Hilbert Transform

Sometimes also called the quadrature function.

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

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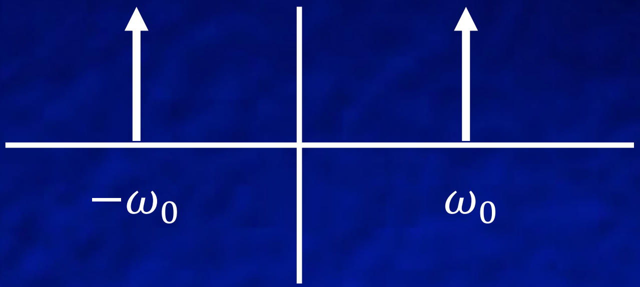
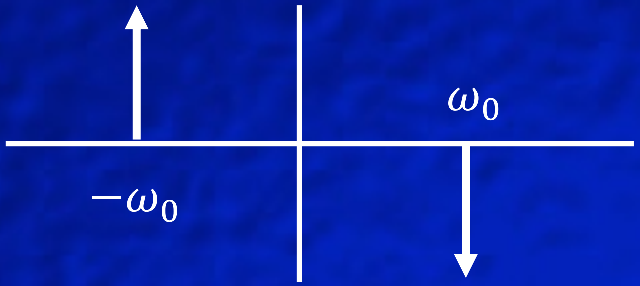
→ A $90°$ phase shift regardless of the original.

Recall

$$F[\cos(\omega_0 t)] = \frac{1}{2}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



$$F[\sin(\omega_0 t)] = \frac{1}{2}i[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$



We can express this operator $H[\,]$ as a convolution with its transfer function,

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→ $H(\omega) = -isgn(\omega)$

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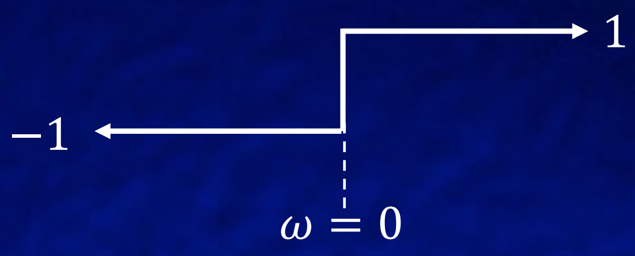
Signum function, $\mathrm{sgn}(\omega)$



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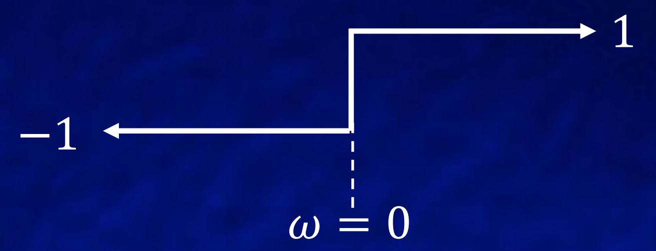
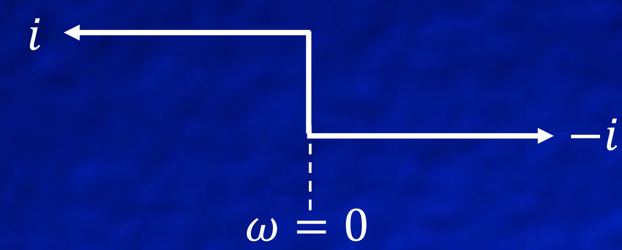


Frequency domain transfer function for our 90° phase shifter, $H(\omega) = -i sgn(\omega)$



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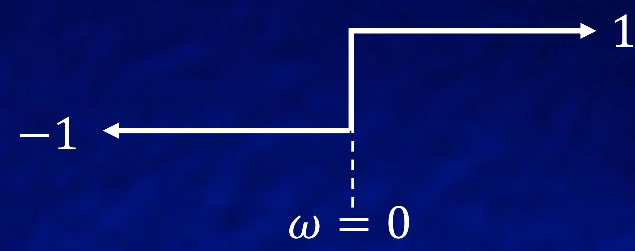

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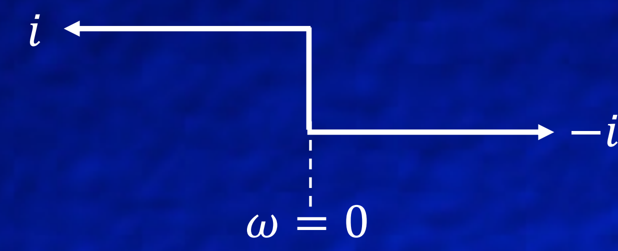

$$h(t) = F^{-1}[H(\omega)] = F^{-1}[-isgn(\omega)] = -iF^{-1}[\mathrm{sgn}(\omega)]$$

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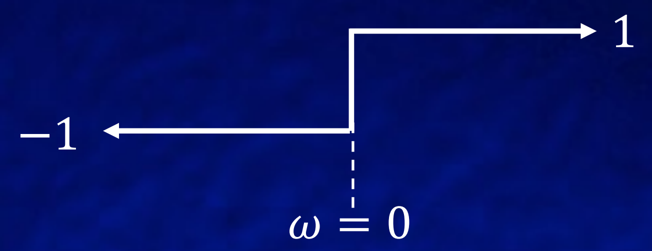

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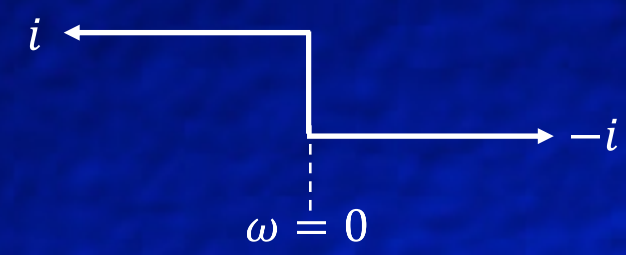

$$h(t) = F^{-1}[H(\omega)] = F^{-1}[-isgn(\omega)] = -iF^{-1}[\mathrm{sgn}(\omega)] = \frac{1}{\pi t}$$

From tables.

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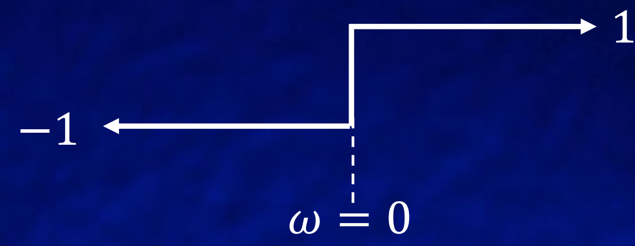

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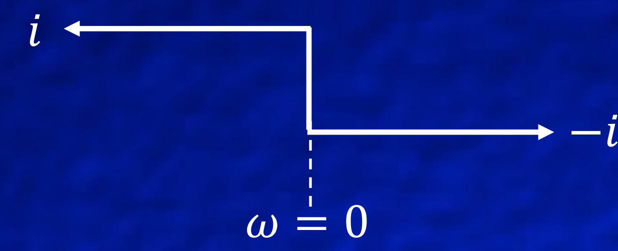

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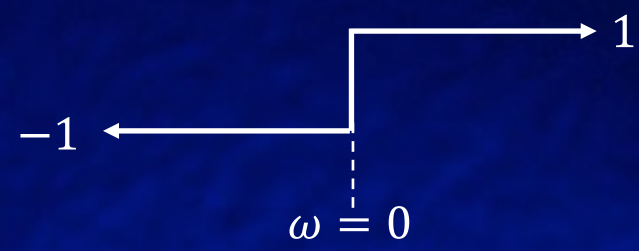

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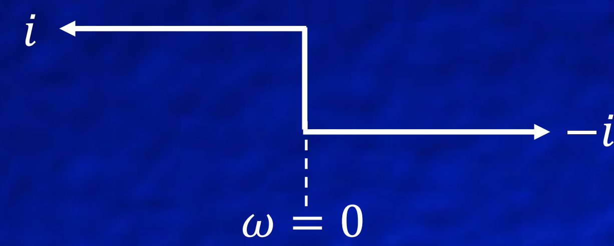

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$$h(t) * f(t) = \int_{-\infty}^{\infty} \frac{f(t-\tau)}{\pi\tau} d\tau = \int_{-\infty}^{\infty} \frac{f(\tau)}{\pi(t-\tau)} d\tau \equiv$$ Hilbert transform.

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Or one could use tables or the matlab *hilbert* function.

The matlab *hilbert* command returns the analytic signal rather than the Hilbert transform.

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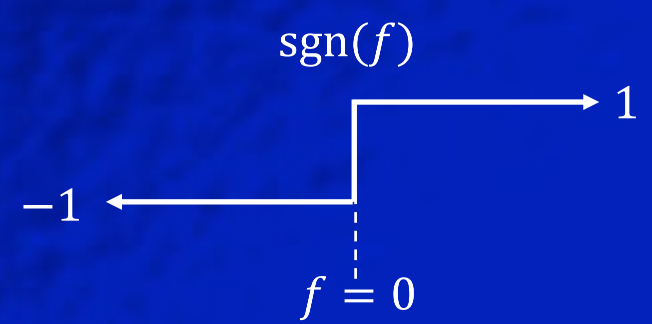

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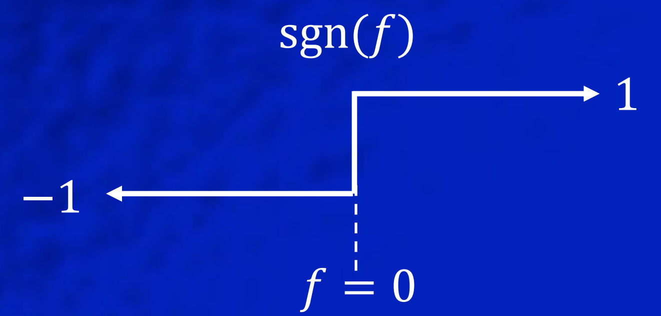

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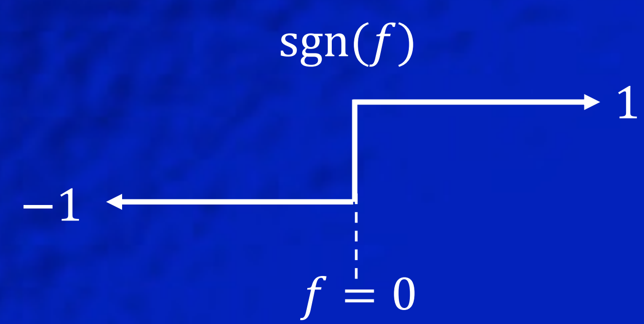

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$$= \Phi(f) - i[isgn(f)\Phi(f)]$$

$$= \Phi(f) + \mathrm{sgn}(f)\,\Phi(f)$$

$$= \begin{cases} 0, & f < 0 \\ 2\Phi(f), & f > 0 \end{cases}$$

Recall $H[\phi(t)] = h(t) * \phi(t)$

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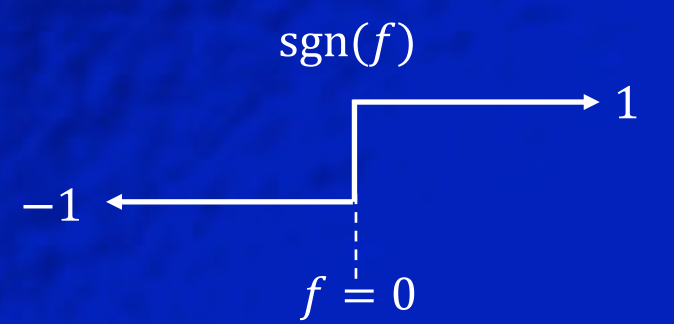

Then for a real valued signal, the analytic function is the inverse fourier transform of the one-sided spectrum of the real-valued signal.

That is, given $\phi(t)$ and it's one sided spectrum, $\begin{cases} 0, & f < 0 \\ 2\Phi(f), & f > 0 \end{cases}$

The inverse FT of that one sided spectrum is the analytic signal. And the real part of the analytic signal is the original real-valued function.

run matlab program hilber_examp.m

$a(t) = \phi(t) - iH[\phi(t)] = |a(t)|e^{i\theta(t)}$ $|a(t)|$ is the envelope function of $\phi(t)$

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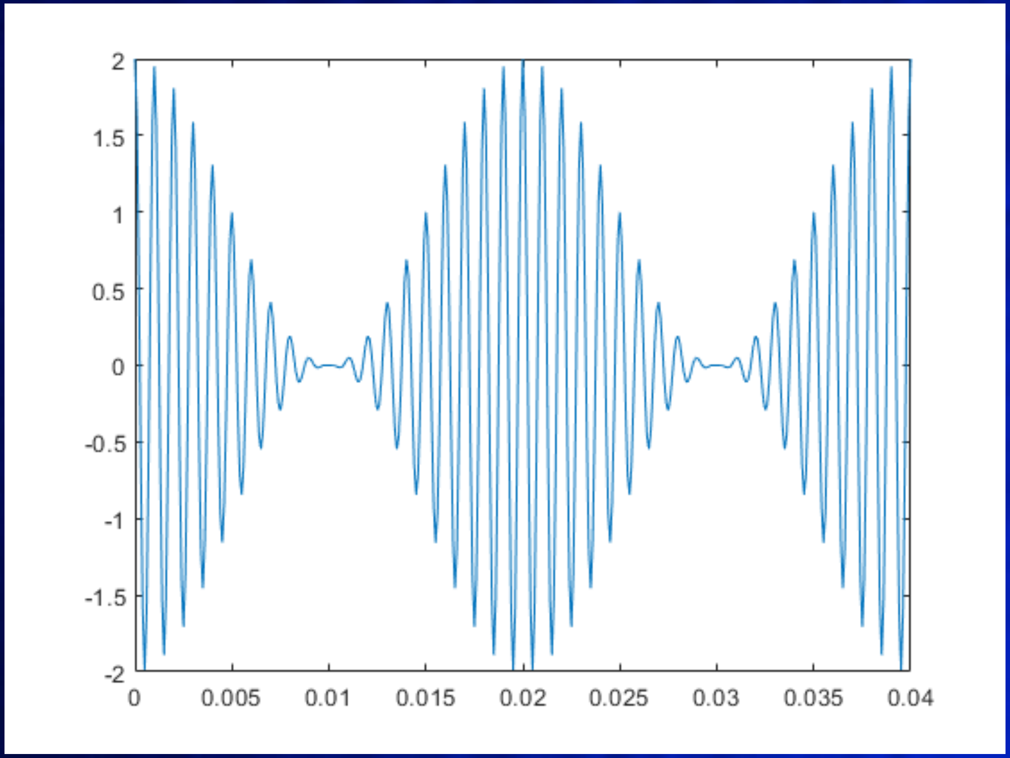
https://www.mathworks.com/help/signal/ug/envelope-extraction-using-the-analytic-signal.html (last accessed July 27, 2020)

```
t = 0:1e-4:0.1;
x = (1+cos(2*pi*50*t)).*cos(2*pi*1000*t);
plot(t,x) xlim([0 0.04])
```



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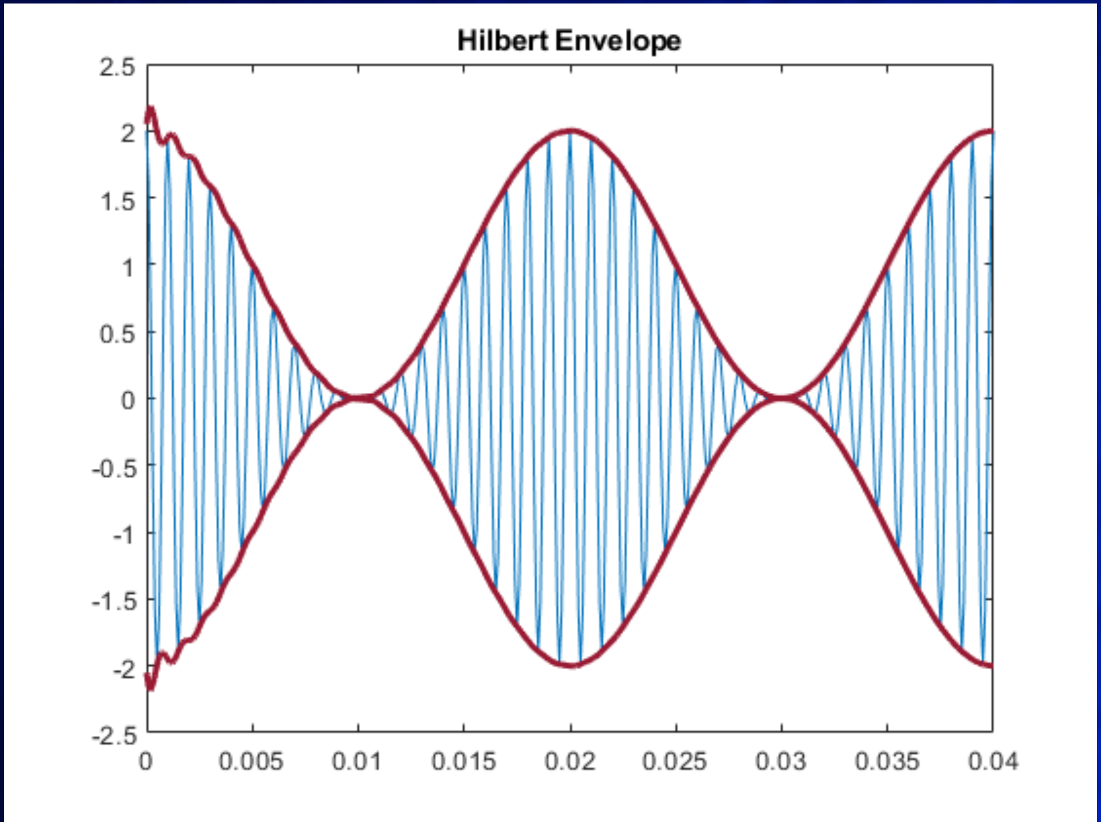
```
y = hilbert(x);
env = abs(y);
plot_param = {'Color', [0.6 0.1 0.2],'Linewidth',2};
plot(t,x)
hold on
plot(t,[-1;1]*env,plot_param{:})
hold off
xlim([0 0.04])
title('Hilbert Envelope')
```

← Returns the analytic signal not the Hilbert transform.



The magnitude of the analytic signal captures the slowly varying features of the signal, while the phase contains information about the higher frequencies.

$$a(t) = \phi(t) - iH[\phi(t)] = |a(t)|e^{i\theta(t)}$$

$$\theta(t) = \tan^{-1}\left[\frac{H[\phi(t)]}{\phi(t)}\right]$$

Instantaneous phase

$$w(t) = \frac{d\theta}{dt}$$

Instantaneous frequency

The instantaneous frequency tells us how the frequency of a signal varies with time. This is more useful for tracking quasi-monochromatic signals, or monochromatic changes in time, e.g. FM signals superimposed on a carrier wave or chirp functions used in reflection seismology.

You will see the analytic function in several places including developing a causal transfer function for dispersive media and in exploration seismology.