Earth Science Applications of Space Based Geodesy

DES-7355

Tu-Th 9:40-11:05

Seminar Room in 3892 Central Ave. (Long building)

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http://www.ceri.memphis.edu/people/smalley/ESCI7355/ESCI_7355_Applications_of_Space_Based_Geodesy.html

Class 2

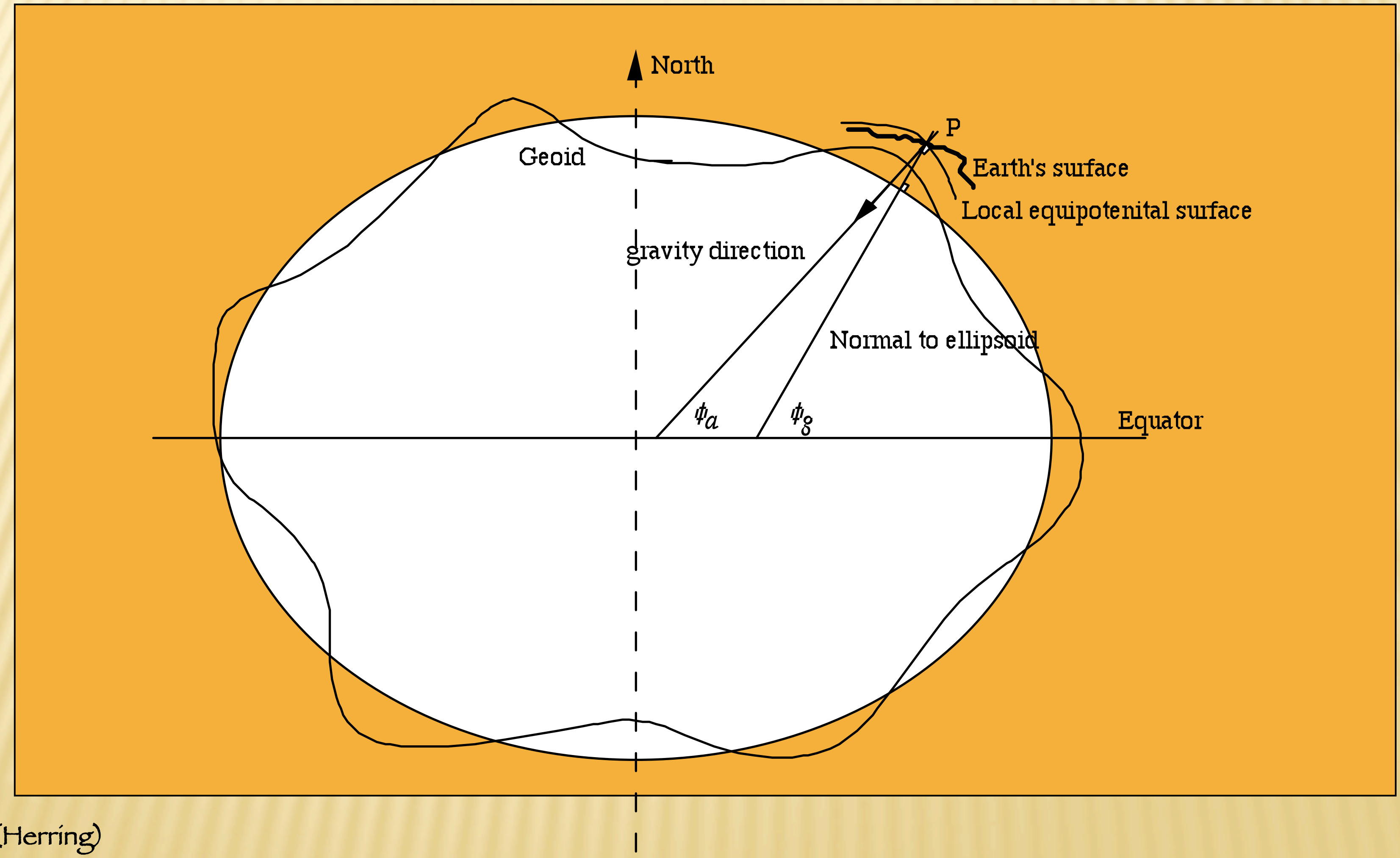
Coordinate systems

Simple spherical

Geodetic – with respect to ellipsoid normal to surface does not intersect origin [in general]

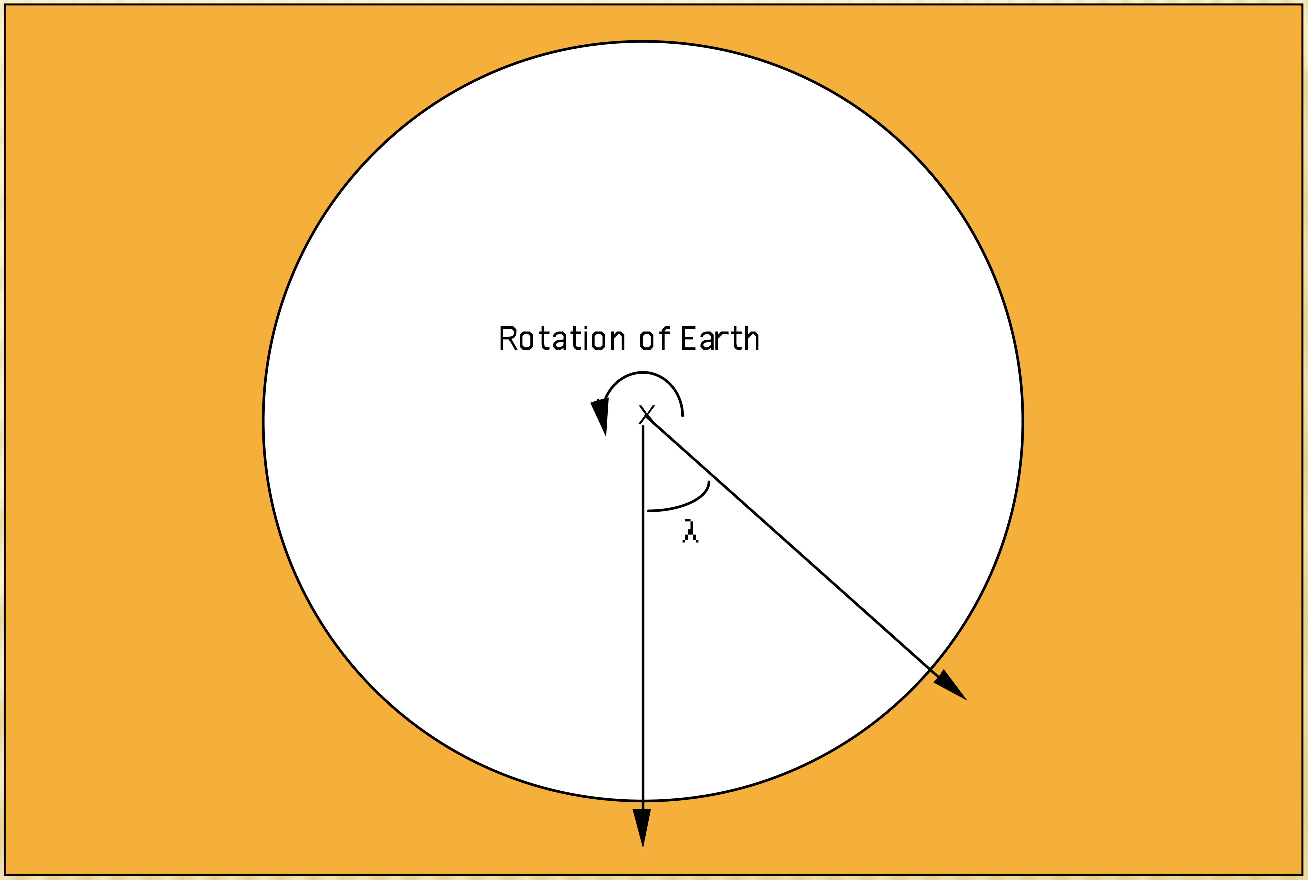
ECEF XYZ – earth centered, earth fixed xyz. Is what it says.

GEODETIC COORDINATES: LATITUDE



(Herring)

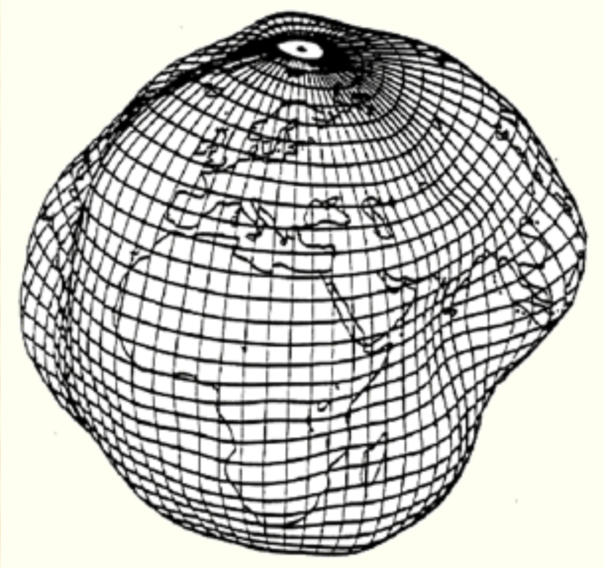
LONGITUDE



Longitude measured by time difference of astronomical events

(Herring)

The "problem" arises because we're defining the "location" (latitude) based on the orientation of the surface <u>at the point</u> where we want to determine the location.

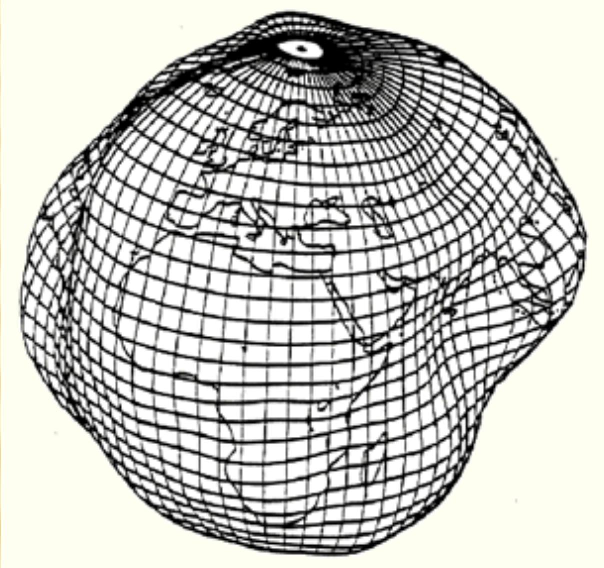


(Assume gravity perpendicular to surface – which is not really the case - since measurements made with a level.)

"shape" of the surface of the earth - with the variations greatly exaggerated. For now we're not being very specific about what the surface represents/how it is defined.

Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

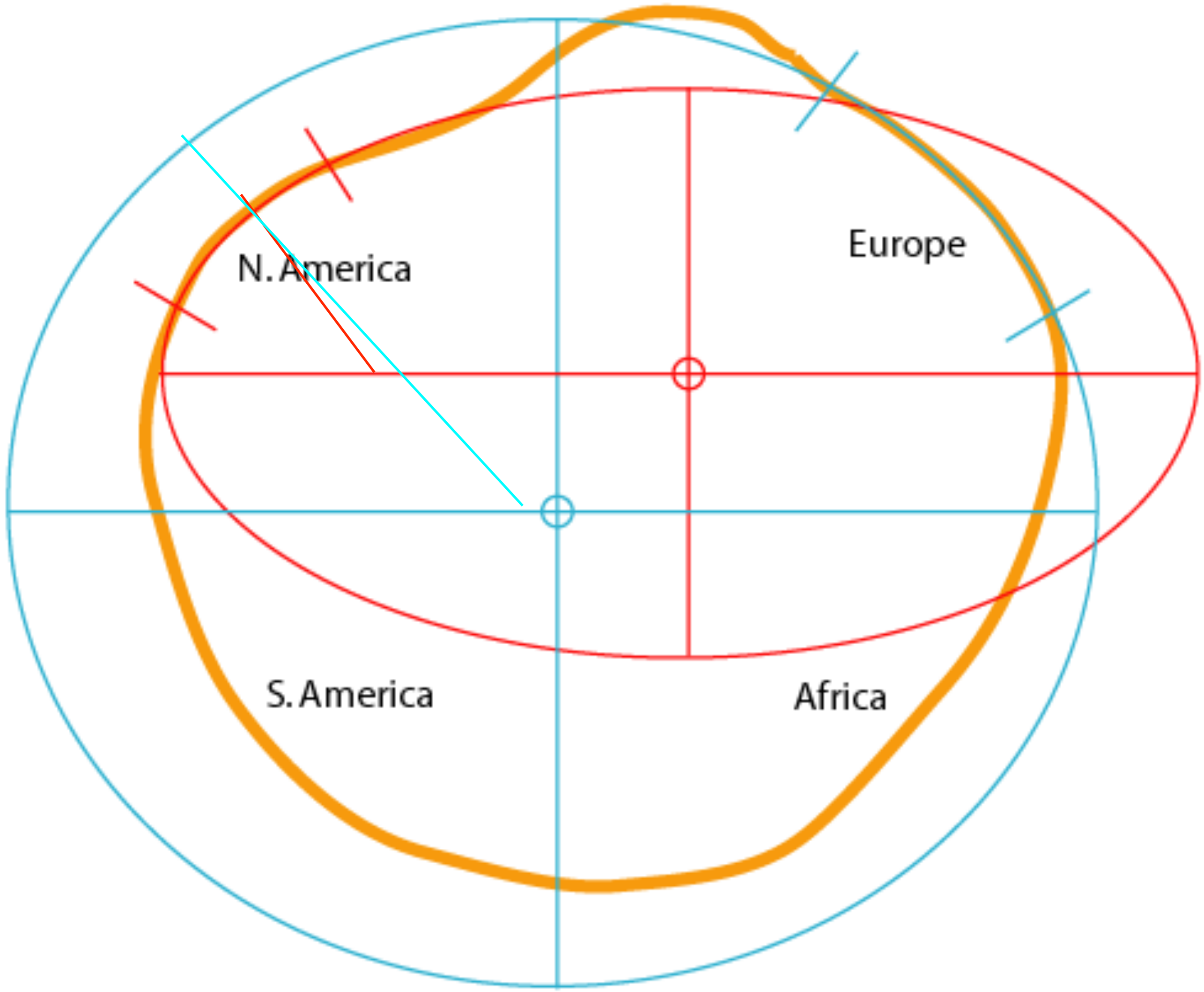
This means that we have to take the "shape" of the surface into account in defining our reference frame.



We are still not even considering the vertical. We're still only discussing the problem of 2-D location on the surface of the earth.

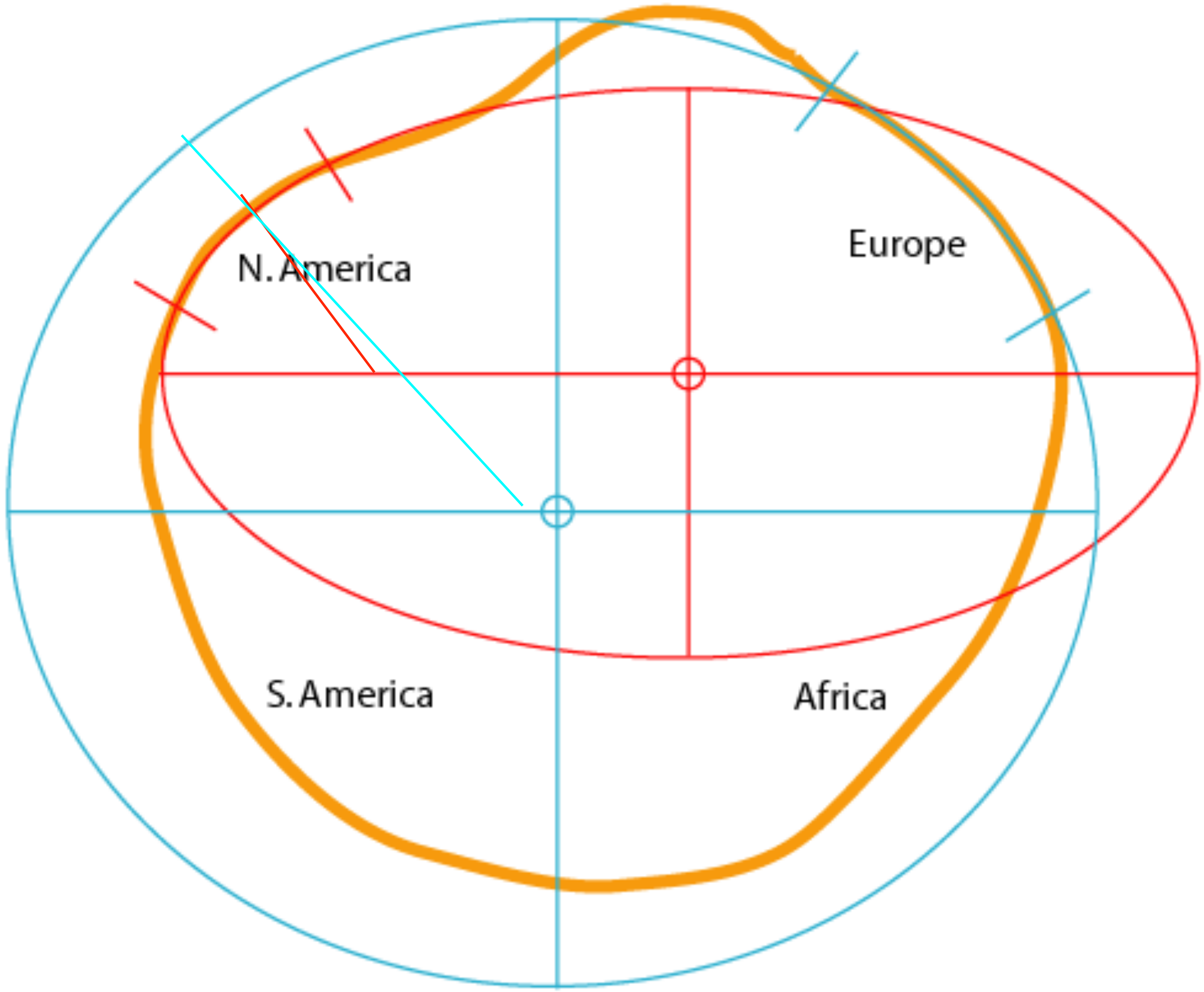
Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

Traditional approach was to define local/regional datums (flattening, size, <u>origin – typically not earth centered,</u> orientation).

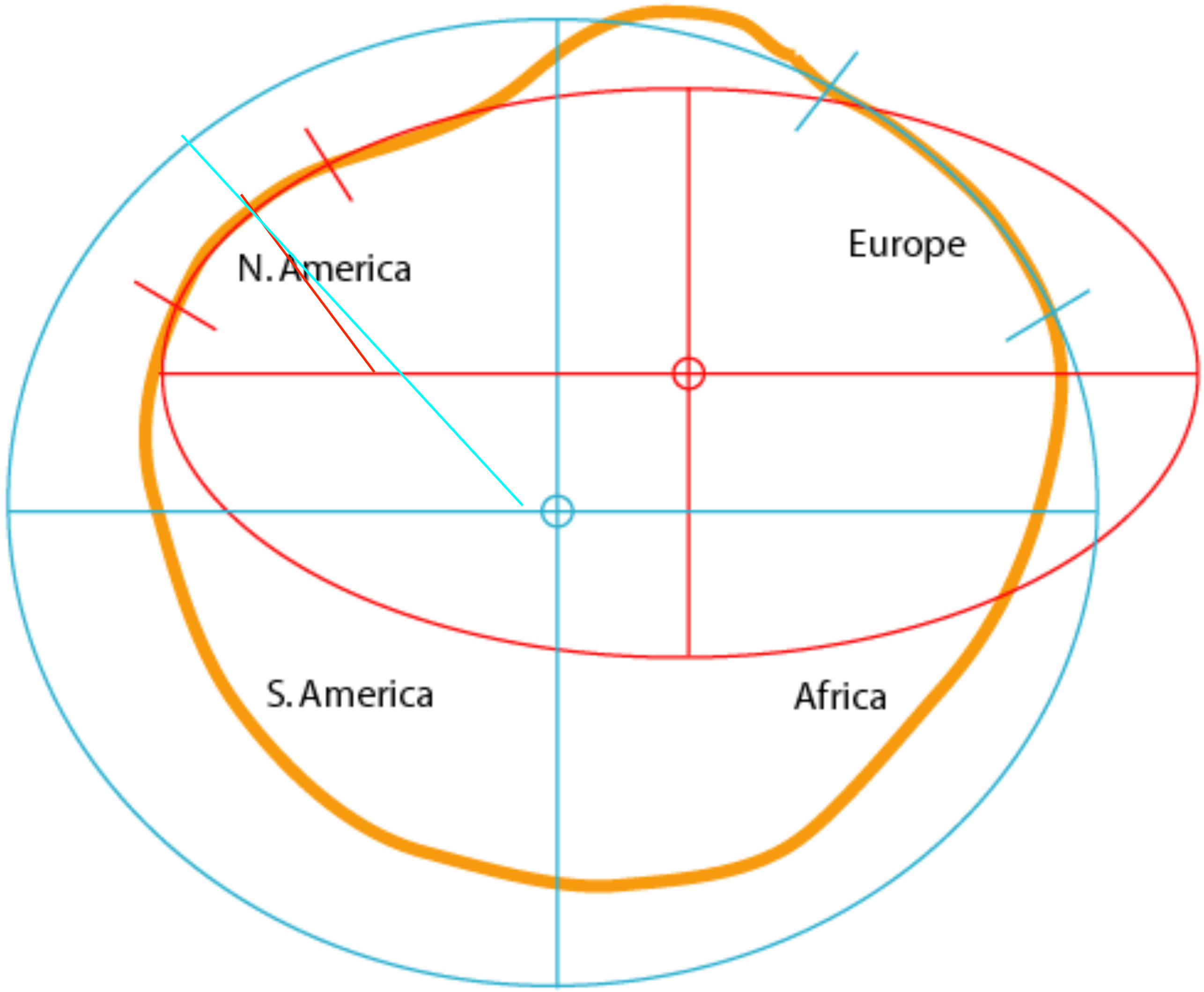


(Assume gravity perpendicular to surface – which is not really the case - since measurements made with a level.)

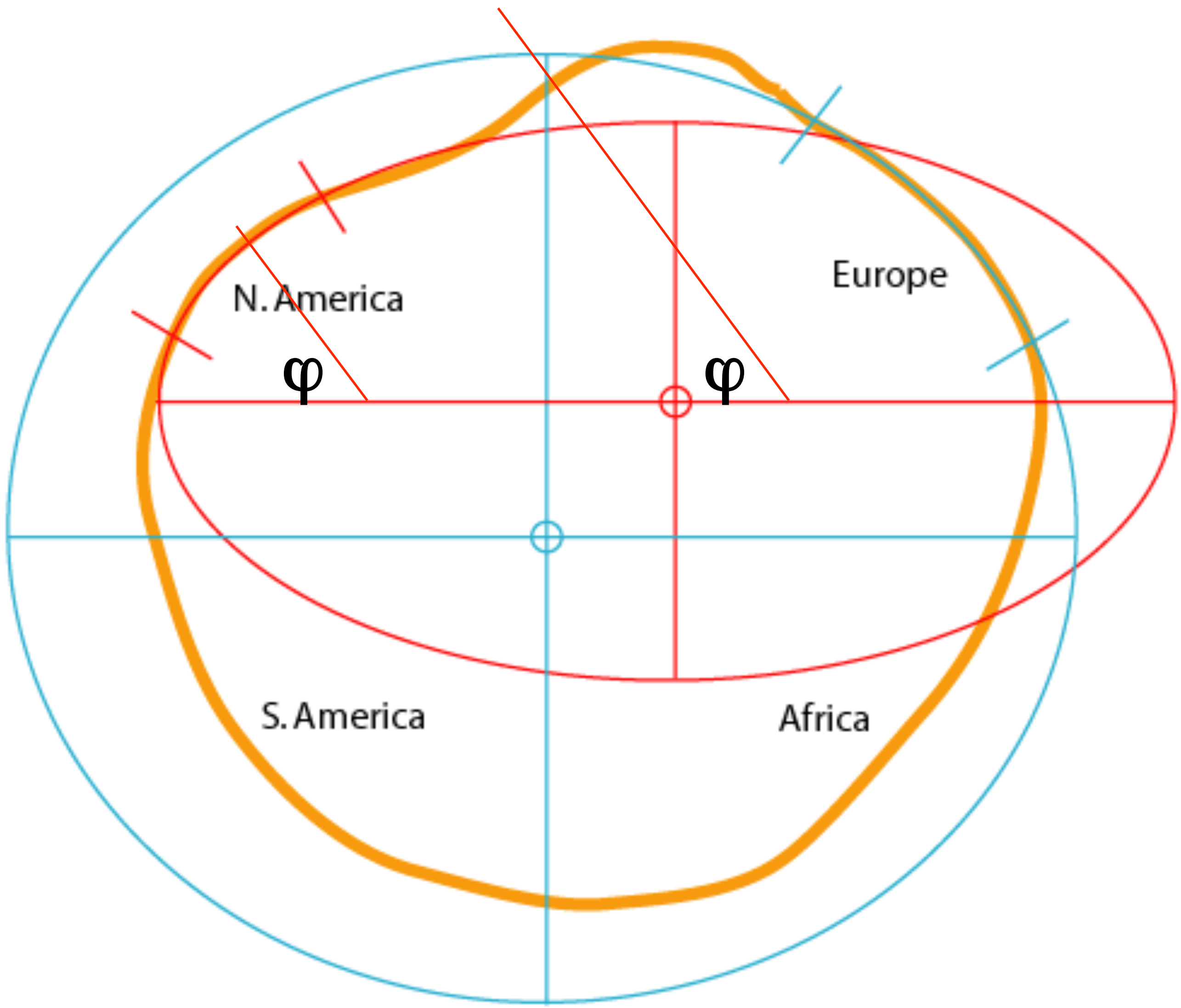
These datums were “best fits” for the regions that they covered. They could be quite bad (up to 1 km error) outside those regions however.



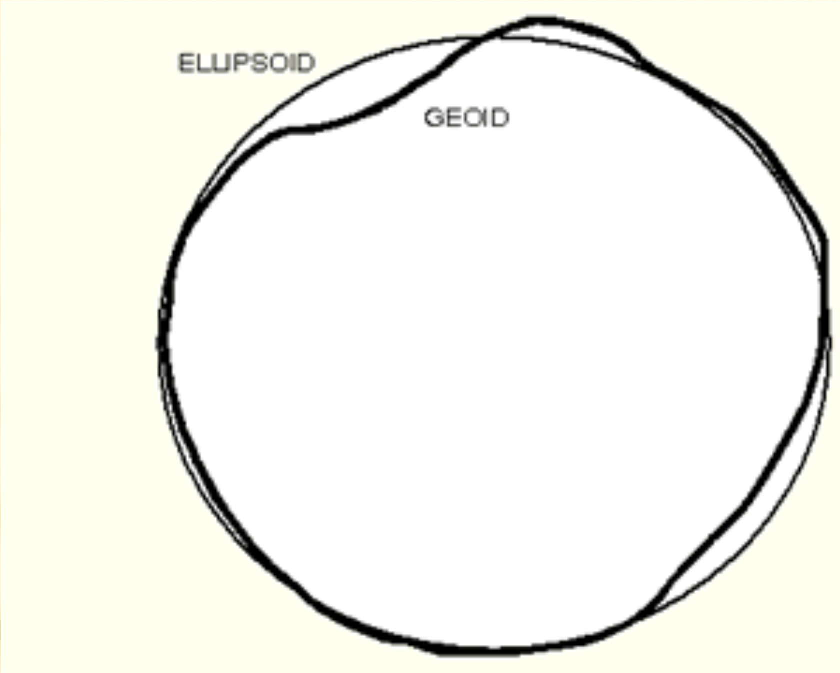
These datums are also not "earth centered" (origin not center of mass of earth). Converting from one to another not trivial in practice.



Can also have uniqueness problem – more than one spot with same "latitude"!



“Modern” solution is an earth centered global “best fit” ellipsoid.

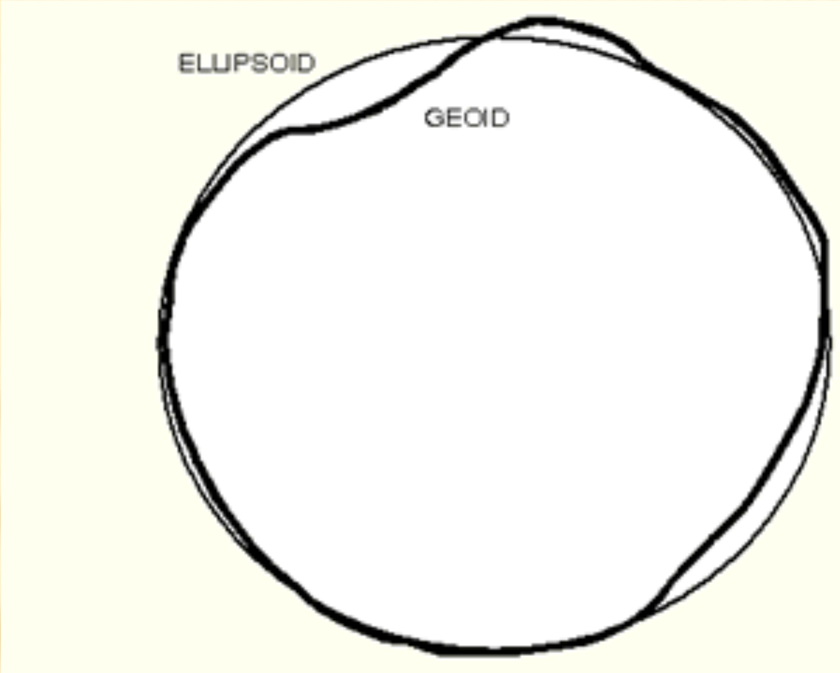


Here we introduce the “thing” that defines the “shape” of the earth – the GEOID.

The geoid is the thing that defines the local vertical.

Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

The geoid is a physical thing – an equipotential of the gravity field.



But we may not be able to "locate" it.

So make "model" for geoid.

Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

Here we can introduce the concept of "physical" vs "geometric" position.

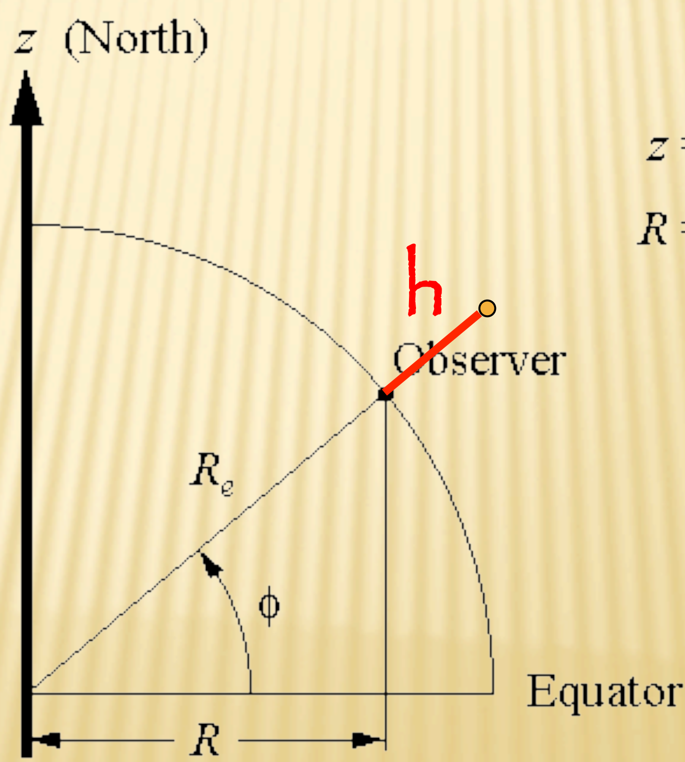
The geoid (since it depends on the actual shape of the earth, and we will see that it directly effects traditional measurements of latitude) gives a physical definition of position.

The ellipsoid gives a geometric definition of position (and we will see that "modern" positioning – GPS for example – works in this system – even though gravity and other physics effects the system).

The horizontal "datum" is a best fit ellipsoid (to a region or the whole earth) used as a coordinate system for specifying position.

WHAT ABOUT HEIGHT

Geocentric coordinates (ϕ, λ, h)
(this is based on standard spherical coordinate system with $h=R-R_e$)



$z = R_e \sin\phi$

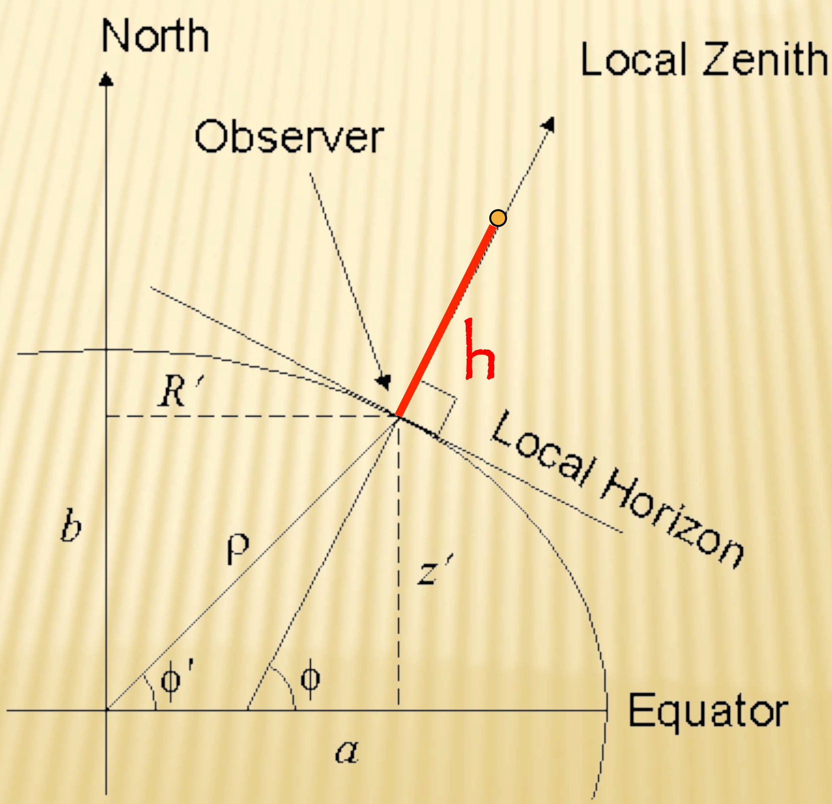
$R = R_e \cos\phi$

From Kelso, Orbital Coordinate Systems, Part I, Satellite Times, Sep/Oct 1995

WHAT ABOUT HEIGHT

For the Ellipsoid coordinates (ϕ, λ, h) – Ellipsoidal/ Geodetic height.

Distance of a point from the ellipsoid measured along the perpendicular from the ellipsoid to this point.



From Kelso, Orbital Coordinate Systems, Part III, Satellite Times, Jan/Feb 1996

What about HEIGHT

For the Geoid things get a little more interesting.

The height is the distance of a point from the geoid measured along the perpendicular from the geoid to this point.

Notice that –

the height above the geoid (red line) may not be/is not the same as the ellipsoid height (blue line)

and that height above the geoid may not be unique

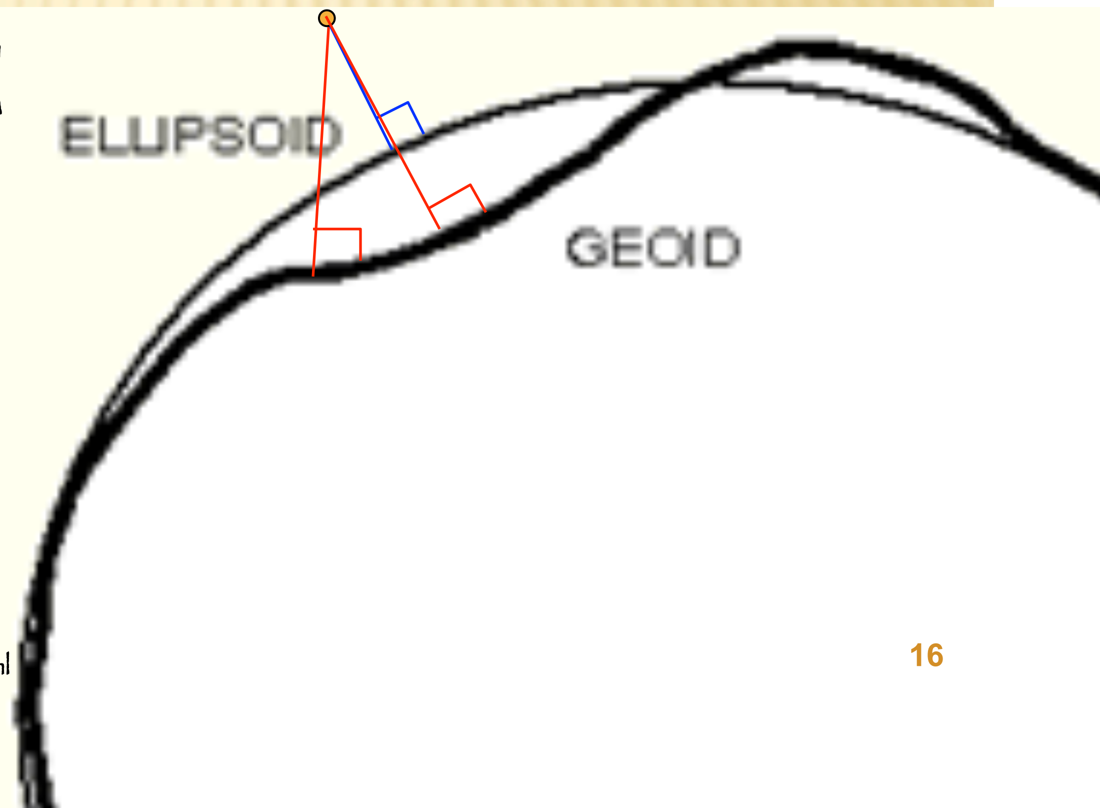


Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

What about HEIGHT

when we use a level to find the vertical (traditional surveying) we are measuring with respect to the geoid (what is the "geoid"?).

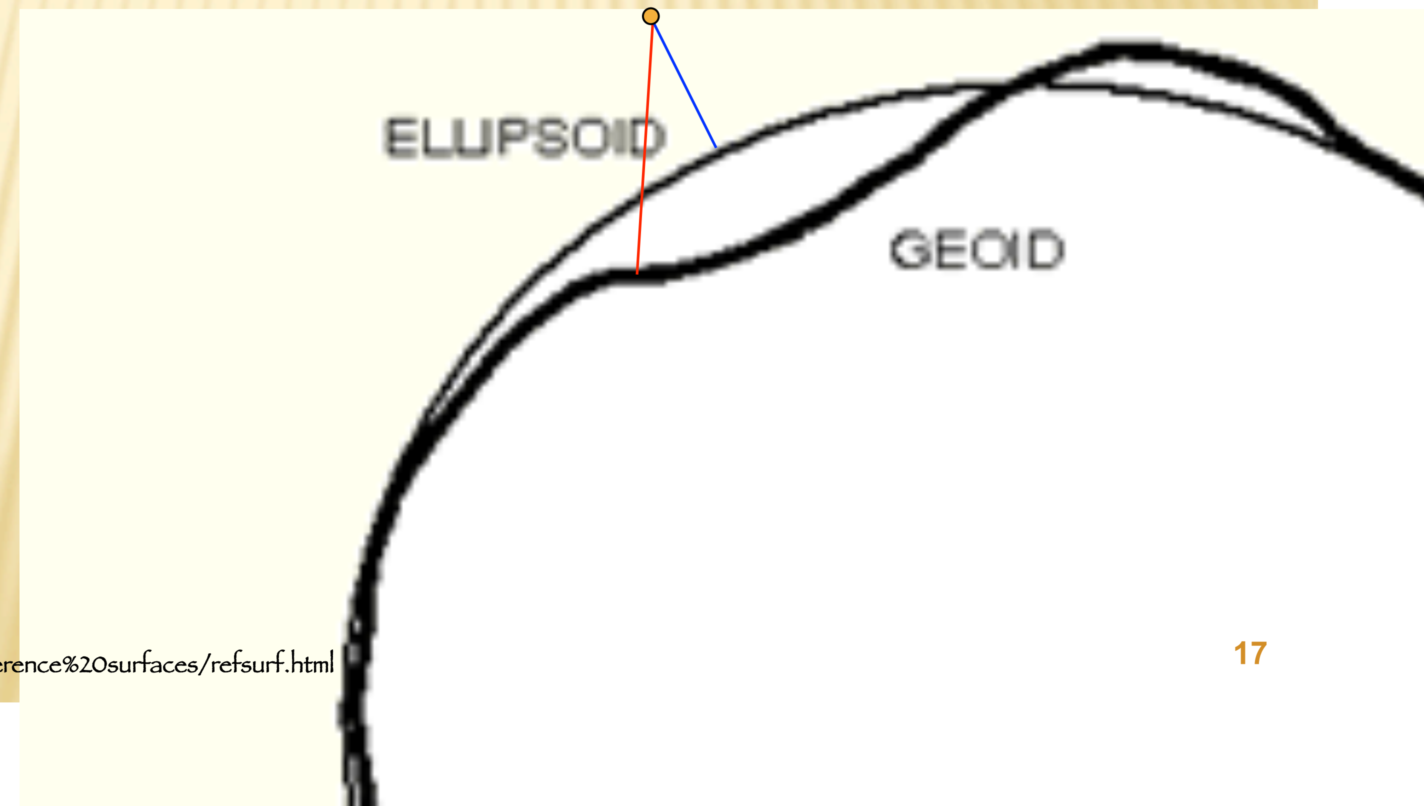


Image from: http://kartoweb.itc.nl/geometrics/Reference%20surfaces/refsurf.html

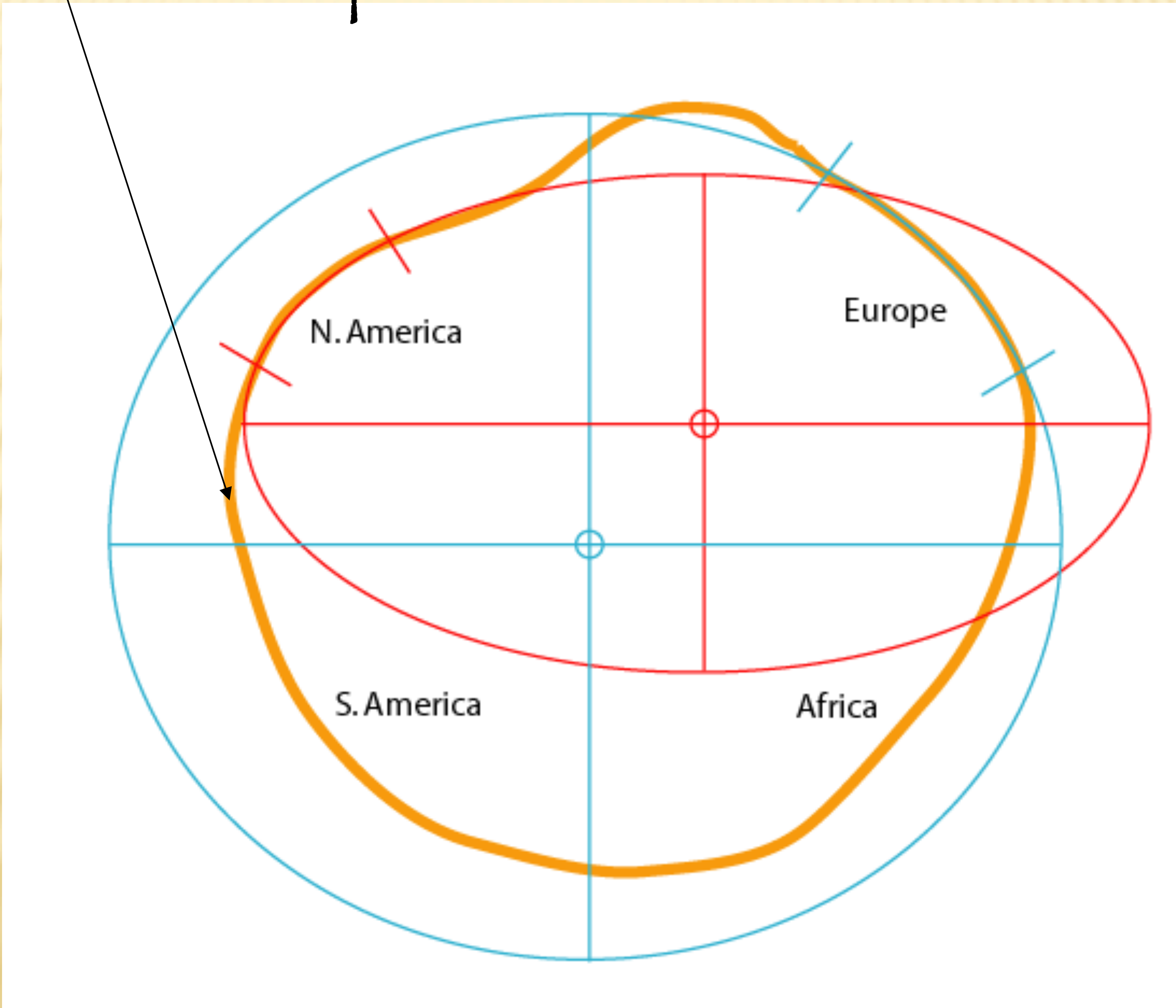
This brings us to a fundamental problem in Geodesy ----

‘"Height" is a common, ordinary everyday word and everyone knows what it means.

Or, more likely, everyone has an idea of what it means, but nailing down an exact definition is surprisingly tricky.‘

Thomas Meyer, University of Connecticut

The geoid is the "actual" shape of the earth. Where the word "actual" is in quotes for a reason!



The geoid is a representation of the surface the earth would have if the sea covered the earth.

This is <u>not</u> the surface one would get if one pours more water on the earth until there is no more dry land!

It is the shape a fluid Earth (of the correct volume) would have if that fluid Earth had exactly the same gravity field as the actual Earth.

Where did this reference to the gravity field sneak in?

Since water is a fluid, it cannot support shear stresses.

This means that the surface of the sea (or of a lake, or of water in a bucket, etc.) will be

<u>-- perpendicular to the force of gravity</u>

<u>-- an equipotential surface</u>

(or else it will flow until the surface of the body of water is everywhere in this state).

So the definition of the "shape" of the earth, the geoid, is <u>intimately and inseparably</u> tied to the earth's gravity field.

This is good

gravity is one of the most well understood branches of Physics.

This is bad

the gravity field of the earth depends on the details of the mass distribution within the earth (which do not depend on the first principles of physics – the mass distribution of the earth is as we find it!).

The geoid is a representation of the surface the earth would have if the sea covered the earth.

Or - it is the shape a fluid Earth would have if it had <u>exactly</u> the same gravity field as the actual Earth.

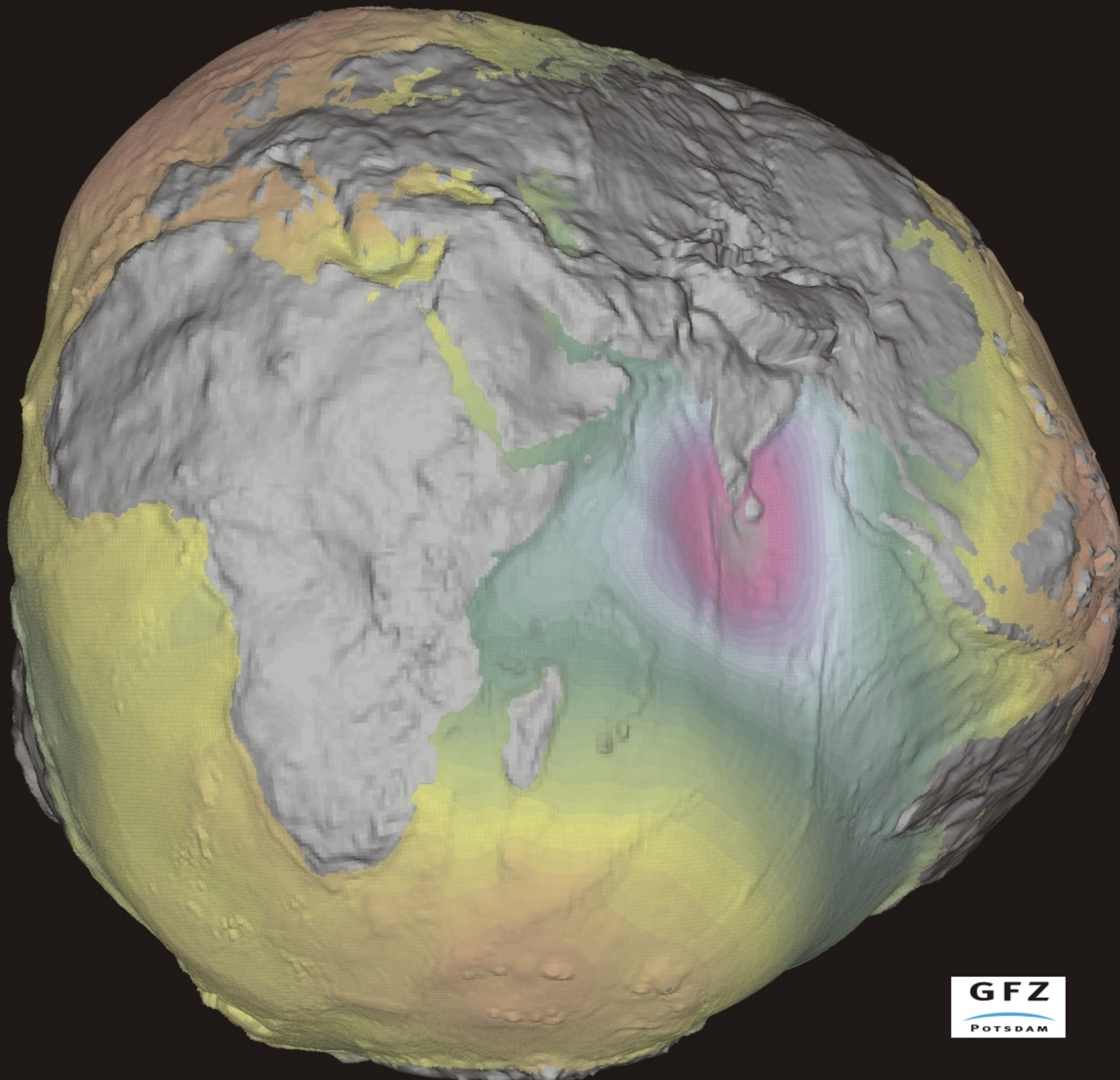
The definition is clear and concise.

Problems arise when trying to find where this surface actually resides due to things like

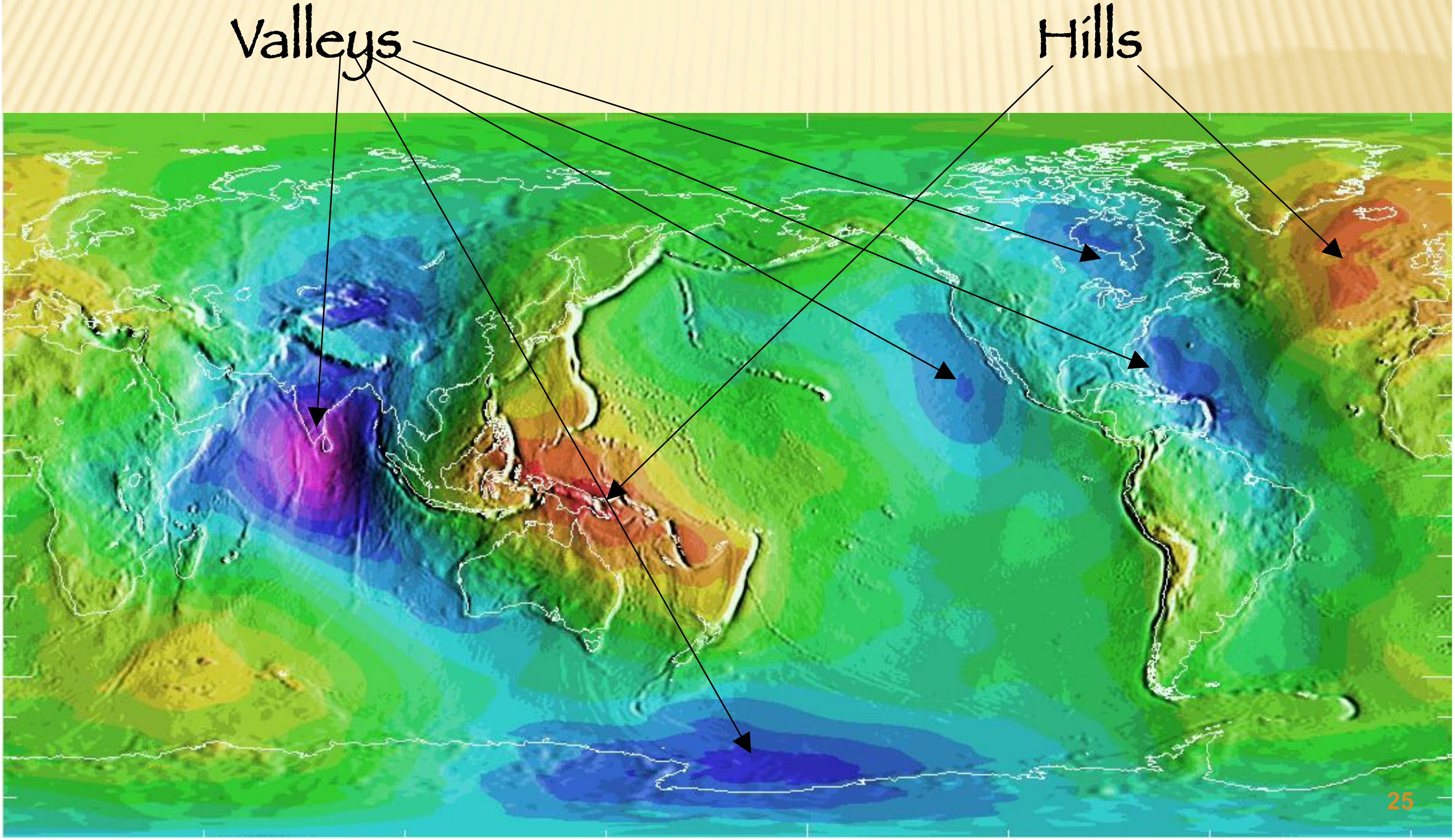
-- currents, winds, tides effecting "sea level"
-- where is this imaginary surface located on land? (must be below the land surface – except where the land surface is below sea level, e.g. Death Valley - it is the level of fluid in channels cut through the land [approximately].)

So – what does this surface – the geoid – actually look like?

(greatly exaggerated in the vertical)



Shaded, color coded “topographic” representation of the geoid



Bad joke for the day

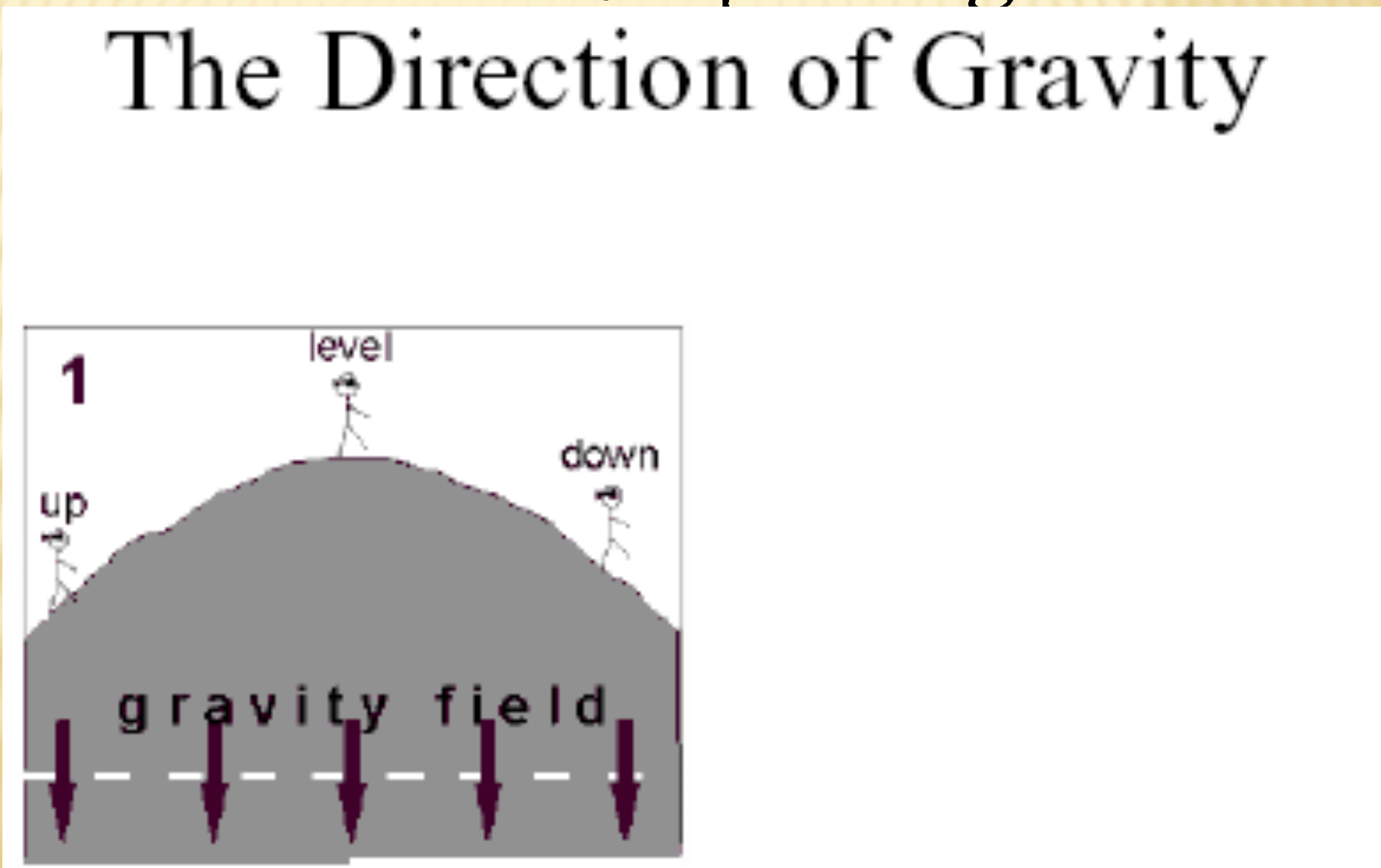
"What's up?"

"Perpendicular to the geoid."

2. Geodesy

Shape of the earth / gravity, geoid (physical)

reference frames, ellipsoids (geometric)



From Mulcare or http://www.ordnancesurvey.co.uk/oswebsite/gps/information/coordinatesystemsinfo/guidecontents/guide2.html

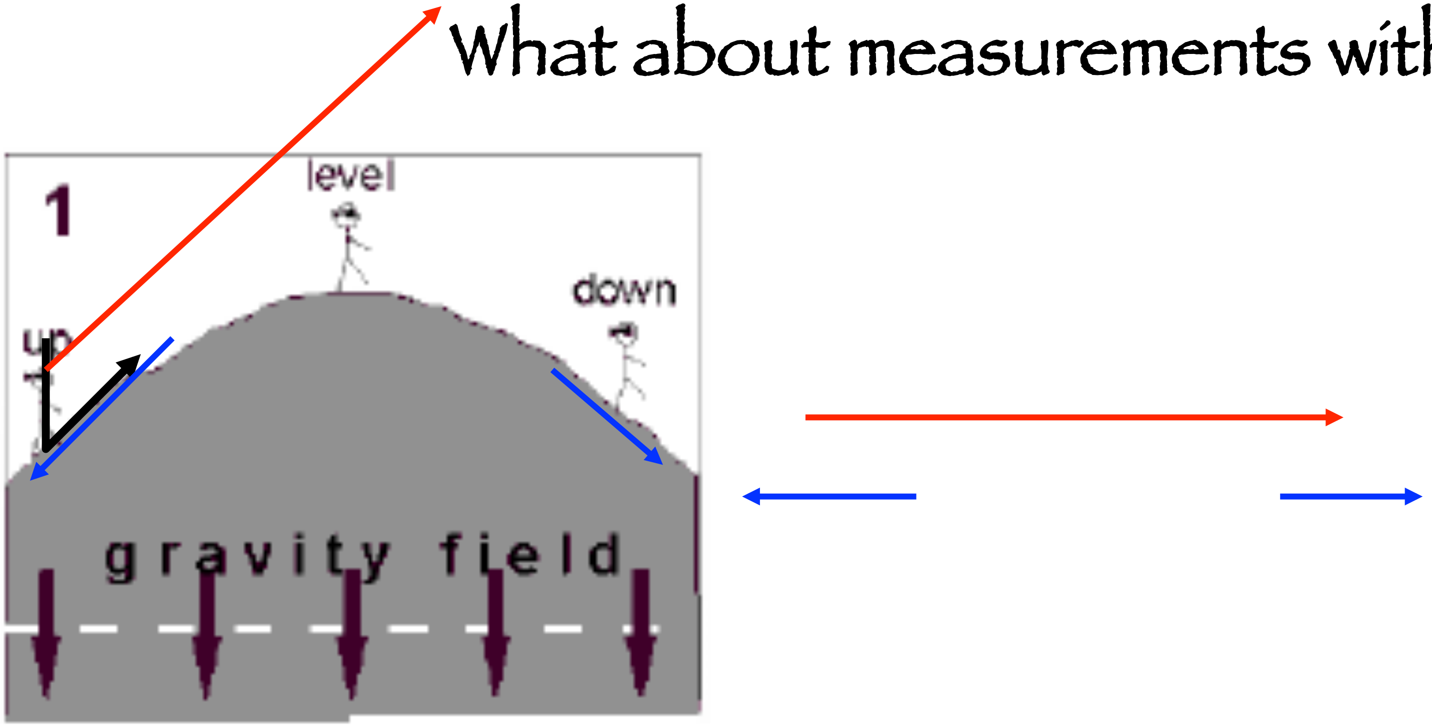
2. Geodesy

How gravity makes it "interesting"

Which way is "up"? (how does water flow?)

The Direction of Gravity

What about measurements with light?



From Mulcare

What is the Geoid?

Since the geoid is a complicated physical entity that is practically indescribable –

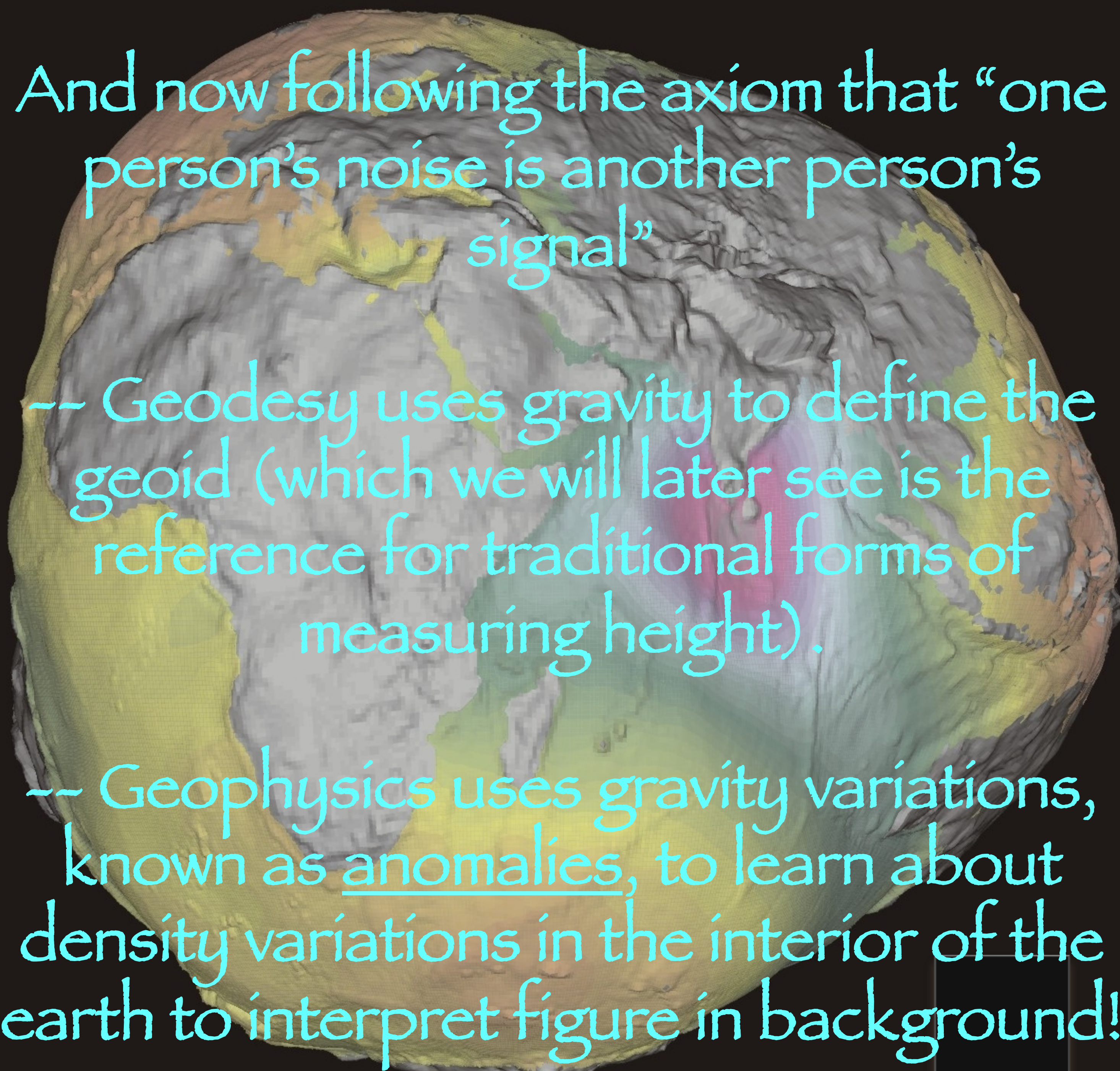
Find a "best fit" ellipsoid

(and look at variations with respect to this ellipsoid).

Current NGS definition

The equipotential surface of the Earth's gravity field which best fits, in a least squares sense, global mean sea level.

From Mulcare



And now following the axiom that "one person's noise is another person's signal"

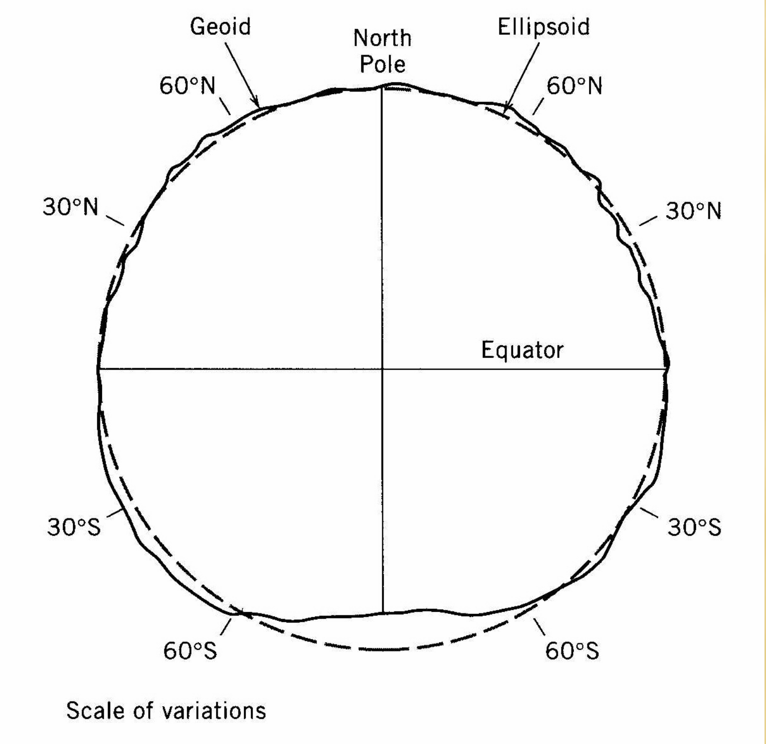
-- Geodesy uses gravity to define the geoid (which we will later see is the reference for traditional forms of measuring height) .

-- Geophysics uses gravity variations, known as <u>anomalies</u>, to learn about density variations in the interior of the earth to interpret figure in background!

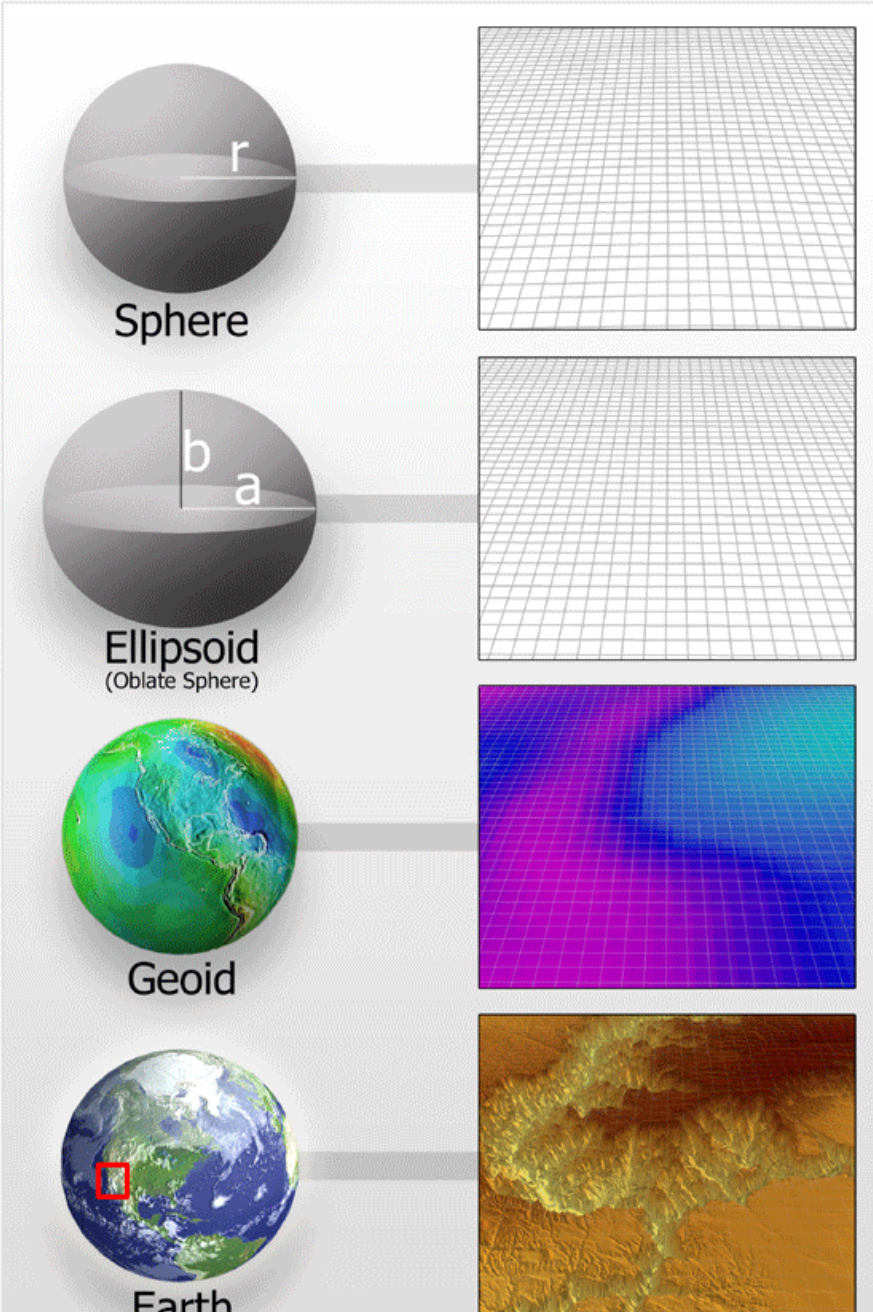
One can (some people do) make a career of modeling the "actual "geoid by using spherical harmonic expansions of the geoid with respect to the ellipsoidal best fit geoid.

There are ~40,000 terms in the "best" expansions.

Famous "pear" shape of earth.



16. Relation of Geoid to Ellipsoid (After King-Hele, 1963)



The Grand Canyon, Arizona

Geodetic Reference Surfaces

A beachball globe

Mathematical best fit to Earth's surface: used for defining Latitude and Longitude

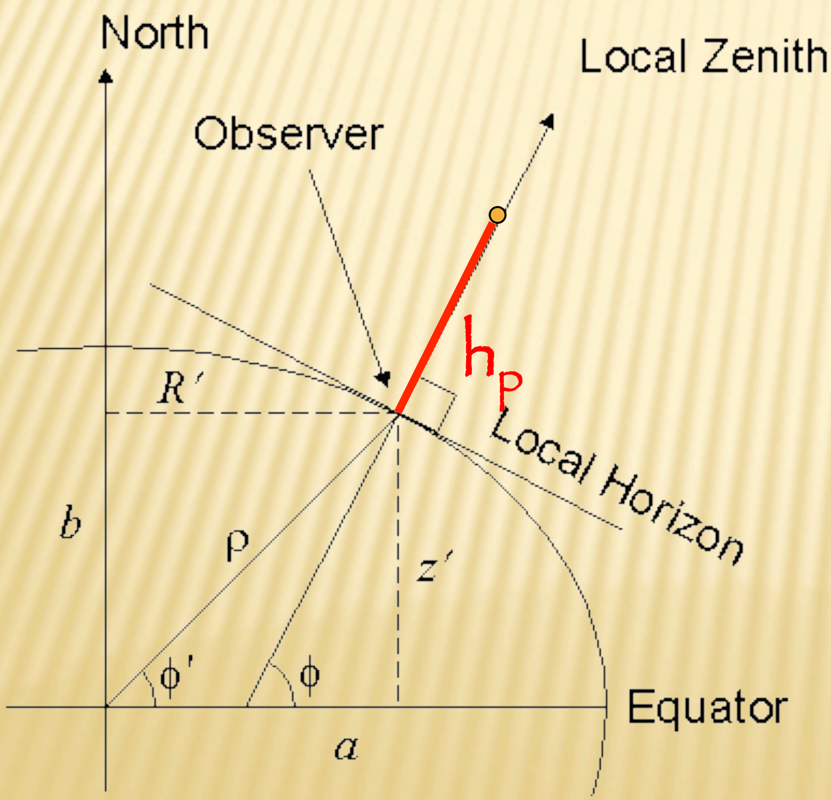
Modeled best fit to "sea surface" *equipotential gravity field* used for defining Elevation

The real deal

Fig from NGS: file:///C:/Documents%20and%20Settings/Bob/My%20Documents/geodesy/noaa/geo03_figure.html

Heights and Vertical Datums

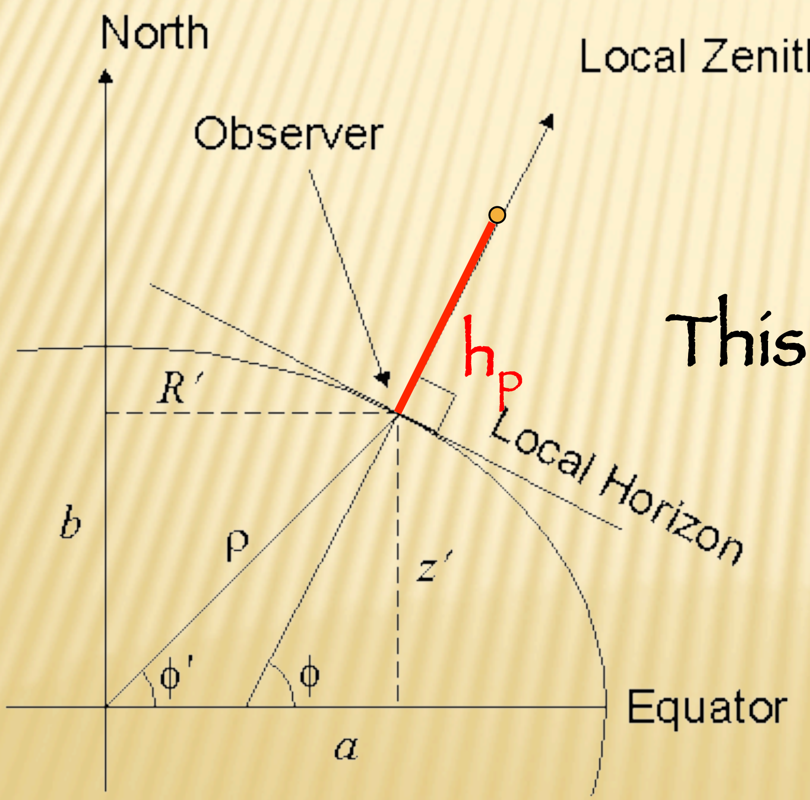
Define location by triplet - (latitude, longitude, height)



Heights and Vertical Datums

More precisely - Geodetic latitude and longitude – referred to oblate ellipsoid.

Height referred to perpendicular to oblate ellipsoid. (geometrical, is "accessible" by GPS for example).



This is called "ellipsoidal" height, h_P

In traditional surveying – height is measured with respect to <u>mean sea level</u>, which serves as the vertical datum (and is accessible at the origin point).

Height is measured as distance along the "plumb" line (which is not actually straight) and is called orthometric height (H_P)

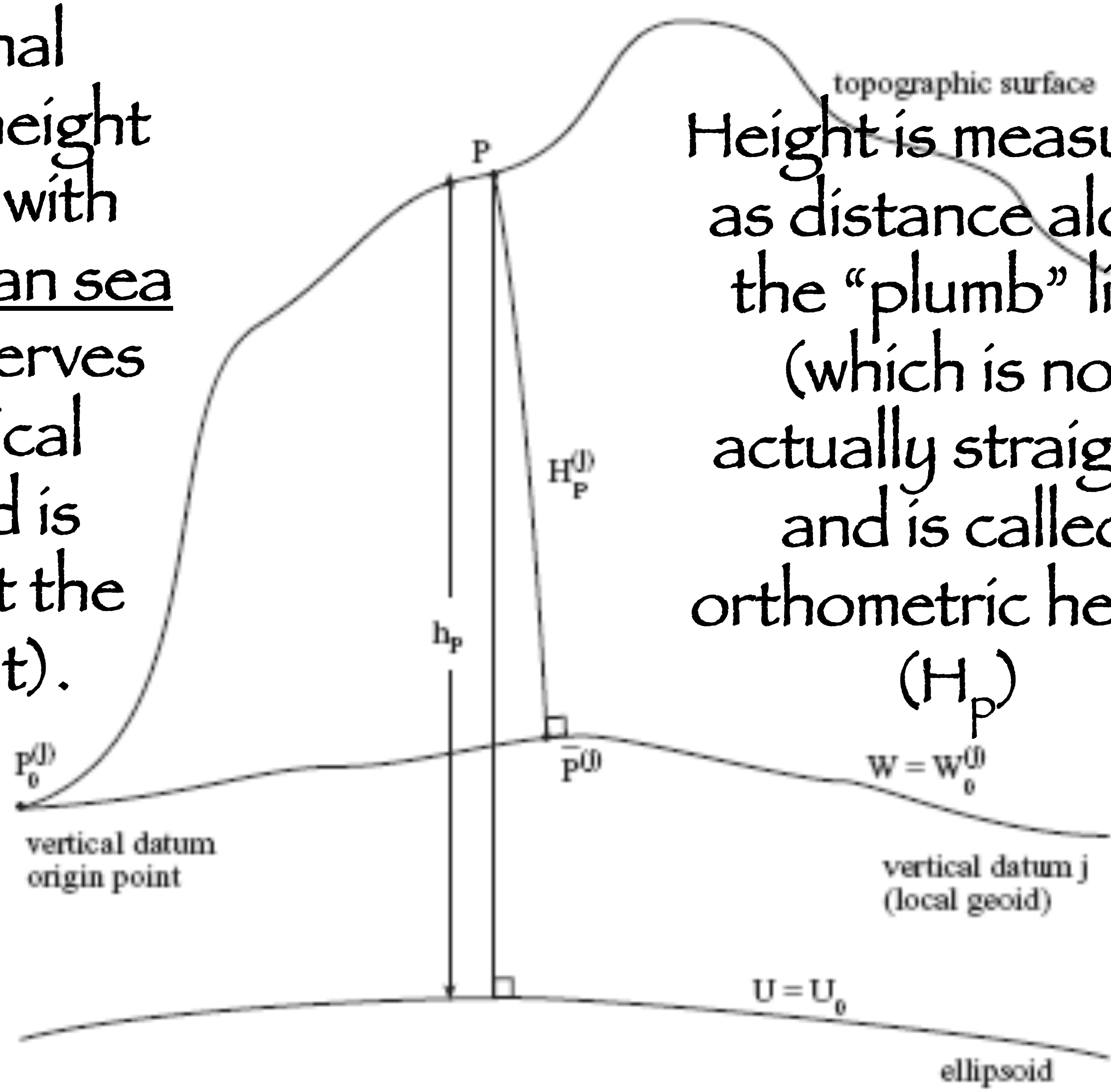
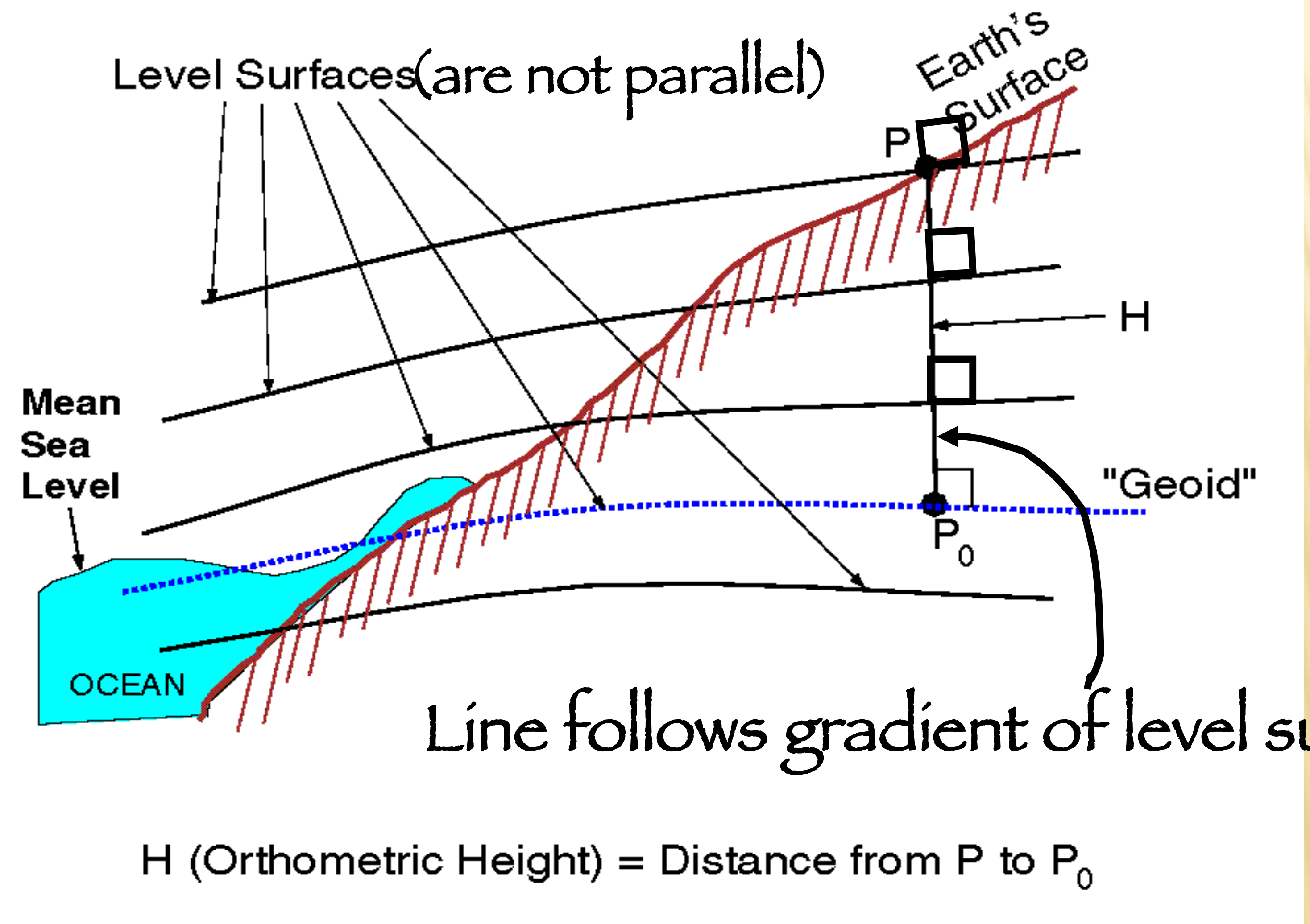


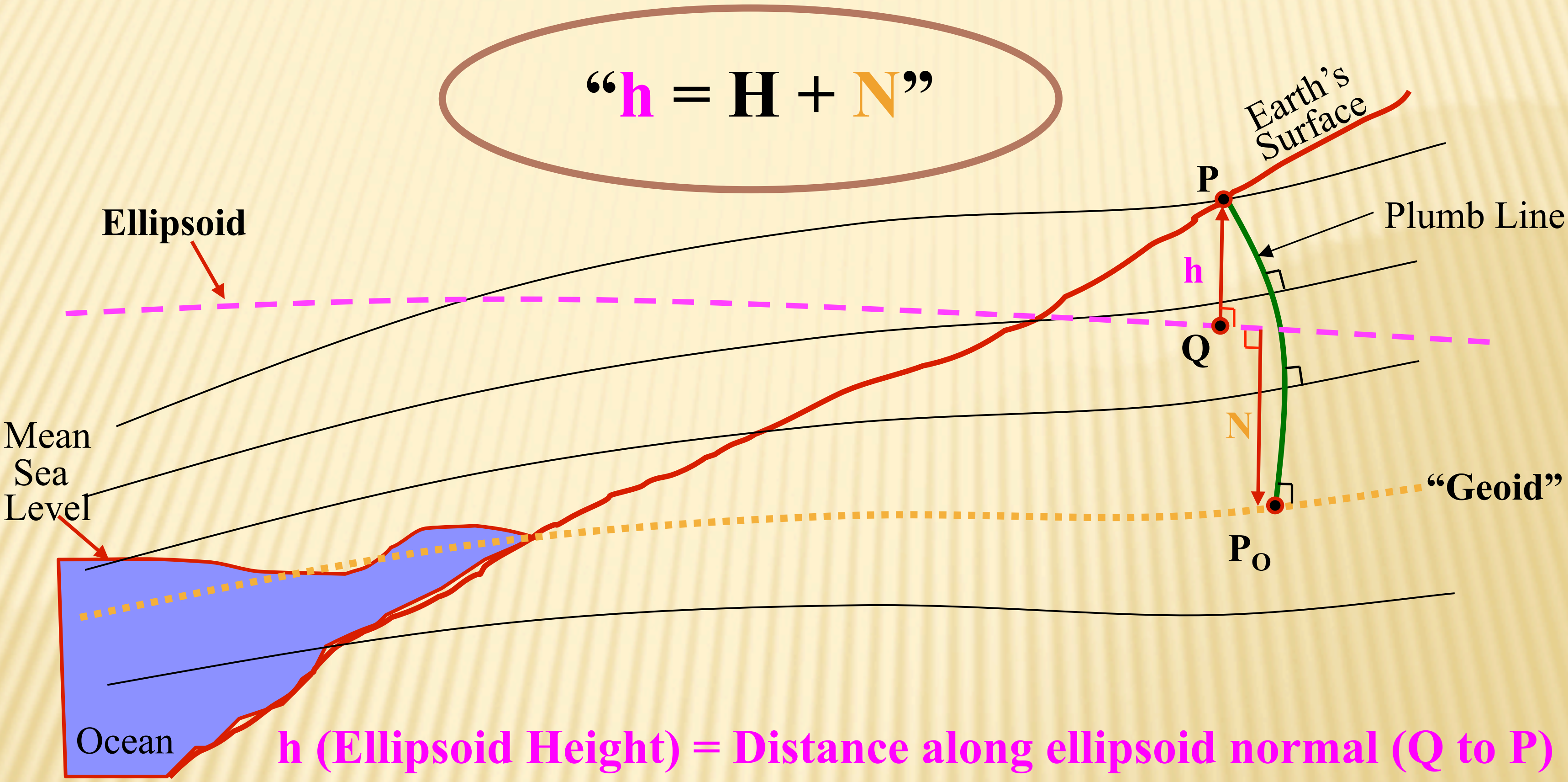
Figure 1: Ellipsoidal height versus orthometric height with respect to vertical datum, j.

Jekeli, 2002: http://www.fgg.uni-lj.si/~/mkuhar/Zalozba/Heights_Jekeli.pdf



Little problem – geoid defined by equipotential surface, can't measure where this is on continents (sometimes even have problems in oceans), can only measure direction of <u>perpendicular</u> to this surface and <u>force of gravity</u>.

Ellipsoid, Geoid, and Orthometric Heights



h (Ellipsoid Height) = Distance along ellipsoid normal (Q to P)

N (Geoid Height) = Distance along ellipsoid normal (Q to P_O)

H (Orthometric Height) = Distance along plumb line (P_O to P)

David B. Zilkoski
138.23.217.17/jwilbur/student_files/ Spatial%20Reference%20Seminar/dzilkoski.ppt

Two questions –

1

Given density distribution, can we calculate the gravitational field?

Yes – Newton's law of universal gravitation

2

Given volume V, bounded by a surface S, and some information about gravity on S, can you find gravity inside V (where V may or may not contain mass)?

Qualified yes (need g or normal gradient to potential everywhere on surface)

Potential Fields

As was mentioned earlier, the geoid/mean sea level is defined with respect to an equipotential surface.

So how do we connect what we need (the equipotential surface) with what we have/can measure (direction and magnitude of the force of gravity)

Use potential field theory

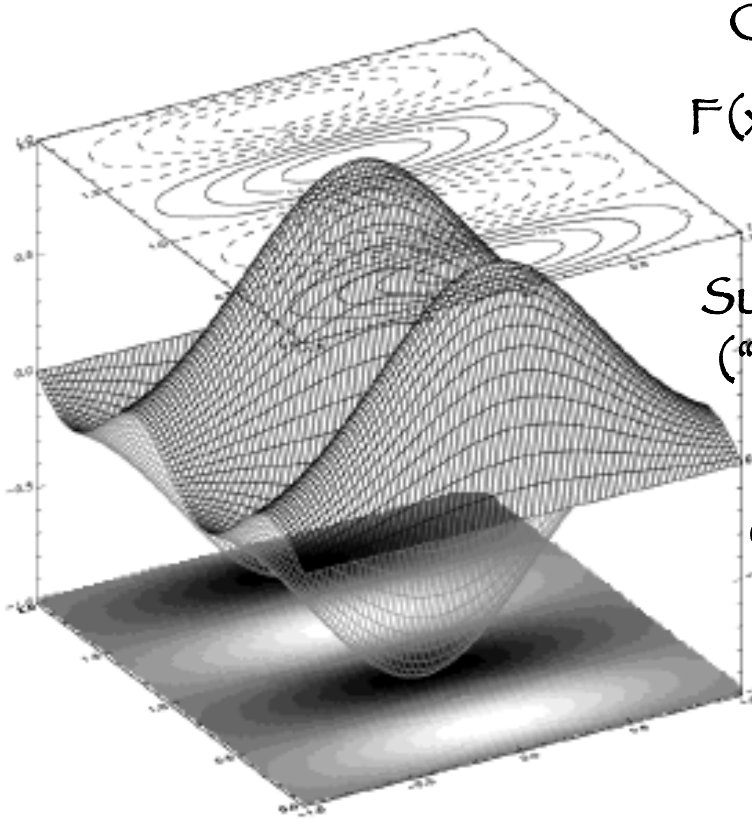
So, first what are Fields?

A field is a function of space and/or time.

Examples of scalar fields

temperature

topography

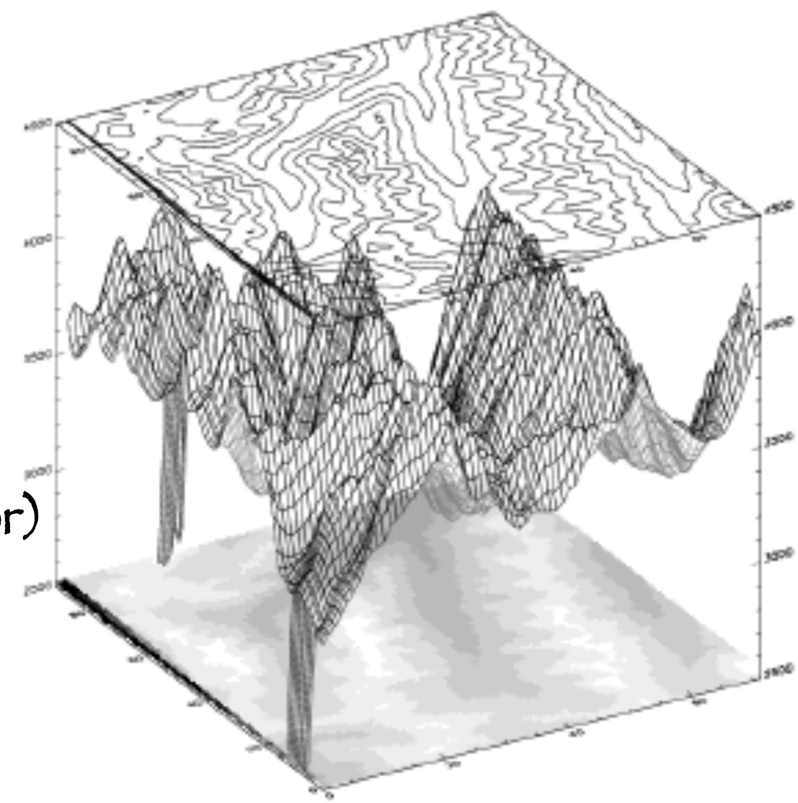


Contours

F(x,y)=const

Surface plot
("drawing")

Grey (color)
scale



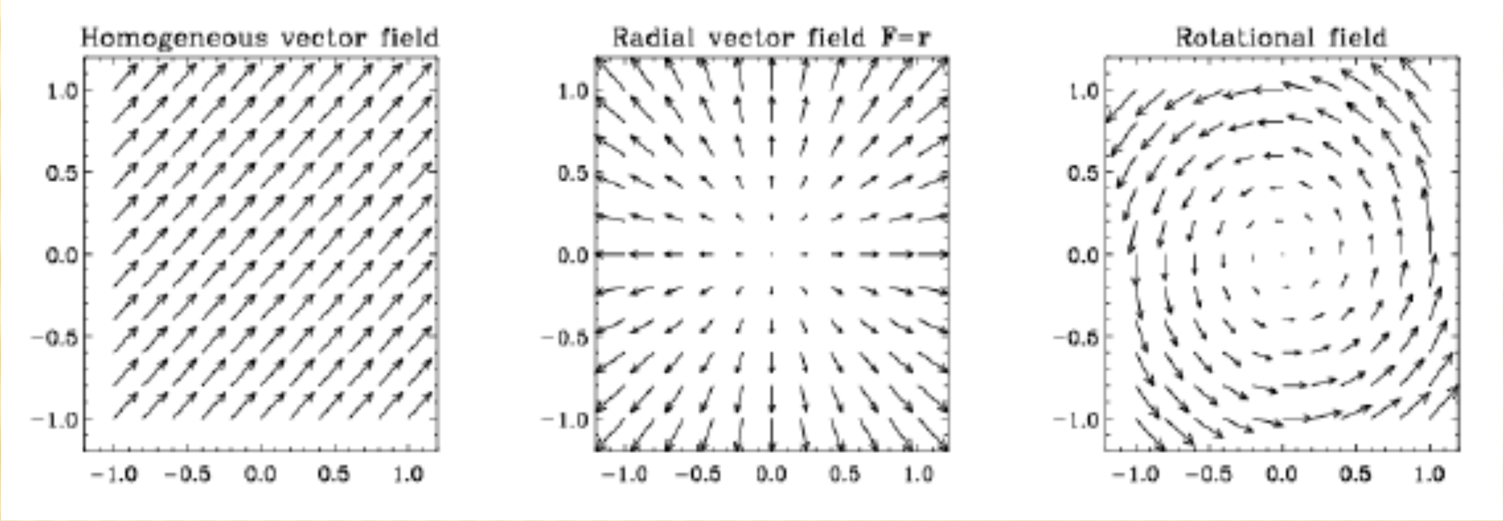
J vogt -- http://www.faculty.iu-bremen.de/jvogt/edu/spring03/NatSciLab2-GeoAstro/nslga2-lecture2.pdf

Examples of vector fields

streamlines

slopes

Vector map



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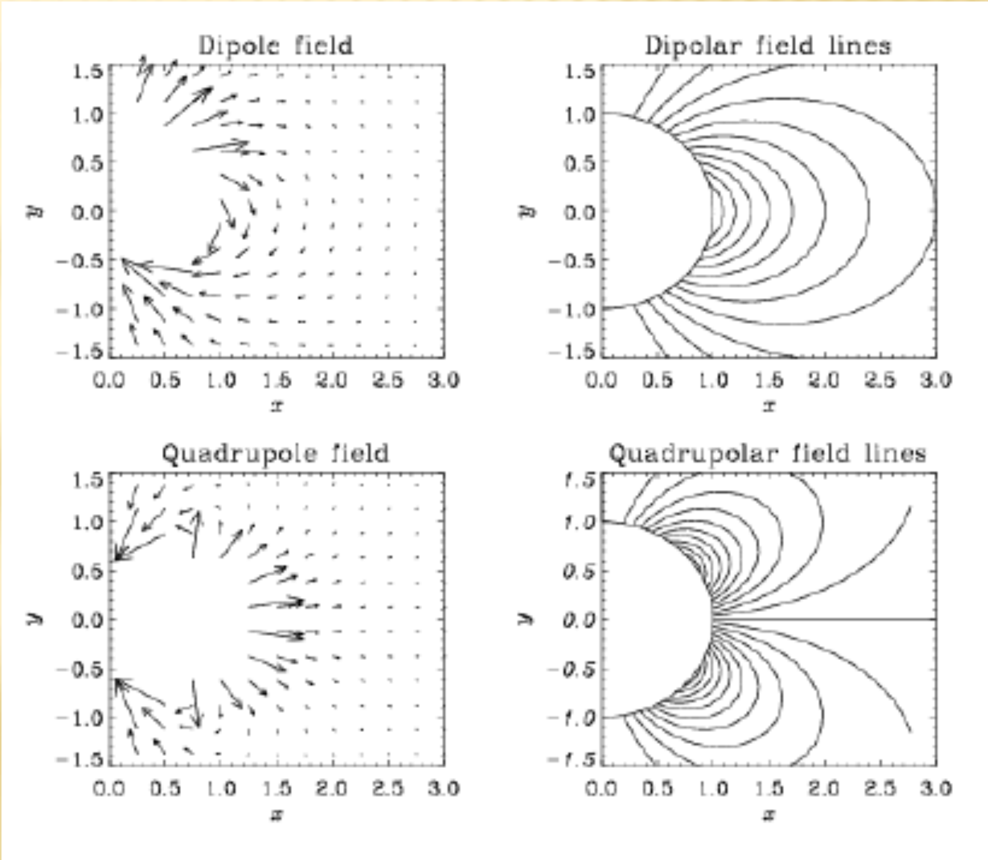
Examples of vector fields

streamlines

slopes

Plot streamlines



J vogt -- http://www.faculty.iu-bremen.de/jvogt/edu/spring03/NatSciLab2-GeoAstro/nslga2-lecture2.pdf

We are interested in

Force fields

describe forces acting at each point of space at a given time

Examples:

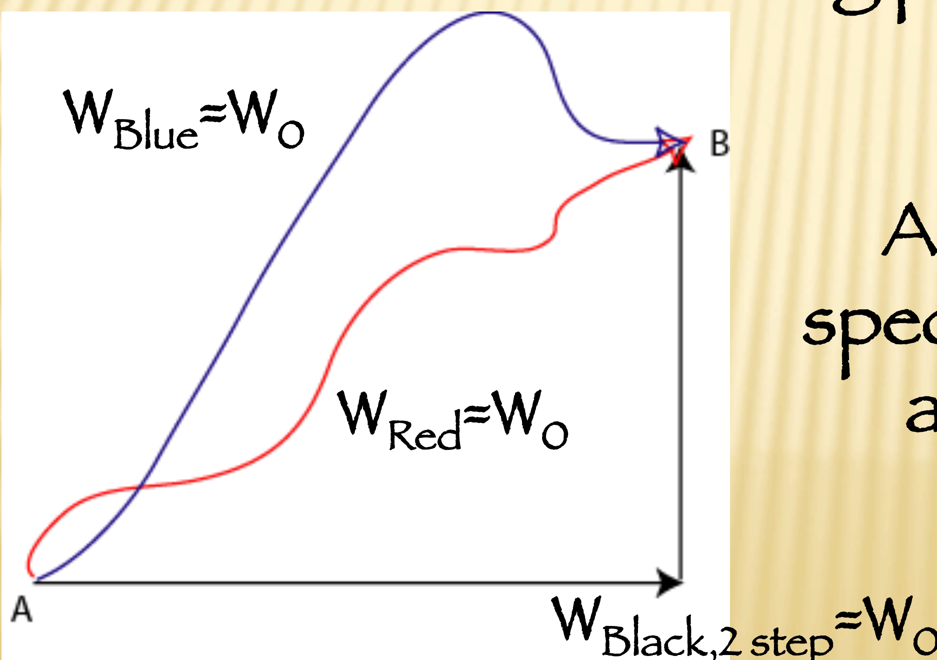
gravity field
magnetic field
electrostatic field

Fields can be scalar, vector or tensor

We know that work is the product of a force applied through a distance.

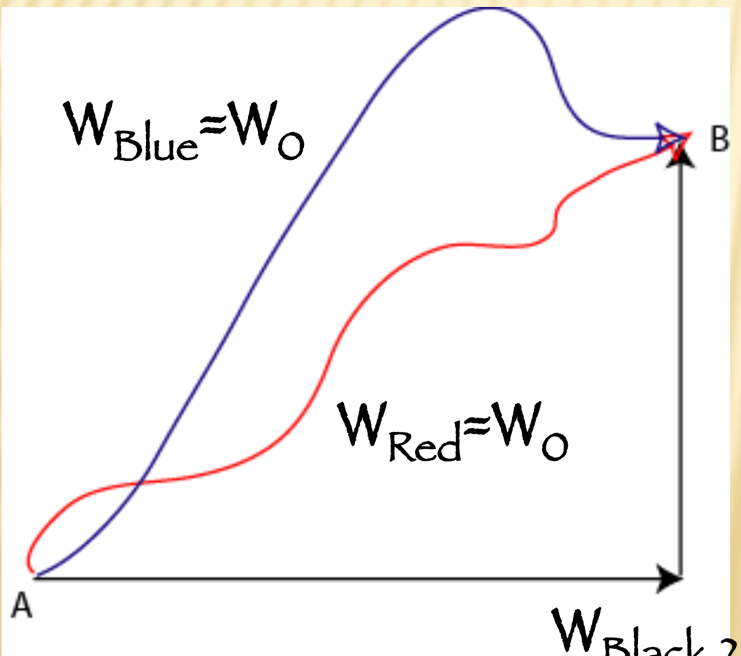
$$W = -\int_{x_0}^{x_1} F(x)\,dx$$

If the work done is independent of the path taken from x_0 to x_1, the work done depends only on the <u>starting</u> and <u>ending</u> positions.



A force with this type of special property is said to be a "conservative" force.

If we move around in a conservative force field and return to the starting point – by using the blue path to go from A to B and then return to A using the red path for example – the work is zero.



$W_{Black,2\ step}=W_0$

We can write this as

$$\oint_c F \bullet dl = 0$$

Important implication of conservative force field

$$\oint_{closed\ path} \vec{F} \bullet d\vec{l} = \iint_S \left(\nabla \times \vec{F}\right) \bullet d\vec{S} \quad \text{Stoke's theorem}$$

$$\text{if} \quad \oint_{closed\ path} \vec{F} \bullet d\vec{l} = 0 \text{ for all closed paths, then}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet d\vec{S} = 0 \quad \text{for all surfaces means}$$

$$\nabla \times \vec{F} = 0 \text{ everywhere (curl free vector field)}$$

and since $\nabla \times \nabla U = 0$ for all scalar functions U, then

we can write $\vec{F} = \nabla U$

A conservative force field is the derivative (gradient in 3-D) of a scalar field (function)!

This means our work integral is the solution to the differential equation

$$F(x) = -\frac{dU(x)}{dx}$$

Where we can define a scalar "potential" function U(x) that is a <u>function of position</u> only and

$$U(x) = -\int_{x_0}^{x} F(x)dx + U(x_0)$$

Where we have now included an arbitrary constant of integration. The potential function U(x) is only defined to within a constant – this means we can put the position where U(x)=0 where we want. It also makes it hard to determine it's "absolute", as opposed to "relative" value.

So now we have the pair of equations

$$g(x) = -\frac{dU(x)}{dx}$$

$$U(x) = -\int_{x_0}^{x} g(x)dx + U(x_0)$$

If you know U(x), you can compute g(x), where I have changed the letter for force to "g" for gravity.

If you know the force g(x) and that it is conservative, then you can computer U(x) - to within a constant.

$$U(x) = -\int_{x_0}^{x} g(x)dx + U(x_0)$$

U(x) is potential, the negative of the work done to get to that point.

So to put this to use we now have to ---

1) Show that gravity is a conservative force and therefore has an associated potential energy function.

2) Determine the gravity potential and gravity force fields for the earth

(first approximation – spherical
next approximation – ellipsoidal shape due to rotation
and then adjust for rotation)

3) Compare with real earth

Newton's Universal Law of Gravitation

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$

where $\vec{F}$ is the force

m_1 and m_2 are the masses

r is the distance between them

$\hat{r}$ is the unit vector seperating them

and G is the universal gravitational constant

In geophysics one of the masses is usually the earth so

$$\vec{F} = G\frac{M_e m}{r^2}\hat{r}$$

using

$$\vec{F} = m\vec{a}$$

we can define

$$\vec{a} = G\frac{M_e}{r^2}\hat{r} = -\vec{g}$$ the acceleration due to gravity

(the minus sign is to make the force attractive, with $\hat{r}$ pointing outwards)

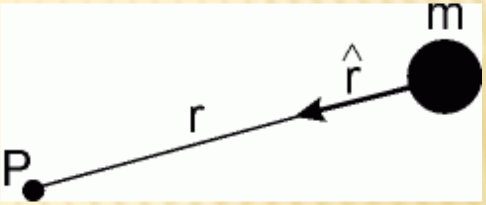
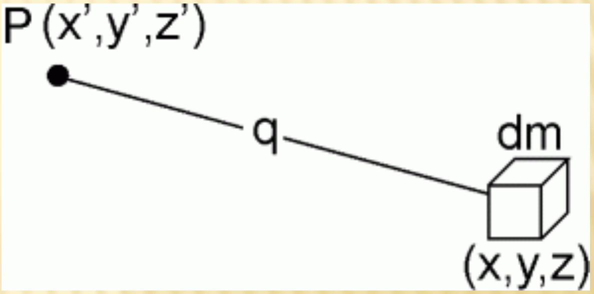


Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

Is gravity conservative?

for a mass distribution the acceleration becomes



$$\vec{a} = G\frac{\rho\, dV}{r^2}\hat{r}$$

check

$$\oint_{closed\ path} \vec{g} \bullet d\vec{r} = \oint_{closed\ path} G\frac{m}{r^2}\hat{r} \bullet d\vec{r} = \oint_{closed\ path} G\frac{m}{r^2}dr$$

use $d\left(\frac{1}{r}\right) = -\frac{1}{r^2}$, and throwing out "-" sign now

$$\oint_{closed\ path} G\frac{m}{r^2}dr = Gm\oint_{closed\ path} d\left(\frac{1}{r}\right) = Gm\frac{1}{r}\Big|_{x_0}^{x_0} = 0$$

so gravity is a conservative force (in general any central force field which depends only on *r* is conservative)

Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

Now we can define the potential as the work done to bring a unit mass from infinity to a distance r (set the work at infinity to zero)

$$work = U = \int_{\infty}^{r_0} \vec{g} \bullet d\vec{r} = \int_{\infty}^{r_0} G \frac{m}{r^2} \hat{r} \bullet d\vec{r} = Gm \frac{1}{r} \bigg|_{\infty}^{r_0} = \frac{Gm}{r_0}$$

for infintesimal mass

$$work = U = \frac{Gdm}{r}$$

So we can write the force field as the derivative of a scalar potential field in 1-D

$$g(x) = -\frac{dU(x)}{dx}$$

going to 3-D, it becomes a vector equation and we have

$$\vec{g}(\vec{x}) = -\nabla U(\vec{x})$$

Which in spherical coordinates is

$$\vec{g}(r,\theta,\phi) = -\nabla U(r,\theta,\phi) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)U(r,\theta,\phi)$$

$$\vec{g}(r,\theta,\phi) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)\left(\frac{Gdm}{r}\right)$$

$$\vec{g}(r) = \frac{Gdm}{r^2}\hat{r}$$

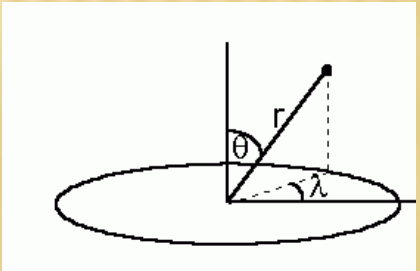


Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

Apply to our expression for the gravity potential

$$\vec{g}(r,\theta,\phi) = -\nabla U(r,\theta,\phi) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)U(r,\theta,\phi)$$

$$U = \frac{Gdm}{r}$$ (a scalar)

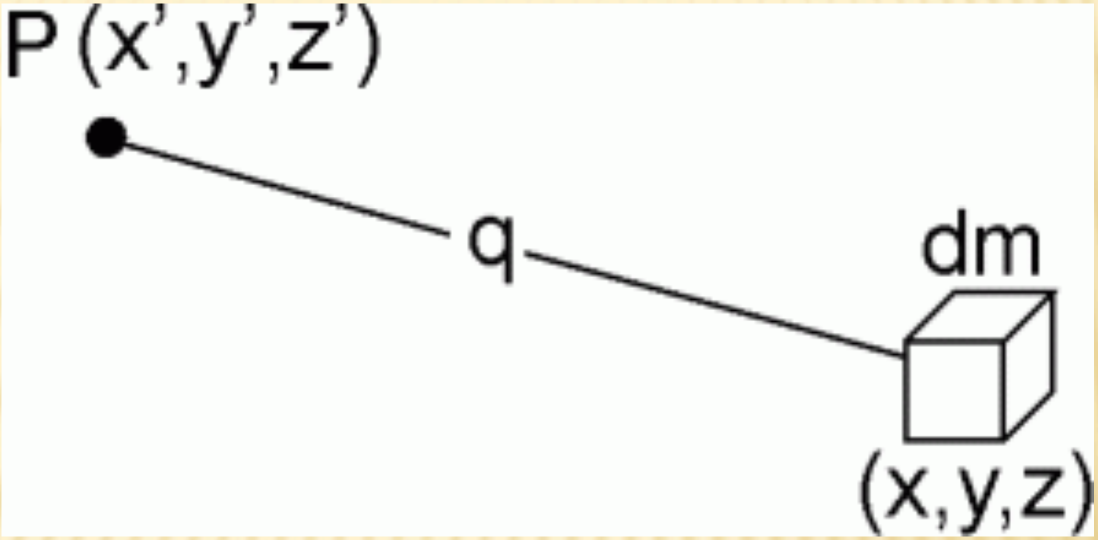
$$\vec{g}(r,\theta,\phi) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)\left(\frac{Gdm}{r}\right)$$

$$\vec{g}(r) = \frac{Gdm}{r^2}\hat{r}$$

Which agrees with what we know

Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

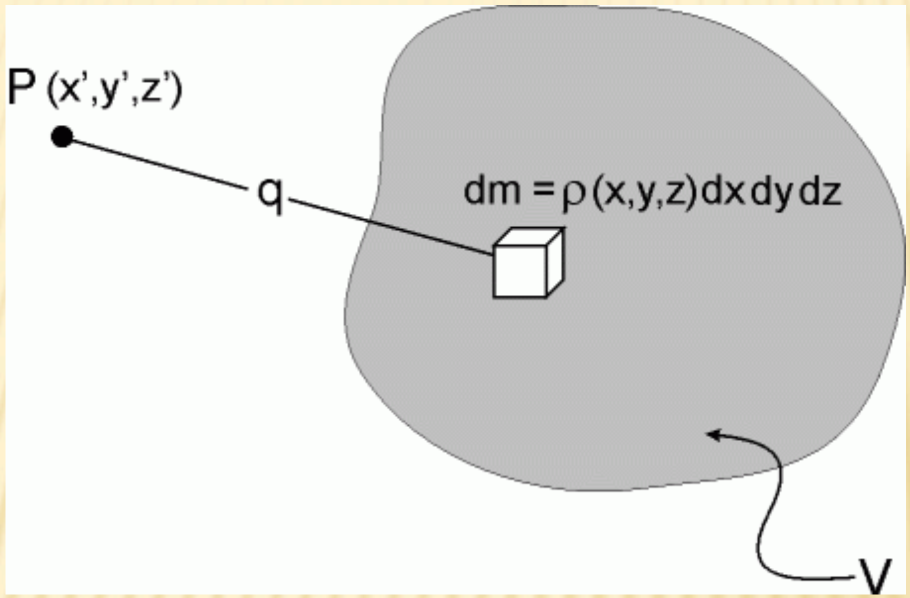
To find the total potential of gravity we have to integrate over all the point masses in a volume.



$$U(x',y',z') = \frac{Gdm}{q} = \frac{Gdm}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

To find the total potential of gravity we have to integrate over all the point masses in a volume.

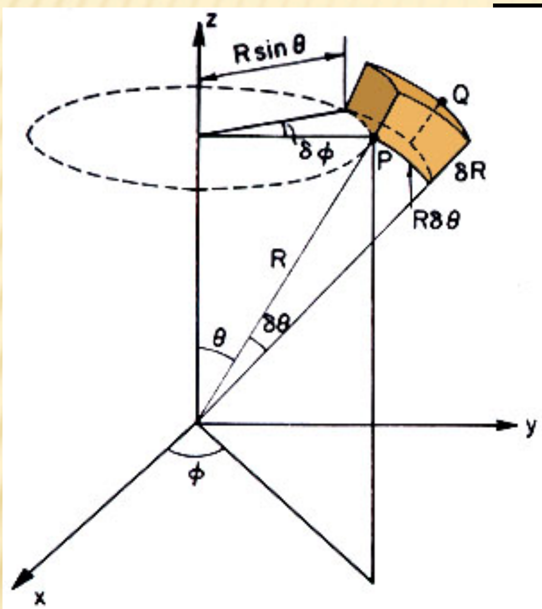


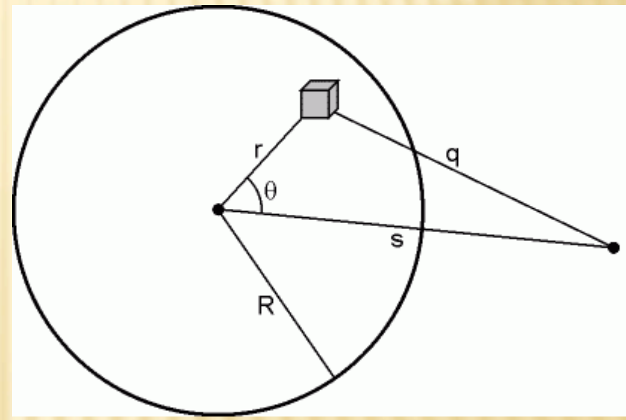
$$U(x',y',z') = \iiint_V \frac{Gdm}{q} dV = G \iiint_V \frac{dm}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} dxdydz$$

Figure from Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

If things are spherically symmetric it is easier to work in spherical coordinates

Ex: uniform density sphere





$$U(P) = \iiint\limits_V \frac{G\rho}{q}\left(r^2 \sin\theta \, dr d\theta d\phi\right), \text{ with constant } \rho$$

$$U(P) = G\rho \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{q} r^2 \sin\theta \, dr d\theta d\phi$$

Figures from: right - Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton, left - http://www.siu.edu/~cafs/surface/file13.html

Grinding thorugh

$$U(P) = G\rho \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{q} r^2 \sin\theta \, dr d\theta d\phi$$

use/substitute $\quad q^2 = r^2 + s^2 - 2rs\cos\theta$

$$U(P) = 2\pi G\rho \int_0^R \int_0^\pi \frac{1}{\sqrt{r^2 + s^2 - 2rs\cos\theta}} r^2 \sin\theta \, dr d\theta$$

$$U(P) = 2\pi G\rho \int_0^R \int_1^{-1} \frac{1}{\sqrt{r^2 + s^2 - 2rsu}} r^2 \, dr du \quad (\text{use } u = \cos\theta)$$

Grinding thorugh

$$U(P) = 2\pi G\rho \int_0^R \int_1^{-1} \frac{1}{\sqrt{r^2 + s^2 - 2rsu}} r^2\, dr du, u = \cos\theta$$

$$\text{use} \int \frac{\mathrm{dx}}{\sqrt{\mathrm{a} + \mathrm{bx}}} = \frac{2\sqrt{a + bx}}{b}$$

$$U(P) = \frac{2\pi G\rho}{s} \int_0^R \frac{\left|\sqrt{r^2 + s^2 + 2rs} - \sqrt{r^2 + s^2 - 2rs}\right|}{r} r^2 dr$$

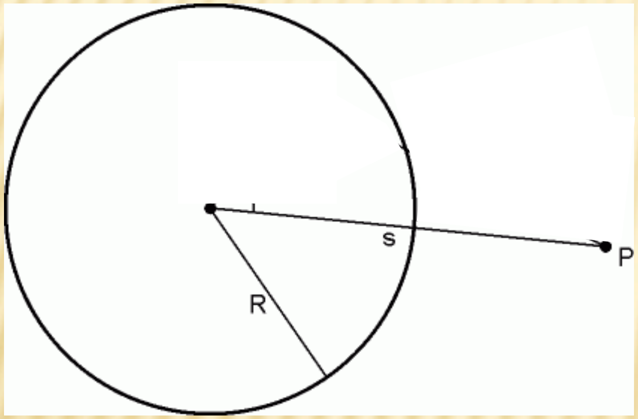
$$U(P) = \frac{2\pi G\rho}{s} \int_0^R \frac{\left|(s + r) - (s - r)\right|}{r} r^2 dr$$

$$U(P) = \frac{4\pi G\rho}{s} \int_0^R r^2 dr = G \frac{4\pi R^3}{3} \frac{\rho}{s} = \frac{G(V\rho)}{s} = \frac{GM}{s}$$

$$g = -\frac{\partial U}{\partial s} = \frac{GM}{s^2}$$

So for a uniform density sphere

The potential and force of gravity at a point P, a distance s≥R from the center of the sphere, are



$$U(P) = \frac{GM}{s}$$

$$\vec{g}(P) = -\frac{\partial U}{\partial s} = \frac{GM}{s^2}\hat{s}$$

Figure after Ahern, http://geophysics.ou.edu/gravmag/potential/gravity_potential.html#newton

Note that in seismology the vector displacement field solution for P waves is also curl free.

This means it is the gradient of a scalar field – call it the P wave potential.

So one can work with a scalar wave equation for P waves, which is easier than a vector wave equation, and take the gradient at the end to get the physical P wave displacement vector field.

(This is how it is presented in many introductory Seismology books such as Stein and Wysession.)

Unfortunately, unlike with gravity, there is no physical interpretation of the P wave potential function.

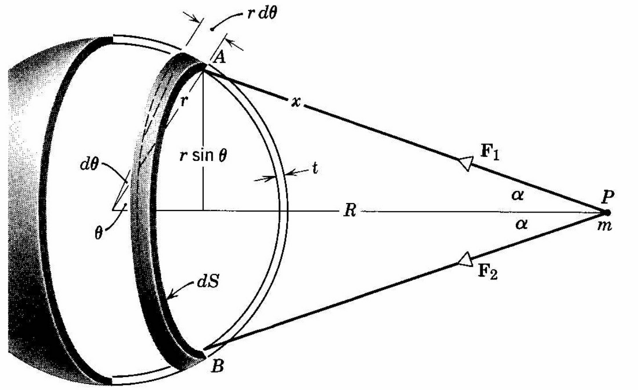


FIGURE 14–4 Gravitational attraction of a section dS of a spherical shell of matter on a particle of mass m.

Next ex:

Force of gravity from spherical shell

$$dV = 2\pi\, t r^2 \sin\theta\, d\theta, \qquad Dm = \rho\, dV$$

$$dF = G\frac{m dM}{x^2}\cos\alpha = 2\pi\, G m\rho\, t r^2 \frac{\sin\theta\, d\theta}{x^2}\cos\alpha$$

from geometry $\quad \cos\alpha = \dfrac{R - r\cos\theta}{x}$

law of cosines $x^2 = R^2 + r^2 - 2Rr\cos\theta$ gives $r\cos\theta = \dfrac{R^2 + r^2 - x^2}{2R}$

$2x dx = 2R\sin\theta\, d\theta$ so $\sin\theta\, d\theta = \dfrac{x}{Rr} dx$

combine $\quad dF = \dfrac{\pi\, G t\rho\, m r}{R^2}\left(\dfrac{R^2 - r^2}{x^2} + 1\right) dx$

After Halliday and Resnick, Fundamentals of Physics

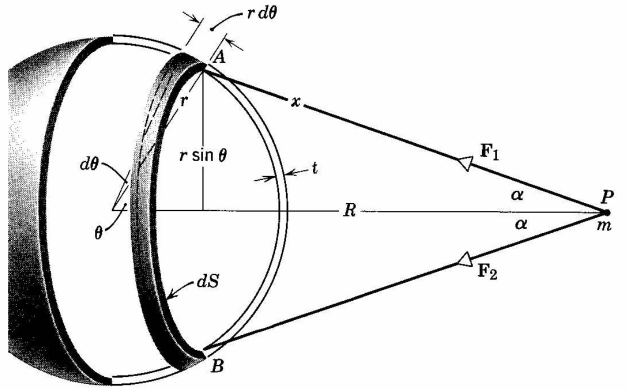


FIGURE 14–4 Gravitational attraction of a section dS of a spherical shell of matter on a particle of mass m.

Force of gravity from spherical shell

$$dF = \frac{\pi G t \rho m r}{R^2}\left(\frac{R^2 - r^2}{x^2} + 1\right)dx$$

integrate over all circular strips (is integral over x)

$$\mathrm{F} = \frac{\pi G t \rho m r}{R^2}\int_{\mathrm{R-r}}^{\mathrm{R+r}}\left(\frac{R^2 - r^2}{x^2} + 1\right)dx = \frac{\pi G t \rho m r}{R^2}\left[-\frac{R^2 - r^2}{x} + x\right]_{R-r}^{R+r} = \frac{\pi G t \rho m r}{R^2}4r$$

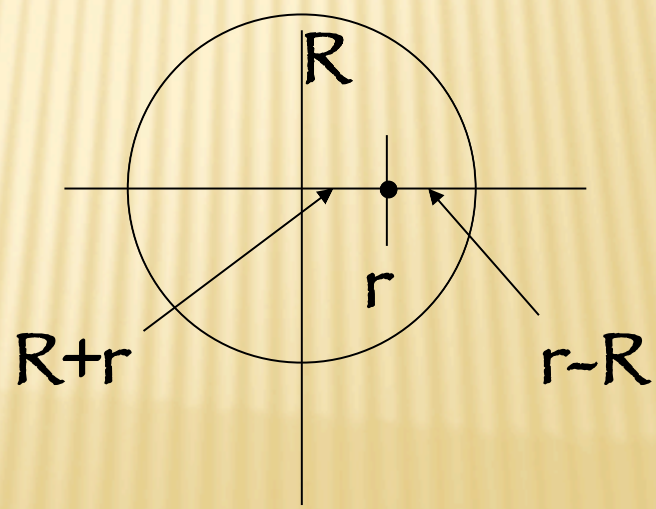
but $M = 4\pi\rho r^2$ so $F = G\dfrac{Mm}{R^2}$

Uniformly dense spherical shell attracts external mass as if all its mass were concentrated at its center.

After Halliday and Resnick, Fundamentals of Physics

From inside a shell, the lower limit of integration changes to r-R and we get zero.

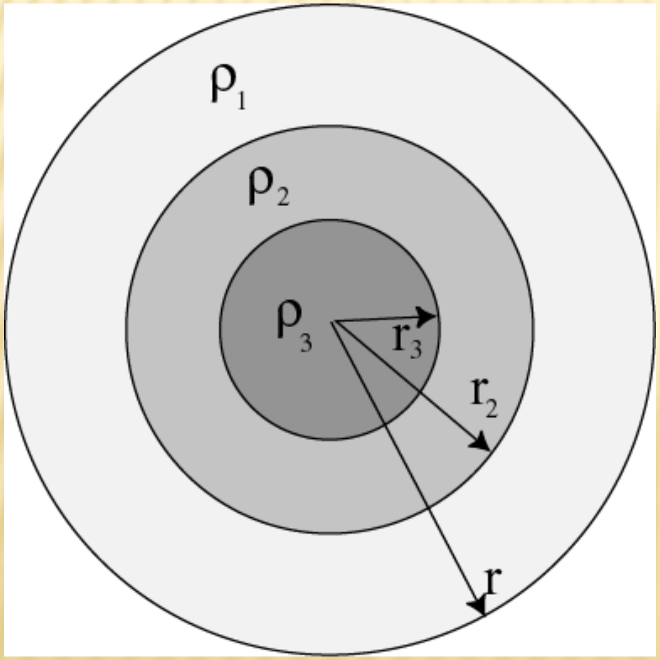
$$F = \frac{\pi G t \rho m r}{R^2} \int_{r-R}^{R+r} \left(\frac{R^2 - r^2}{x^2} + 1 \right) dx = \frac{\pi G t \rho m r}{R^2} \left[-\frac{R^2 - r^2}{x} + x \right]_{r-R}^{R+r} = 0$$



After Halliday and Resnick, Fundamentals of Physics

For a solid sphere – we can make it up of concentric shells.

Each shell has to have a uniform density, although different shells can have different densities (density a function of radius only – think "earth").



From outside – we can consider all the mass to be concentrated at the center.

Now we need to find the potential and force for our ellispsoid of revolution (a nearly spherical body).

(note that we are not starting from scratch with a spinning, self gravitating fluid body and figuring out its equilibrium shape – we're going to find the gravitational potential and force for an almost, but not quite spherical body.)

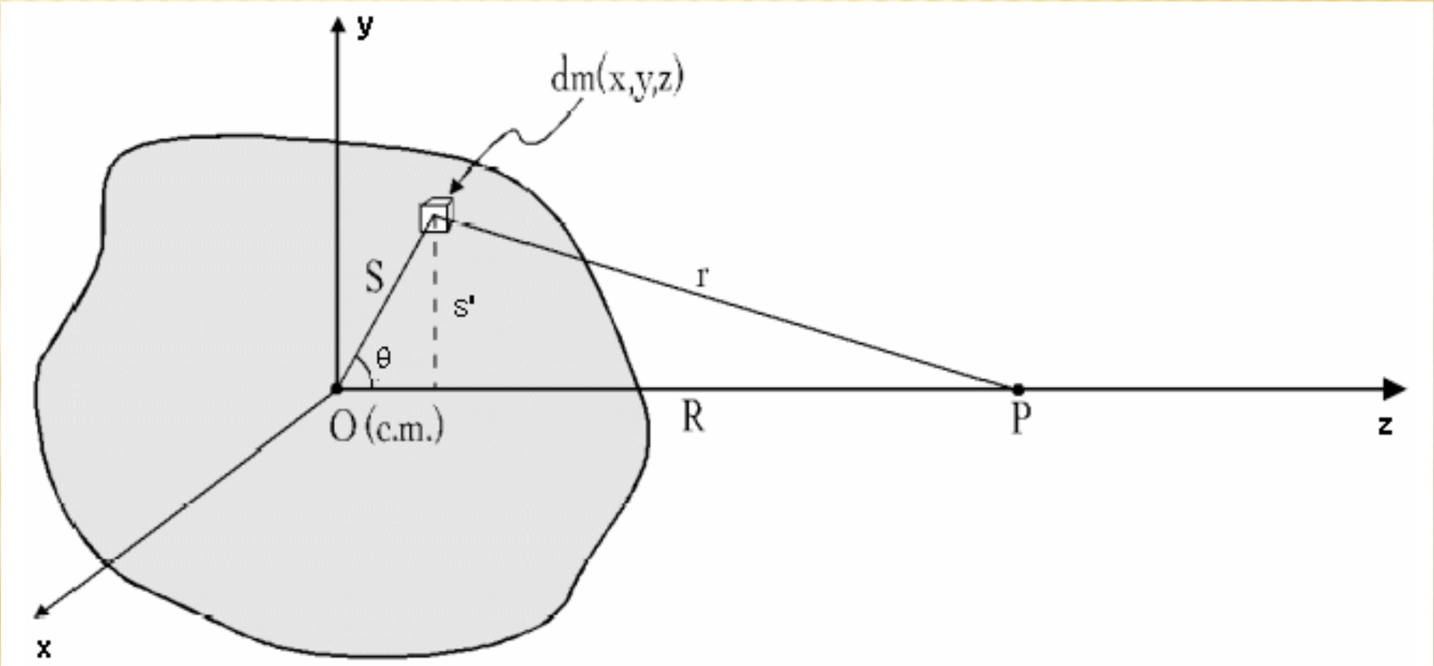
Discussion after Turcotte, Ahern and Nerem

EARTH'S GRAVITY FIELD

Calculate the potential at a point P (outside) due to a nearly spherical body (the earth).

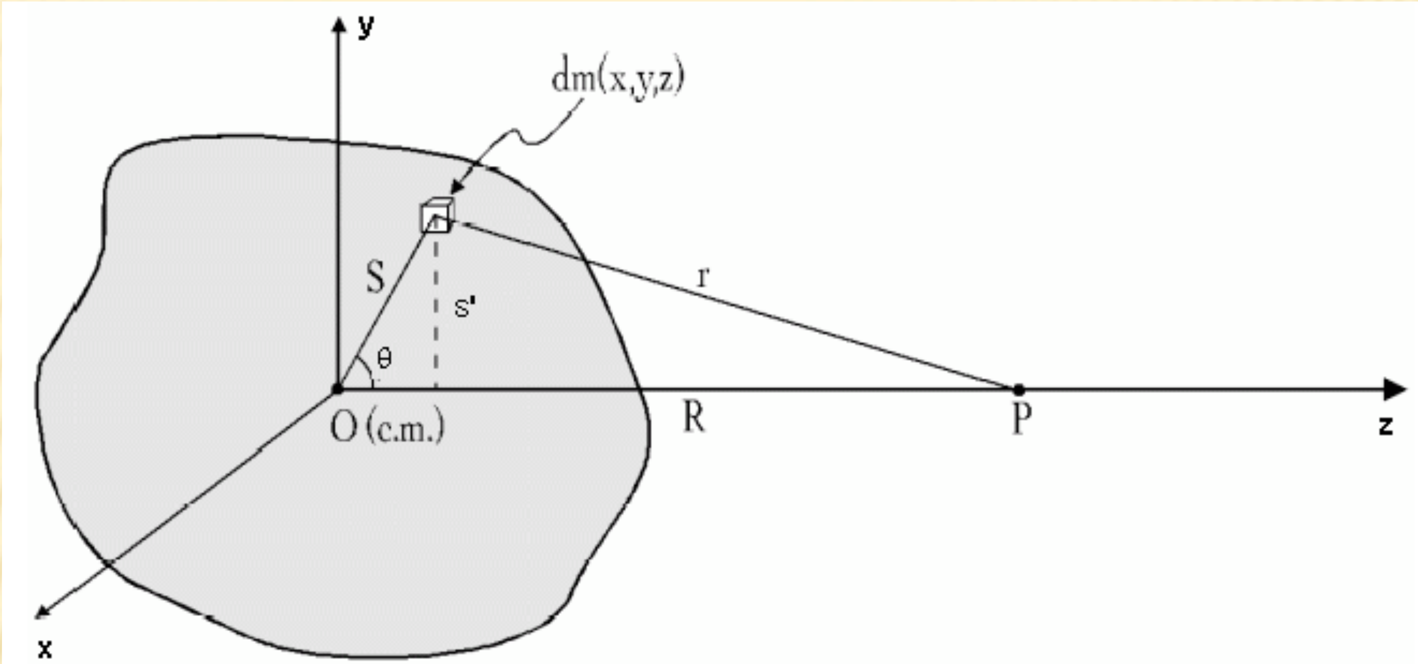
Set up the geometry for the problem:

For simplicity - put the origin at the center of mass of the body and let P be on an axis.



Discussion after Nerem , Turcotte, and Ahern

Calculate the potential at a point P due to a nearly spherical body.



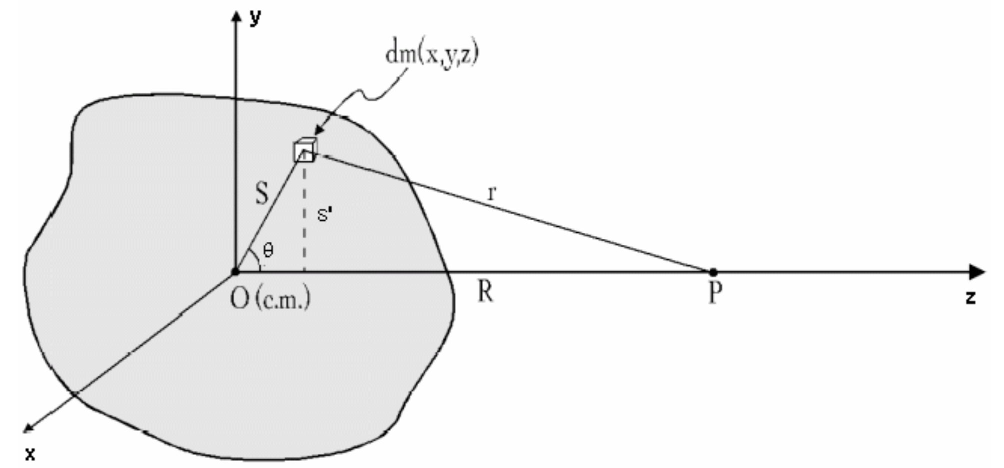
we start with expression for potential

$$U = -G \int_{Volume} \frac{dm}{r}$$

use law of cosines $r^2 = R^2 + S^2 - 2RS\cos\theta$

$$U = -G \int_{Volume} \frac{dm}{\left(R^2 + S^2 - 2RS\cos\theta\right)^{1/2}} = -\frac{G}{R} \int_{Volume} \left(1 + \frac{S^2}{R^2} - 2\frac{S}{R}\cos\theta\right)^{-1/2} dm$$

Calculate the potential at a point P due to a nearly spherical body.

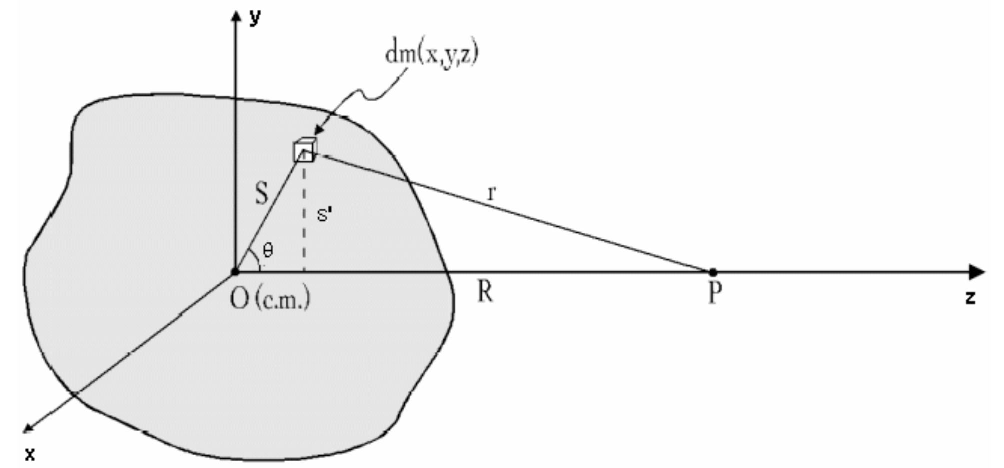


$$U = -\frac{G}{R}\int_{Volume}\left(1+\frac{S^2}{R^2}-2\frac{S}{R}\cos\theta\right)^{-1/2} dm$$

$$\text{for} \quad \varepsilon << 1, \quad (1+\varepsilon)^{-1/2} \approx 1-\frac{\varepsilon}{2}+\frac{3}{8}\varepsilon^2+\cdots, \qquad \frac{S}{R} << 1, \quad \varepsilon = +\frac{S^2}{R^2}-2\frac{S}{R}\cos\theta$$

$$U = -\frac{G}{R}\int_{Volume}\left(1-\frac{\left(\frac{S^2}{R^2}-2\frac{S}{R}\cos\theta\right)}{2}+\frac{3\left(\frac{S^2}{R^2}-2\frac{S}{R}\cos\theta\right)^2}{8}\right)dm$$

Calculate the potential at a point P due to a nearly spherical body.



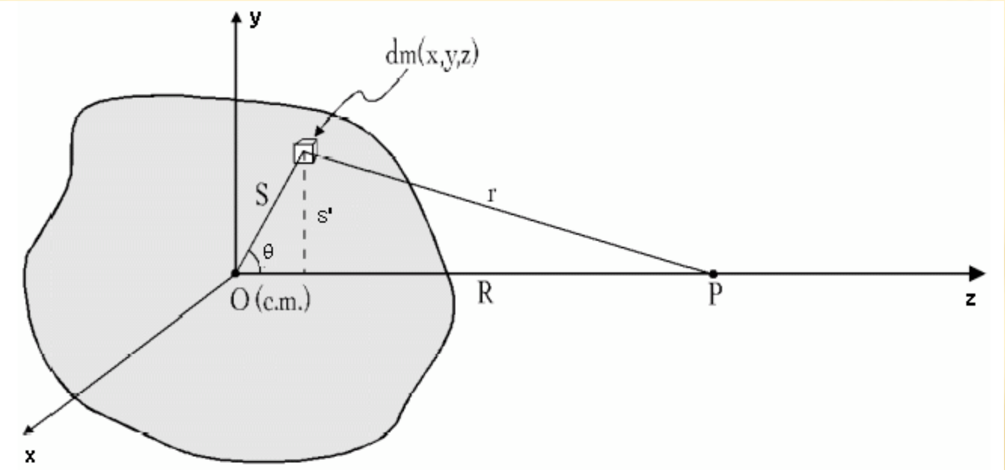
$$U \approx -\frac{G}{R}\int_{Volume}\left(1-\frac{S^2}{2R^2}+\frac{S}{R}\cos\theta+\frac{3S^2}{2R^2}\cos^2\theta+O\left(\frac{S}{R}\right)^3\right)dm$$

$$U \approx -\frac{G}{R}\left[\int_{Volume} dm+\frac{1}{R}\int_{Volume} S\cos\theta\, dm+\frac{1}{2R^2}\int_{Volume}\left(-S^2+3S^2\cos^2\theta\right)dm\right]$$

use $\cos^2\theta = 1-\sin^2\theta$

$$U \approx -\frac{G}{R}\left[\int_{Volume} dm+\frac{1}{R}\int_{Volume} S\cos\theta\, dm+\frac{1}{2R^2}\int_{Volume}\left(2S^2-3S^2\cos^2\theta\right)dm\right]$$

Calculate the potential at a point P due to a nearly spherical body.



$$V \approx -\left(\frac{G}{R} \int_{Volume} dm + \frac{G}{R^2} \int_{Volume} S\cos\theta \, dm + \frac{G}{2R^3} \int_{Volume} \left(2S^2 - 3S^2\cos^2\theta\right) dm \right)$$

where each integral is multiplied by a different order of $\frac{1}{R}$, so rename

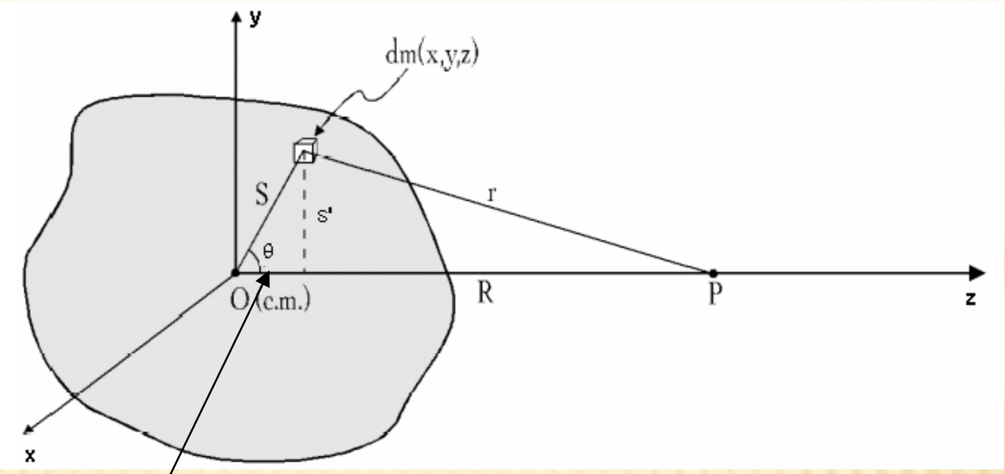
$$U \approx U_0 + U_1 + U_2$$

now

$$U_0 = -\frac{G}{R} \int_{Volume} dm = -\frac{GM}{R}$$ same result for sphere with uniform density

(all mass at point at center)

Calculate the potential at a point P due to a nearly spherical body.



$$V \approx -\left(\frac{GM}{R} + \frac{G}{R^2} \int_{Volume} S\cos\theta \, dm + \frac{G}{2R^3} \int_{Volume} \left(2S^2 - 3S^2\cos^2\theta\right) dm \right)$$

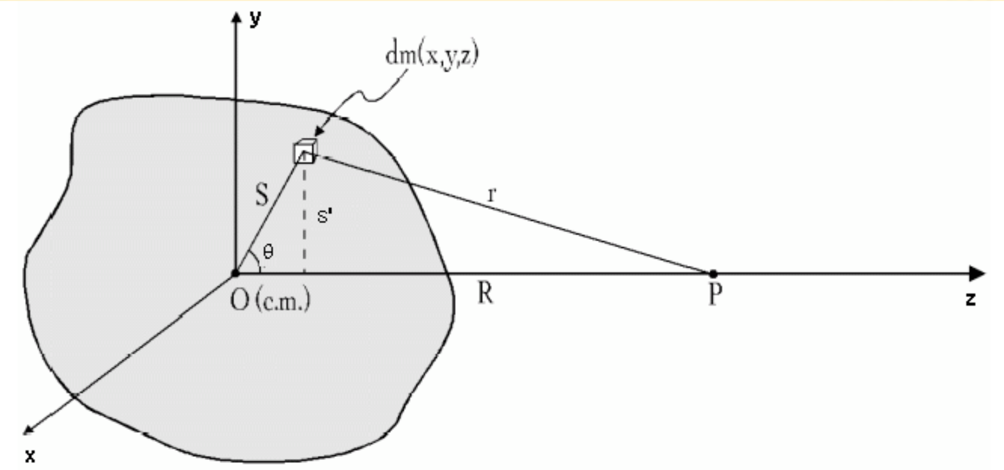
now

$$U_1 = \frac{G}{R^2} \int_{Volume} S\cos\theta \, dm = \frac{G}{R^2} \int_{Volume} z \, dm$$

z is projection of S on z axis.

This is the equation for the center of mass (first moment), but we have placed the origin at the center of mass, so this integral is zero. $U_1 = \dfrac{G}{R^2} z_{center\ of\ mass} = 0$

Calculate the potential at a point P due to a nearly spherical body.



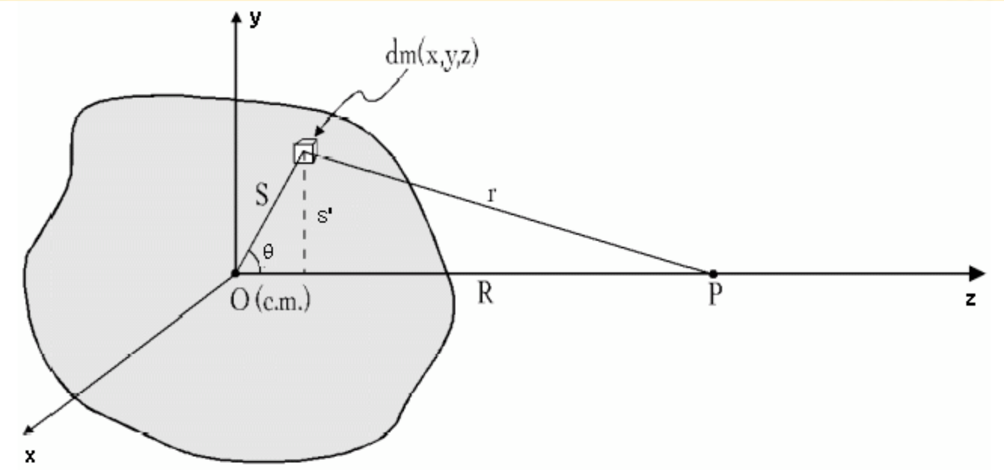
$$U \approx -\left(\frac{GM}{R} + \frac{G}{2R^3} \int_{Volume} \left(2S^2 - 3S^2\cos^2\theta\right) dm \right)$$

$$U_3 = \frac{G}{2R^3}\left[\int_{Volume} 2S^2 dm - \int_{Volume} 3S^2 \cos^2\theta dm \right]$$

use $s' = S\cos\theta$ (see diagram)

$$U_3 = \frac{G}{2R^3}\left[\int_{Volume} 2S^2 dm - \int_{Volume} 3s'^2 dm \right]$$

Calculate the potential at a point P due to a nearly spherical body.

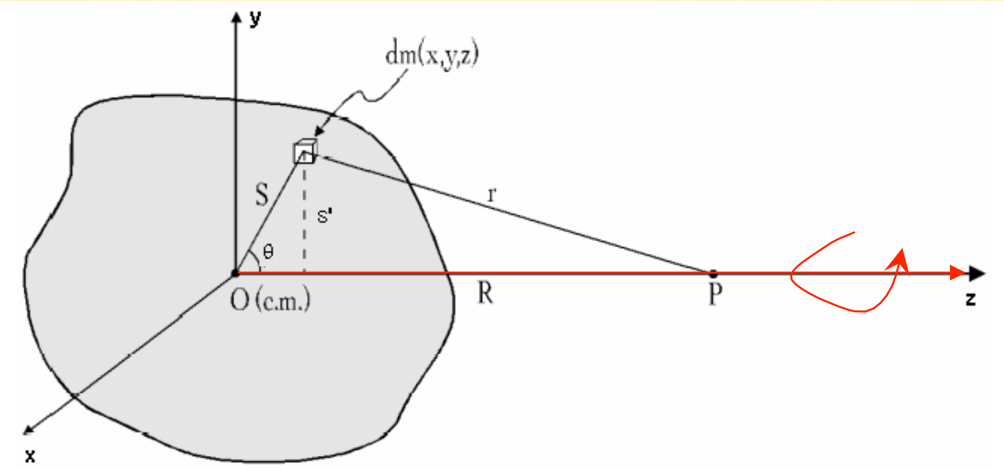


now what are

$$\int_{Volume} 2S^2 dm$$

$$\int_{Volume} 3s'^2 dm$$

Calculate the potential at a point P due to a nearly spherical body.



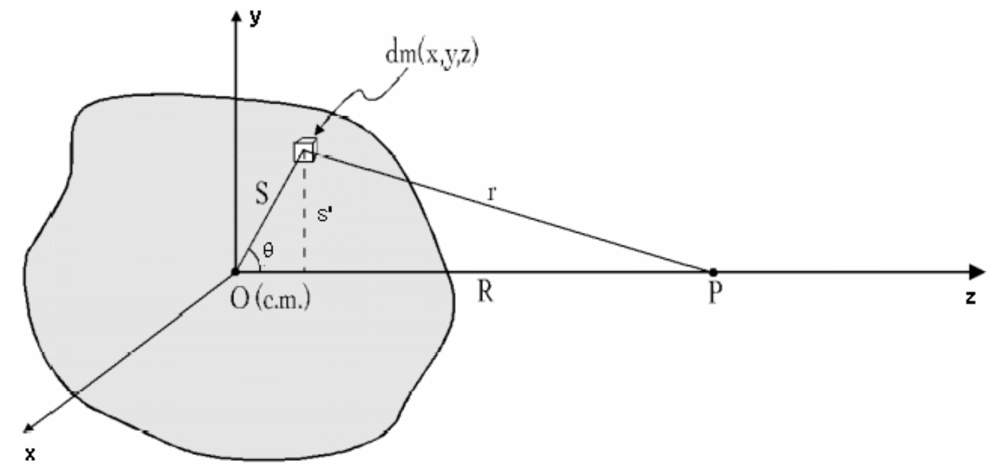
start with

$$3 \int_{Volume} s'^2 dm$$

notice from figure that this is just the moment of intertia about an axis from the origin to the point P

$I_{OP} = \int_{Volume} r^2 dm$, where r is perpendicular distance from rotation axis.

Calculate the potential at a point P due to a nearly spherical body.

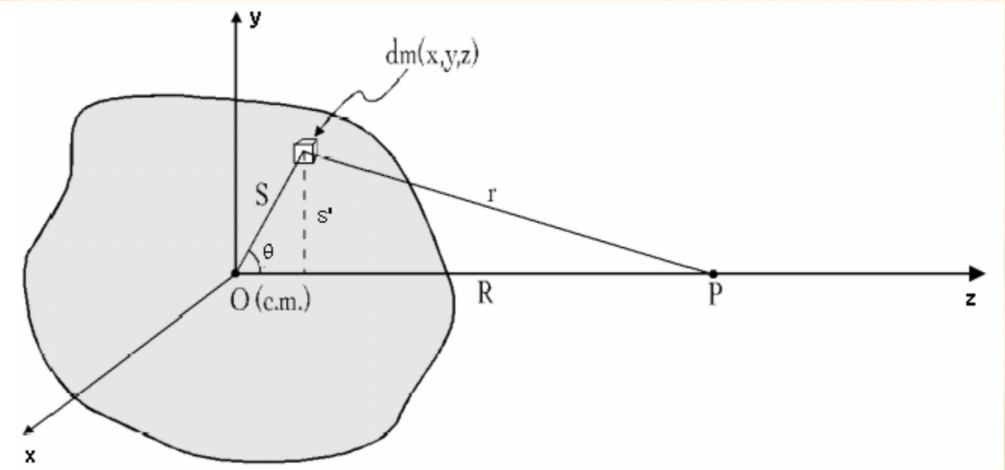


$$U \approx -\left(\frac{GM}{R} - \frac{G}{2R^3} 3I_{OP} + \frac{G}{2R^3} \int_{Volume} 2S^2 dm \right)$$

now for the last term, it too is an integral of distances from the origin to all points in a body. Can we massage this into something that looks like moments of inertia?

(yes, or we would not be asking!)

Calculate the potential at a point P due to a nearly spherical body.



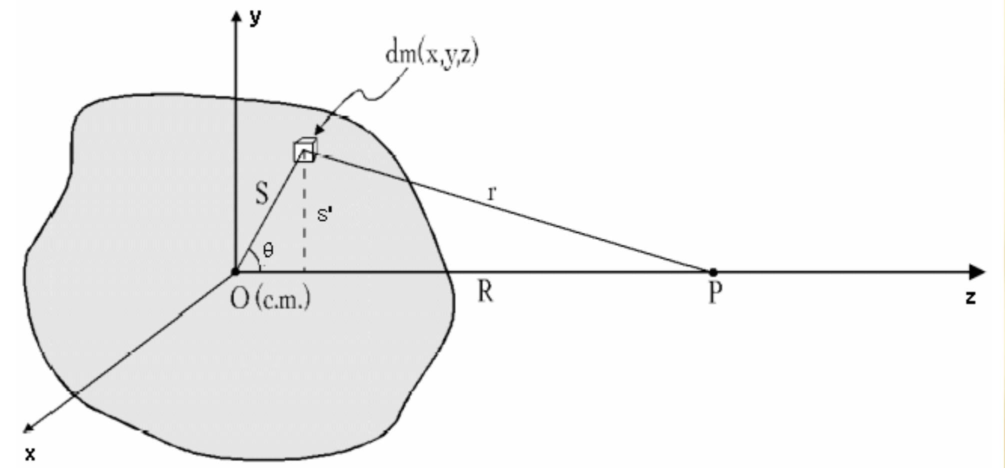
$$U_{3'} = \frac{G}{2R^3} \int_{Volume} 2S^2 dm$$

S is the distance to a point in body, which is invarient under coordinate rotations, so

$$S^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

where the primed values are in the principal coordinate system for the moments of intertia (same idea as principal coordinate system for stress and strain).

Calculate the potential at a point P due to a nearly spherical body.



$$U_{3'} = \frac{G}{2R^3} \int_{Volume} 2S^2 dm = \frac{G}{2R^3} \int_{Volume} 2\left(x'^2 + y'^2 + z'^2\right) dm$$

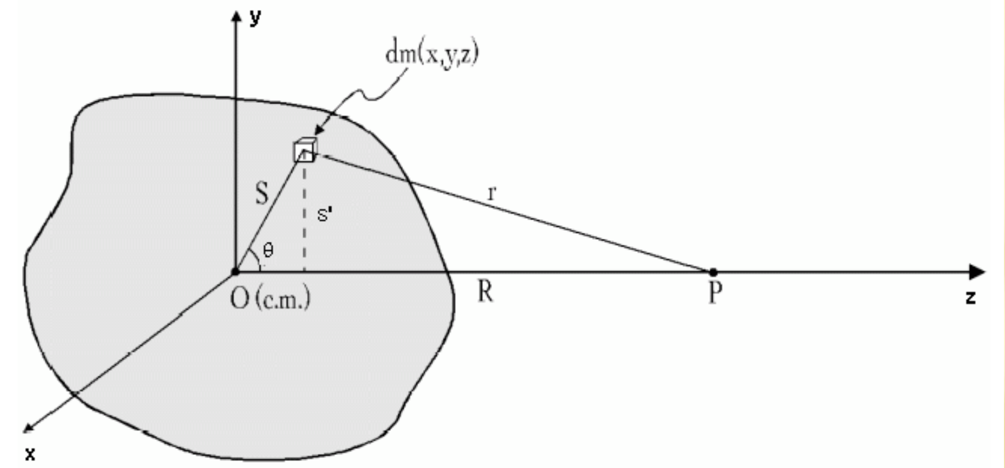
$$U_{3'} = \frac{G}{2R^3} \int_{Volume} \left(x'^2 + y'^2) + (x'^2 + z'^2\right) + (y'^2 + z'^2) dm$$

where each of the terms is the principal moment of inertia about the z', y' and x' axes respectively.

$$U_{3'} = \frac{G}{2R^3}\left(I_1 + I_2 + I_3\right), \text{ so}$$

$$U_3 = \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

Calculate the potential at a point P due to a nearly spherical body.



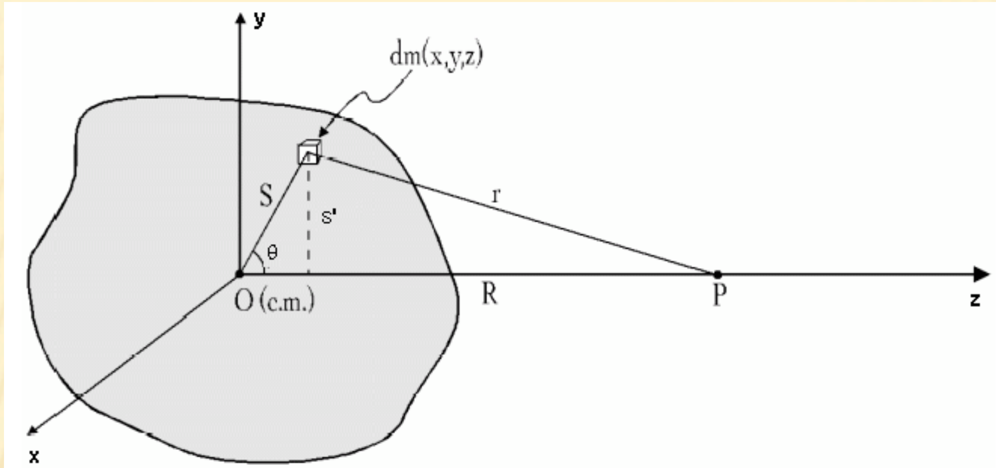
putting it all together

$$U = -G\int_{Volume}\frac{dm}{r} = V_1 + V_3 = -\frac{GM}{R} - \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

Potential for sphere plus adjustments for principal moments of inertia and moment of inertia along axis from origin to point of interest, P.

This is MacCullagh's formula for the potential of a nearly spherical body

Calculate the potential at a point P due to a nearly spherical body.



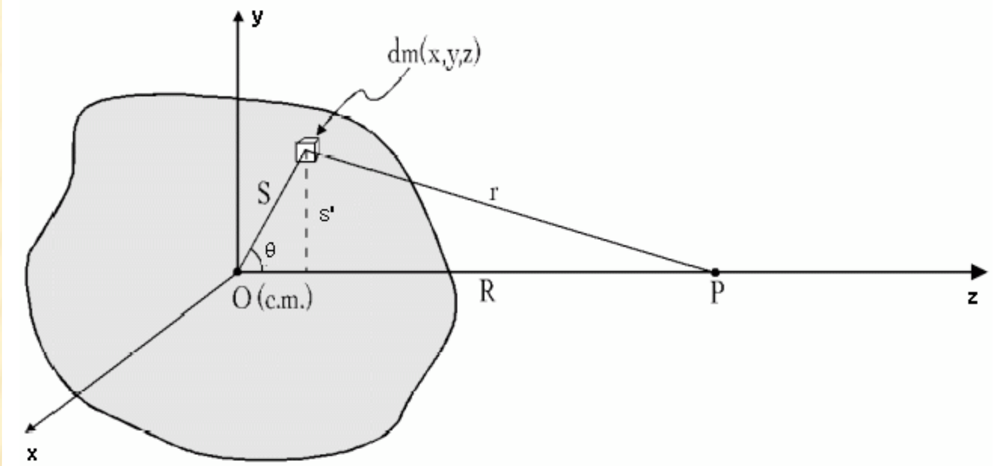
$$U = -G\int_{Volume}\frac{dm}{r} = V_1 + V_3 = -\frac{GM}{R} - \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

For a sphere $I_1 = I_2 = I_3 = I_{op}$ and

$$U = -\frac{GM}{R}$$

(which we knew already)

Calculate the potential at a point P due to a nearly spherical body.

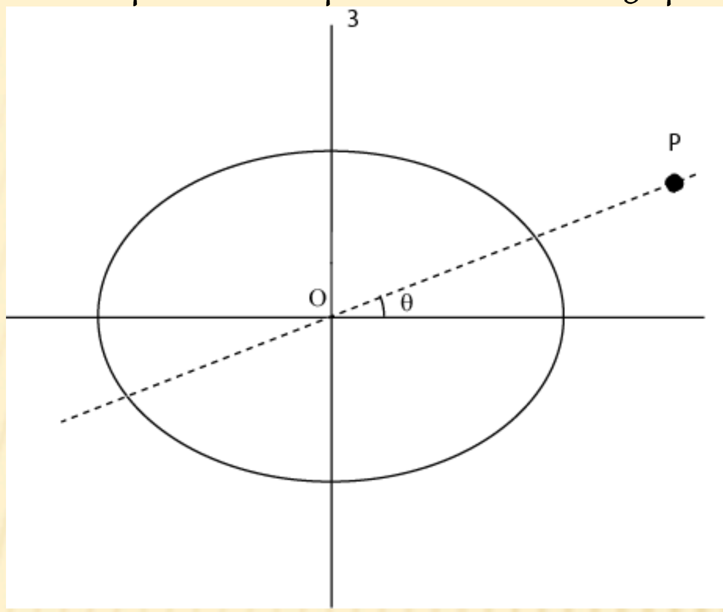


So here's our semi-final result for the potential of an approximately spherical body

$$U = -G\int_{Volume}\frac{dm}{r} = U_1 + U_3 = -\frac{GM}{R} - \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

Now let's look at a particular approximately spherical body – the ellipsoid

Calculate the potential at a point P due to a nearly spherical body.



$$U = -G\int_{Volume}\frac{dm}{r} = U_1 + U_3 = -\frac{GM}{R} - \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

for an ellipsoid $I_1 = I_2 \neq I_3$

$I_{OP} = I_1\cos^2\theta + I_3\sin^2\theta$, where θ is latitude (rotate into prin. coord. sys.)

$$I_{OP} = I_1\left(1 - \sin^2\theta\right) + I_3\sin^2\theta = I_1 + \left(I_3 - I_1\right)\sin^2\theta$$

so $\left(I_1 + I_2 + I_3 - 3I_{OP}\right) = \left(2I_1 + I_3 - 3\left(I_1 + \left(I_3 - I_1\right)\sin^2\theta\right)\right)$

$$\left(I_1 + I_2 + I_3 - 3I_{OP}\right) = \left(I_3 - I_1\right)\left(1 - 3\sin^2\theta\right)$$

Calculate the potential at a point P due to a nearly spherical body.

$$U = -G\int_{Volume} \frac{dm}{r} = U_1 + U_3 = -\frac{GM}{R} - \frac{G}{2R^3}\left(I_1 + I_2 + I_3 - 3I_{OP}\right)$$

so for an ellipsoid this becomes

$$U = -\frac{GM}{R} + \frac{G}{2R^3}\left(I_3 - I_1\right)\left(3\sin^2\theta - 1\right)$$

$$U = -\frac{GM}{R}\left(1 - \frac{\left(I_3 - I_1\right)}{2MR^2}\left(3\sin^2\theta - 1\right)\right)$$

This is MacCullagh's formula for the potential of an an ellipsoid

Calculate the potential at a point P due to a nearly spherical body.

$$U = -\frac{GM}{R}\left(1 - \frac{(I_3 - I_1)}{2MR^2}\left(3\sin^2\theta - 1\right)\right)$$

the term $\left(3\sin^2\theta - 1\right)$ is the Legendre Polynomial

$$P_2(x) = 3x^2 - 1, \quad P_2(\cos\theta) = 3\cos^2\theta - 1$$

letting $J_2 = \dfrac{(I_3 - I_1)}{MR_e^{\ 2}} = 1.08263\ 10^{-3}$

J_2 has various names including "dynamic form factor" and " ellipticity coefficient"

$$U(R,\theta) = -\frac{GM}{R} + \frac{GMR_e^{\ 2}}{2R^3} J_2 P_2(\cos\theta) = -\frac{GM}{R}\left(1 - \frac{R_e^{\ 2}}{2R^2} J_2 P_2(\cos\theta)\right)$$

So the final result for the potential has two parts –

the result for the uniform sphere

plus a correction for the ellipse

Now we can find the force of gravity

$$U(R,\theta) = -\frac{GM}{R}\left(1 - \frac{R_e{}^2}{2R^2} J_2 P_2(\cos\theta)\right)$$

now to find $\vec{g}(r)$, we take the derivative

$$\vec{g}(r,\theta) = -\frac{\partial U(r,\theta)}{\partial r}\hat{r} = \left(\frac{GM}{r^2} - \frac{3GMR_e{}^2}{2r^4} J_2 P_2(\cos\theta)\right)\hat{r}$$

This is MacCullagh's formula for the gravity of an ellipsoid.

Differential form of Newton's law -

So far we've looked at the "integral" form for Newton's gravitational force law.

$$\vec{g}(\vec{x}) = -G \int_V \frac{\rho(\vec{x}')\, d\vec{x}'^3}{d^2}$$

But we also have

$$\vec{g}(\vec{x}) = \nabla U(\vec{x})$$

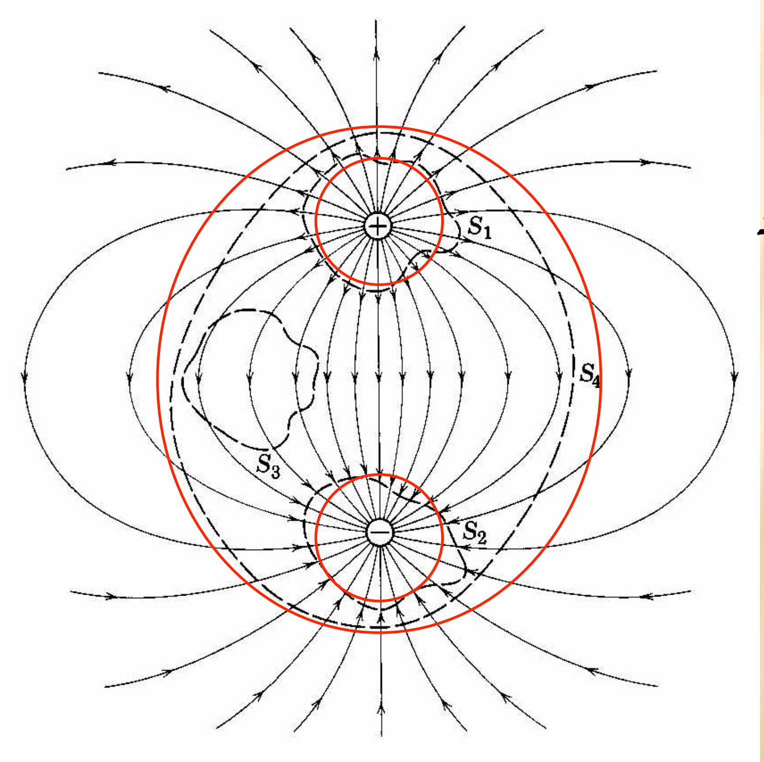
Which is a differential equation for the potential U.

Can we relate U to the density without the integral?

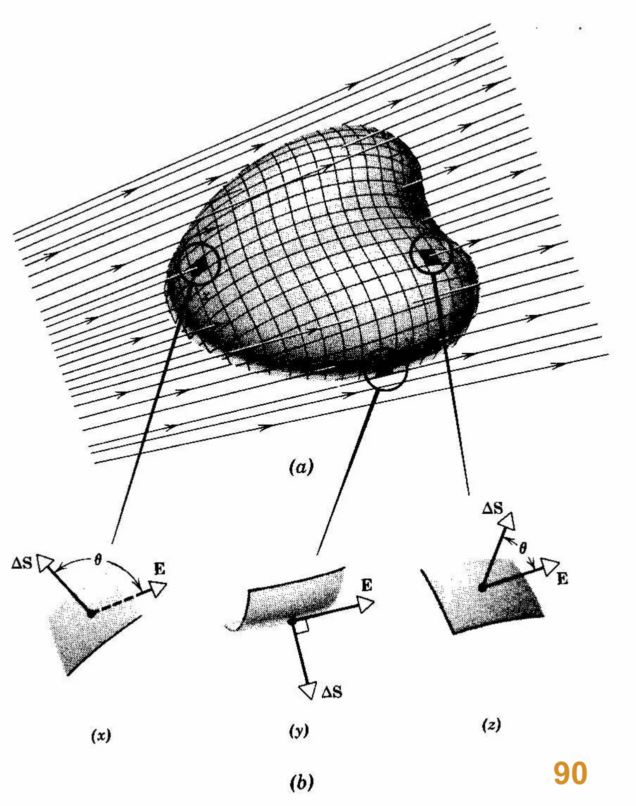
Poisson's and Laplace's equations

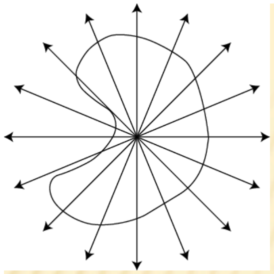
Start with Gauss's/Divergence theorem for vector fields

$$\int_S \vec{F} \bullet d\vec{a} = \int_V \nabla \bullet \vec{F} dV$$

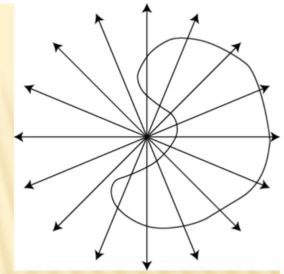


Which says the flux out of a volume equals the divergence throughout the volume.





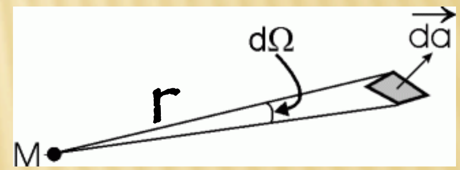
Examine field at point M.



Point M inside volume

Point M outside volume

Gauss's/Divergence Theorem: $\int_S \vec{g} \bullet d\vec{a} = \int_V \nabla \bullet \vec{g}\, dV$

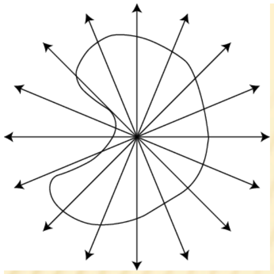


work on left hand side: $\vec{g} = -\dfrac{GM}{r^2}\hat{r}, \quad \dfrac{1}{r^2}\hat{r} \bullet d\vec{a} = d\Omega$

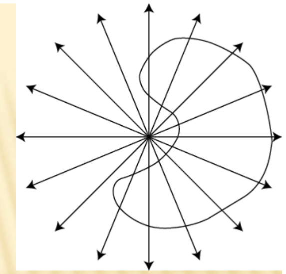
$$\int_S \vec{g} \bullet d\vec{a} = -\int_S GMd\Omega = -4\pi\, GM = -4\pi\, G \int_V \rho\, dV$$

$$-4\pi\, G \int_V \rho\, dV = \int_V \nabla \bullet \vec{g}\, dV$$

Ahern: http://geophysics.ou.edu/solid_earth/notes/laplace/laplace.html



Examine field at point M.



Point M inside volume

Point M outside volume

Gauss's/Divergence Theorem: $\int_S \vec{g} \bullet d\vec{a} = \int_V \nabla \bullet \vec{g}\, dV$

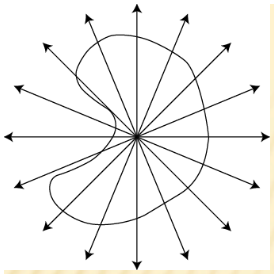
$-4\pi G \int_V \rho\, dV = \int_V \nabla \bullet \vec{g}\, dV$ since this holds for arbitrary volumes, the integrands of the two integrals have to be equal

$\nabla \bullet \vec{g} = -4\pi G \rho$ for M inside volume

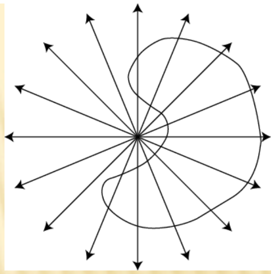
$\nabla \bullet \vec{g} = 0$ for M outside volume

(does not work ON surface where there is a density discontinuity)

Ahern: http://geophysics.ou.edu/solid_earth/notes/laplace/laplace.html



Examine field at point M.



Point M inside volume

Point M outside volume

$$\nabla \bullet \vec{g} = -4\pi G\rho \qquad \text{for M inside volume}$$

$$\nabla \bullet \vec{g} = 0 \qquad \text{for M outside volume}$$

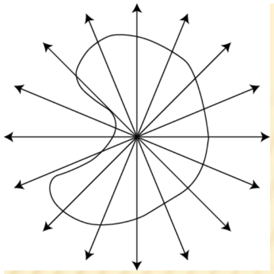
now use $\vec{g} = -\nabla U$

$$\boxed{\begin{aligned} \nabla^2 U &= -4\pi G\rho \\ \nabla^2 U &= 0 \end{aligned}}$$

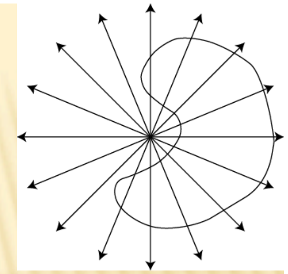
for M inside volume - Poisson's Eq.

for M outside volume - Laplace's Eq.

Ahern: http://geophysics.ou.edu/solid_earth/notes/laplace/laplace.html



Examine field at point M.



Point M inside volume

Point M outside volume

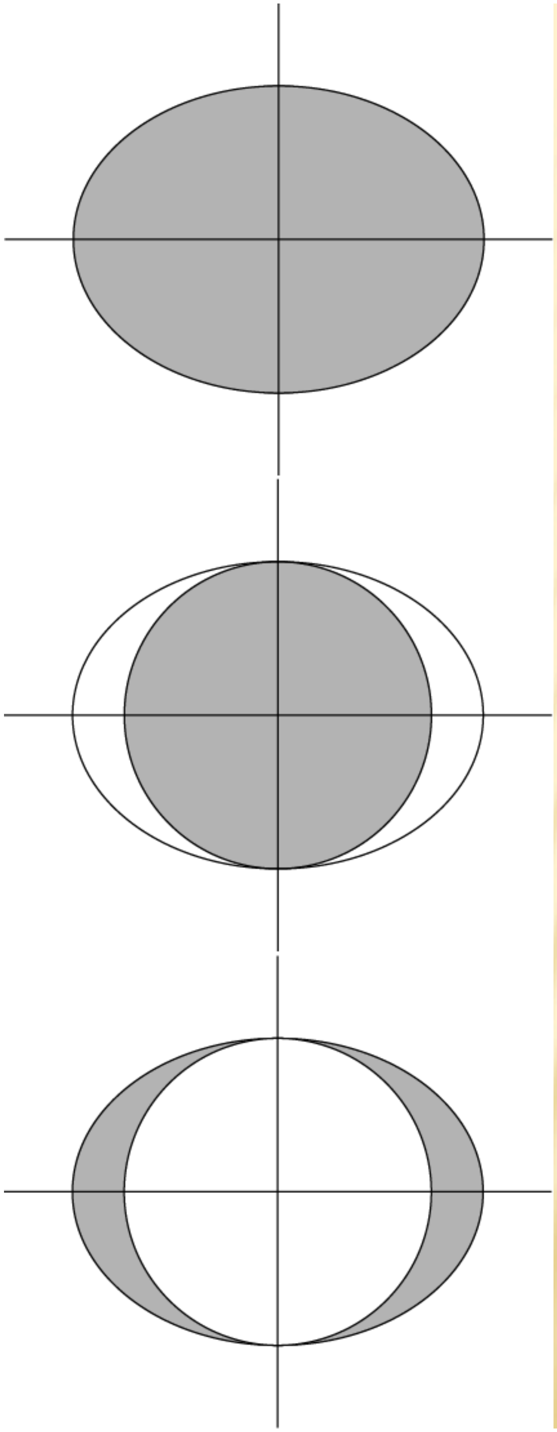
$$\boxed{\begin{aligned}\nabla^2 U &= -4\pi G\rho \\ \nabla^2 U &= 0\end{aligned}}$$

for M inside volume - Poisson's Eq.

for M outside volume - Laplace's Eq.

So the equation for the potential, a scalar field (easier to work with than a vector field) satisfies Poisson's equation (Lapalce's equation is a special case of Poisson's equation). Poisson's equation is <u>linear</u>, so we can <u>superimpose sol'ns – </u>¡importantisimo!

Ahern: http://geophysics.ou.edu/solid_earth/notes/laplace/laplace.html



In the spherical shell example we used the fact that gravity is

"linear"

i.e. we get final result by adding up partial results (this is what integration does!)

So ellipsoidal earth can be represented as a solid sphere plus a hollow elliposid.

Result for the gravity potential and force for an elliposid had two parts –

that for a sphere <u>plus</u> an additional term which is due to the mass in the ellipsoidal shell.

GRAVITY POTENTIAL

- All gravity fields satisfy Laplace's equation in free space or material of density ρ. If V is the gravitational potential then

$$\nabla^2 V = 0$$

$$\nabla^2 V = 4\pi G\rho$$

(Herring)

LINEAR	NON-LINEAR
Superposition: break big problems into pieces	No superposition: solve whole problem at once
Smooth, predictable motions	Erratic, aperiodic motion
Response proportional to stimulus	Response need not be proportional to stimulus
Find detailed trajectories of individual particles	Find global, qualitative description of all possible trajectories

Linearity and Superposition

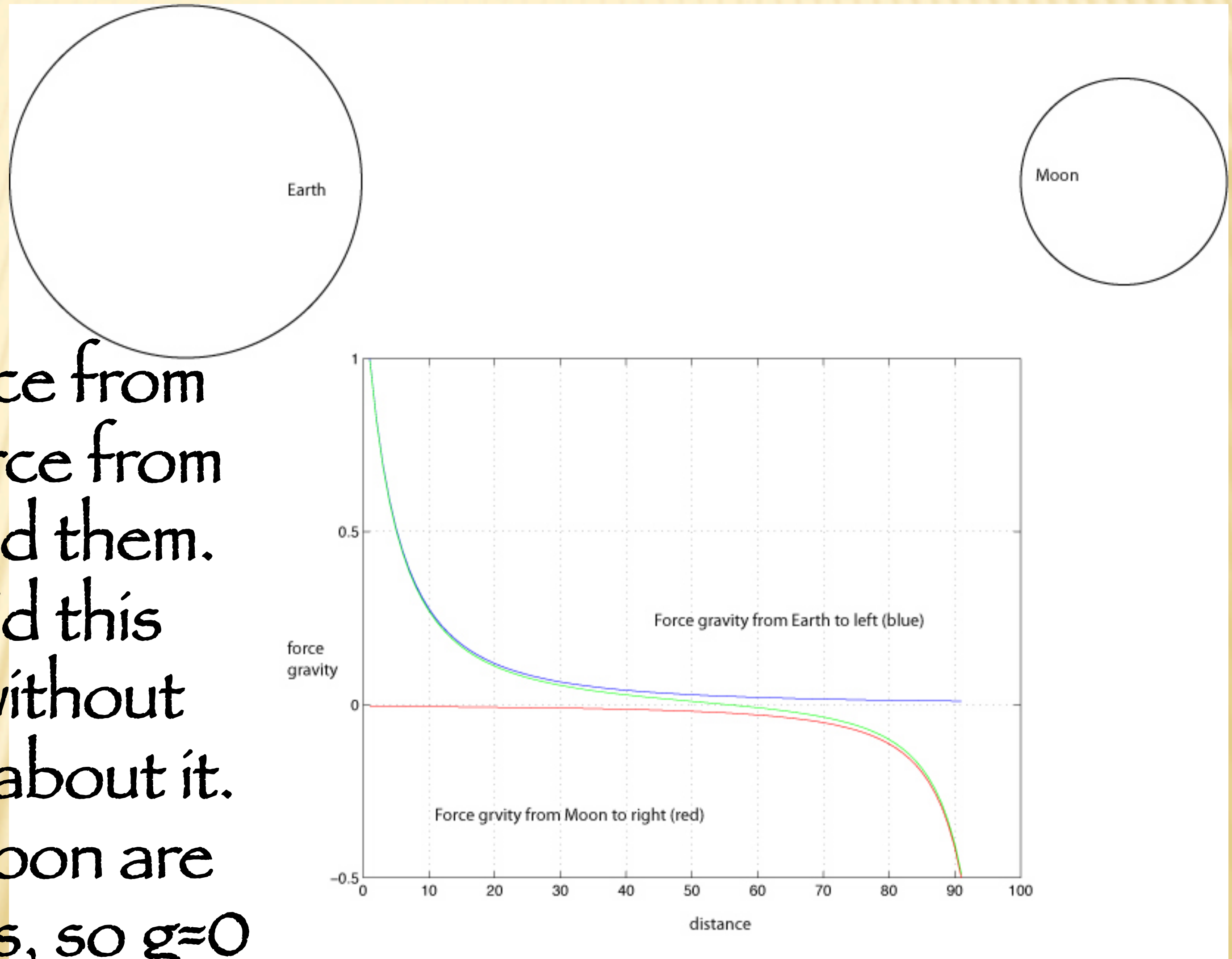
$$L(x) + L(y) = L(x + y)$$

Says order you do the “combination” does not matter.

Very important concept.

If system is linear you can break it down into little parts, solve separately and combine solutions of parts into solution for whole.

Net force of Gravity on line between Earth and Moon



Net force from superposition/linearity (green)

Solve for force from Earth and force from Moon and add them. Probably did this procedure without even thinking about it. (earth and moon are spherical shells, so g=0 inside)

Net force of Gravity for Earth with a Core

Solve for force from Earth and force from Core and add them.
Same procedure as before (and same justification) – but probably had to think about it here. (Earth and core are again spherical shells so g=0 inside)