The Hankel Transform

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$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(xu+yv)}dxdy$$ and,

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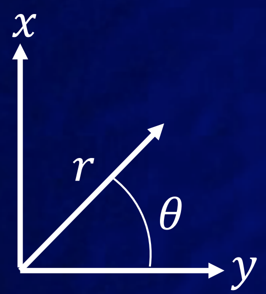

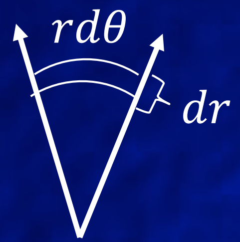

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$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

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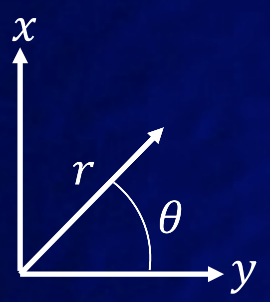

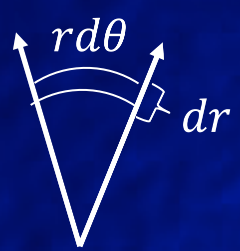

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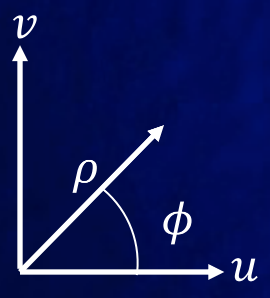

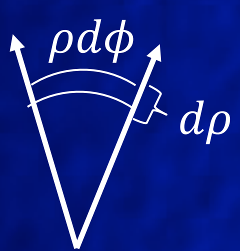

$$\rho^2 = u^2 + v^2 \qquad dudv = \rho d\rho d\phi$$

$$u = \rho\cos\phi \qquad v = \rho\sin\phi$$

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$f(r,\theta) = f(r)$ and from $xu + yv$,

$$g(\phi,\theta) = r\cos(\theta)\rho\cos(\phi) + r\sin(\theta)\rho\sin(\phi)$$

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This is a job for Bessel functions.

$J_n(x)$ is a Bessel function of the first kind of order n. They are sometimes also called cylinder functions or cylindrical harmonics.

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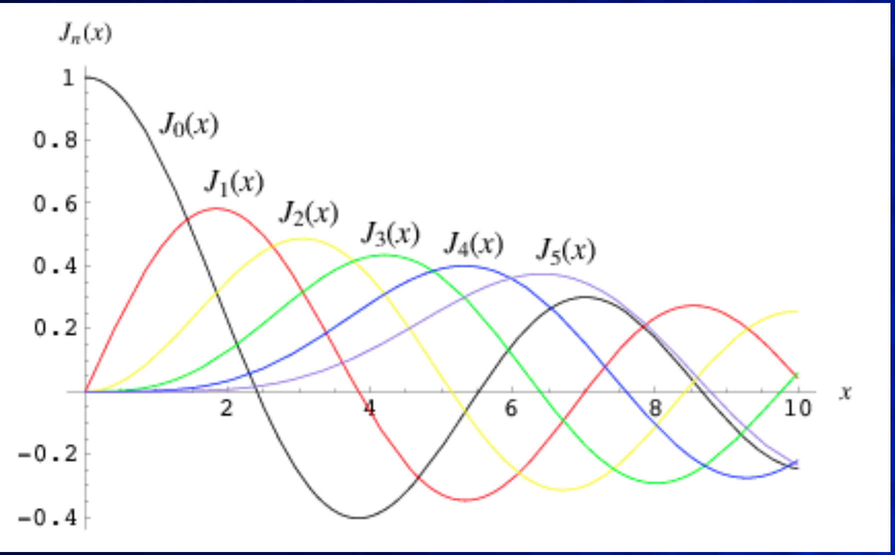

Figure source: https://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html Last accessed July 16, 2020

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So that, $$F(\rho) = 2\pi \int_0^{\infty} r f(r) J_0(2\pi r\rho) dr$$

Similarly, $$f(r) = 2\pi \int_0^{\infty} \rho F(\rho) J_0(2\pi r\rho) d\rho$$

Hankel Transform Pair

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$$F(\rho) = \frac{1}{2\pi\rho^2} \int_0^{2\pi\rho} r' J_0(r') dr'$$

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$$= \frac{1}{2\pi\rho^2} [2\pi\rho J_1(2\pi\rho)] = \frac{J_1(2\pi\rho)}{\rho}$$

A Jinc function. The 2-d polar coordinate analog of the sinc function.

```
x=2*pi*(0.01:0.01:10);
y=besselj(1,x)./x;
plot(x,y)
```