A Short Theory of Rotation

The analysis of deformation in a continuum yields a symmetric, second rank strain tensor and an anti-symmetric rotation tensor (e.g., Love, 1927). For example, differential motions in a continuum are given by

$$\begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} \quad . \qquad (1)$$

$$= \left[\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} + \begin{pmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{pmatrix} \right] \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

The rotation tensor describes rigid body rotations about the three primary axes and the strain tensor internal deformations of the material. Rotation can be simply derived as the curl of the displacement field or

$$\vec{\theta} = \left[\theta_x, \theta_y, \theta_z \right] = \nabla \times \vec{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \hat{e}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \hat{e}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{e}_z \quad (2)$$

(Morse and Feshbach, 1953). Rotation has vector properties such as direction and magnitude yet it does not have the same the properties as polar vectors in terms of coordinate transformations, reflection about the origin, or choice of left or right-handed coordinate systems. Because of this, rotation is defined as an "axial" vector where the rotation sign (or spin direction) depends on the choice of direction of a normal to the reference surface that defines the vector. Here we use the common "right-hand" rule to define rotation directions for a right-handed Cartesian coordinate system in the typical geocentric configuration.

Although, strictly speaking, rotation is an axial vector, it conforms to typical vector addition properties under many circumstances. In the case of infinitesimal rotational motions, two vector rotations can be summed using the usual rule of vector sums. However, it is well known that finite rotations cannot be added in this way (Resnick and Halliday, 1966). On the other hand, instantaneous quantities like angular velocity(e.g., $\dot{\theta}_x$) or angular acceleration($\ddot{\theta}_x$) can always be summed as vectors since they represent infinitesimal changes in rotation, and the rotation derivative with time.

A fundamental assumption used in the derivation of the wave equation in elastic media is that of infinitesimal displacements and strains (e.g., Aki and Richards, 1980). This assumption requires rotations to be infinitesimal as well. Seismological strains of 10^{-4} are quite large and usually represent non-linear deformations of earth materials that may occur near a seismic source or where earth materials fail. It follows that rotations of this magnitude, while still numerically small, will also occur in similar circumstances. Thus, we can expect rotation itself to be accurately handled as a simple vector in most seismological circumstances. For example, superposition of rotation vectors at a field point from different wave equation solutions in numerical simulations should be a valid technique. Thus, seismological rotation fields will usually be infinitesimal and can be treated as typical vectors. Tectonic, other geophysical observations (e.g., the rotation of the earth), and extreme engineering observations such as building failure may involve finite rotations. Angular velocities and accelerations will be vectors in these cases but rotation will not.

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