angle. There is a minimum (but not a zero) near 35.5°.

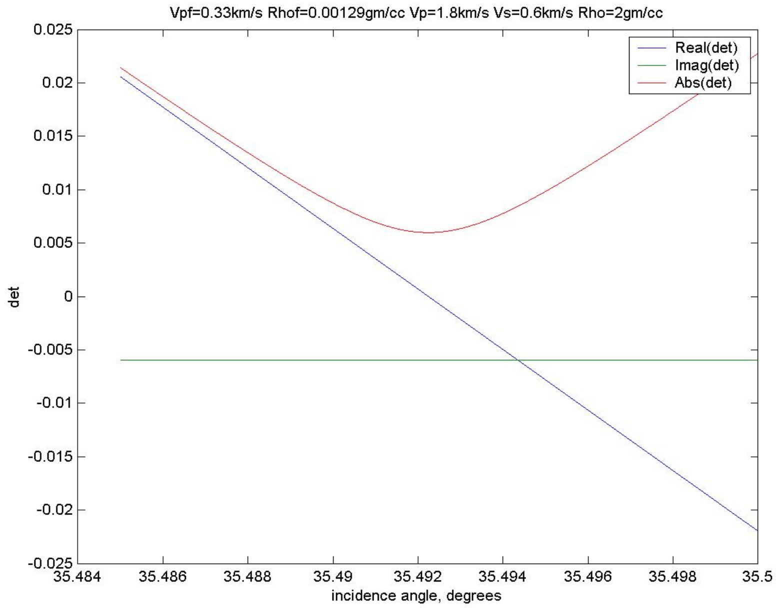

Figure 5: Behavior of $\det \underset{\sim}{D}$ near the minimum. Note that the real part has a zero but that there is small value for the imaginary part.

Figures 6 and 7 show the displacements predicted from equations (22) without the horizontal propagation phase factor or factor of -iω. Figure 8 shows the expected radial to vertical amplitude ratio and phase differences between components. Figure 9 shows the ratio of the displacements to total fluid pressure at the interface.

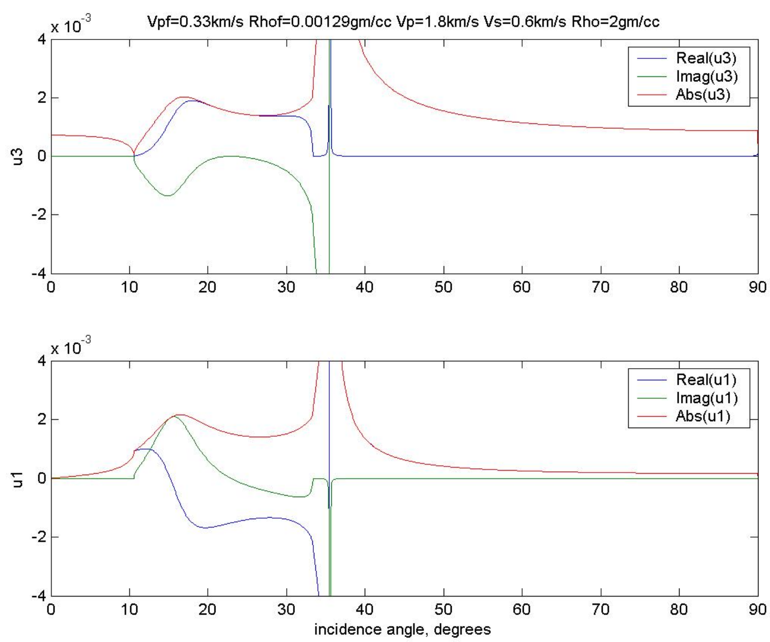

Figure 6: Real and imaginary parts of the vertical and radial displacements in the solid medium.

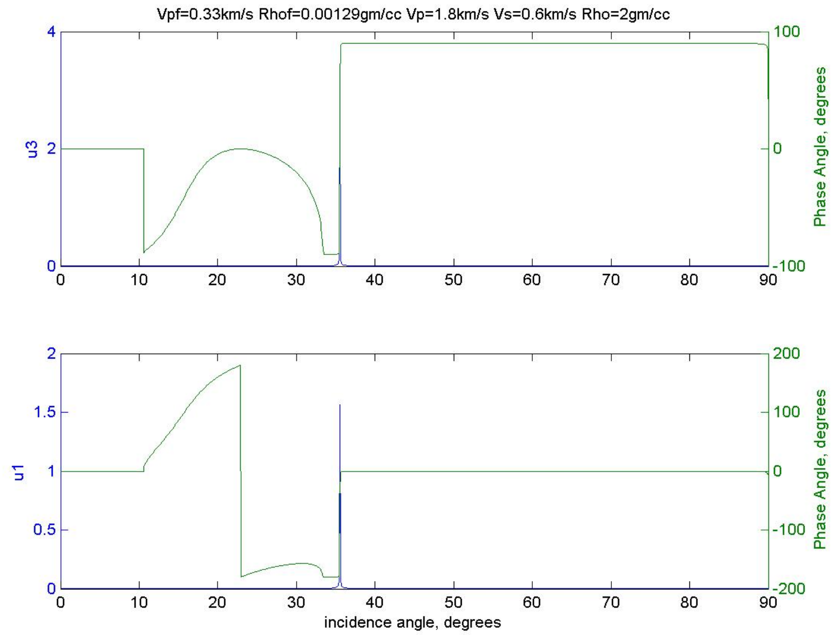

Figure 7: Amplitude and phase at full scale for the vertical and radial displacements in the solid medium.

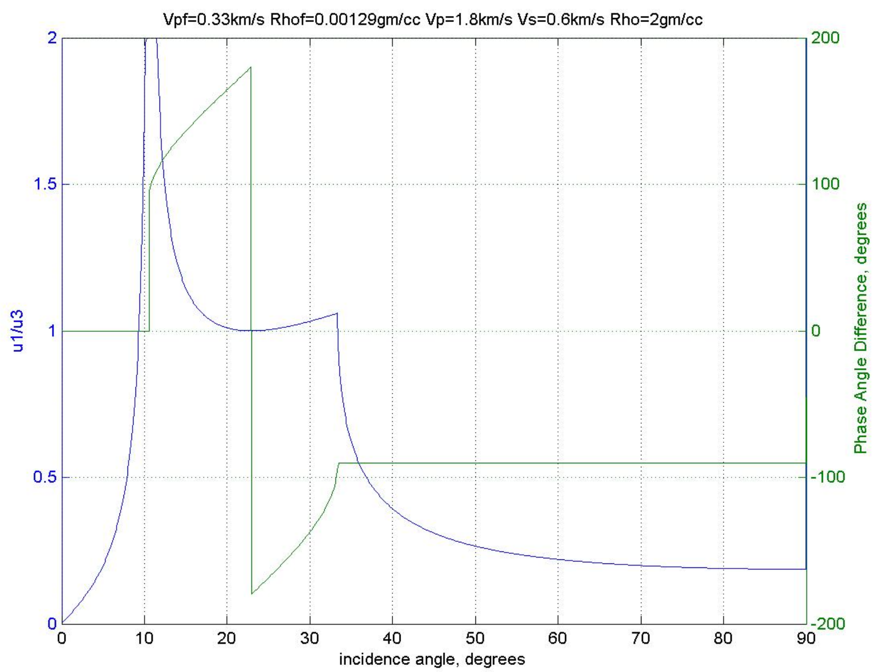

Figure 8: Ratio of the radial to vertical displacement amplitude at the interface. Also shown is the phase difference between the radial and vertical displacements.

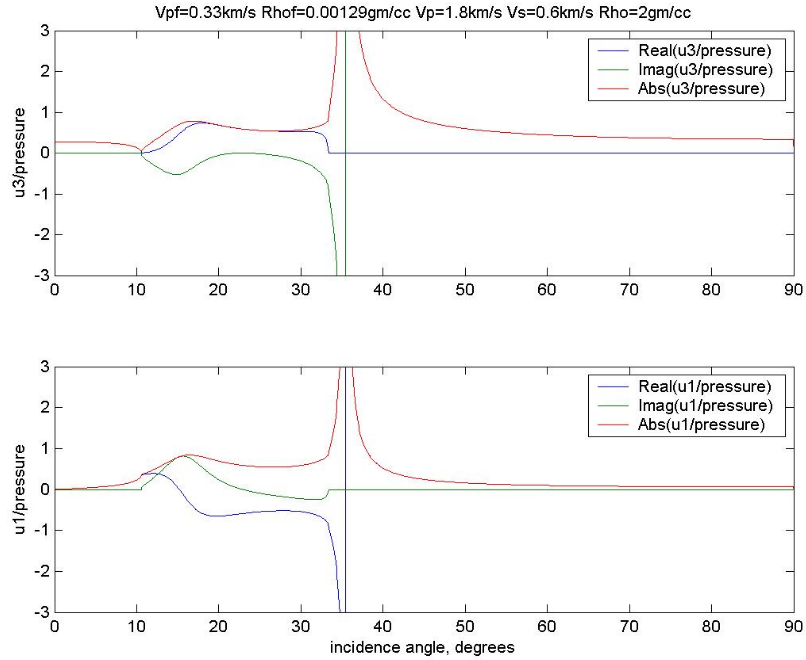

Figure 9: Ratio of the surface displacements to the total pressure field in the fluid. Units are in microns/Pa. The maximum of the ratios at the Stonely wave pseudo-pole are 1243 and 722 for u_3/pressure and u_1/pressure, respectively. The factor of $1/i\omega$ has not been included in these plots and it should be understood that the displacement time histories are the time integral of the incident pressure time history.