IIR Filters

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 5

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$$y_n = \frac{\sum_{m=0}^{M} b_m x_{n-m} - \sum_{k=1}^{K} a_k y_{n-k}}{a_0}$$

The filter coefficients are a and b, the input is x so we find the Z transform of both sides to determine the transfer function of the linear system.

$$\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{K} a_k y_{n-k}\right) z^{-n} = \sum_{n=-\infty}^{\infty}\left(\sum_{m=0}^{M} b_m x_{n-m}\right) z^{-n}$$

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$$Y(z)\sum_{k=0}^{K} a_k z^{-k} = X(z)\sum_{m=0}^{M} b_m z^{-m} \qquad \Phi(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{K} a_k z^{-k}}$$

We require $\lim_{n\to\infty} \phi_n = 0$ for the filter to be stable.

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If we had the weights, a and b for our recursive filter along with the input x, we could use the matlab *filter* command to get the output, y.

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Recall from the Laplace transform that if the poles lie in the left half of the s-plane, then the filter will be stable. Likewise, transfer functions with poles in the z-plane that lie within the unit circle will be stable (decay with time).

$z = e^s = e^{\sigma + i\omega}$ $\sigma < 0$ is the left side of the s-plane, and $|z| < 1$.

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There are numerous ways to design IIR filters and we will explore two. As we did with FIR filters we'll start with an ideal filter response then attempt to match that continuous response to a discrete finite length approximation.

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Impulse Invariance Method of IIR filter design.

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We can find the more familiar frequency response by setting $s = 2\pi i f$

$\Phi(f) = \frac{1}{1 + i2\pi\tau f}$ $\longrightarrow$ 1@f=0 and <1 for f>0

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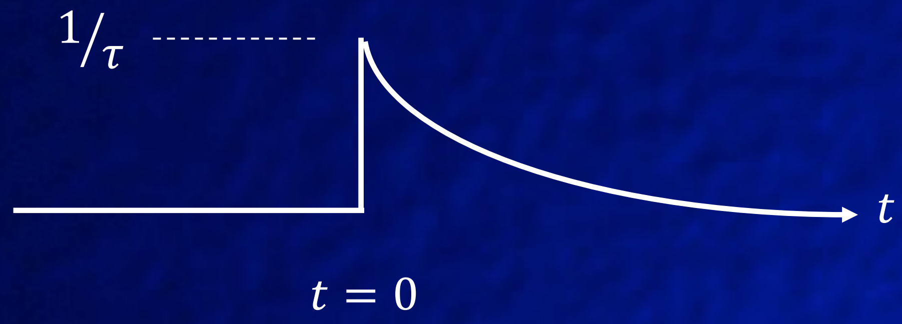

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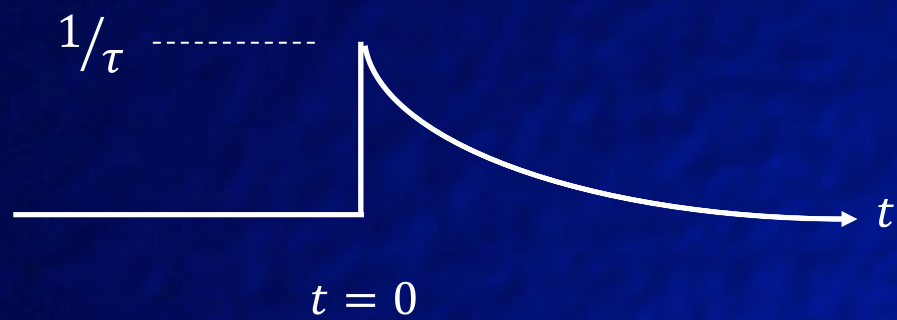

Here τ is the characteristic decay time. $\phi(t)$ has non-zero output for all $t > 0$ which means the length of the time domain response is infinite. Thus it's not easy to model with a FIR filter so instead we use an IIR filter. This is true of the recursive class of filters of which ours is a simple example.

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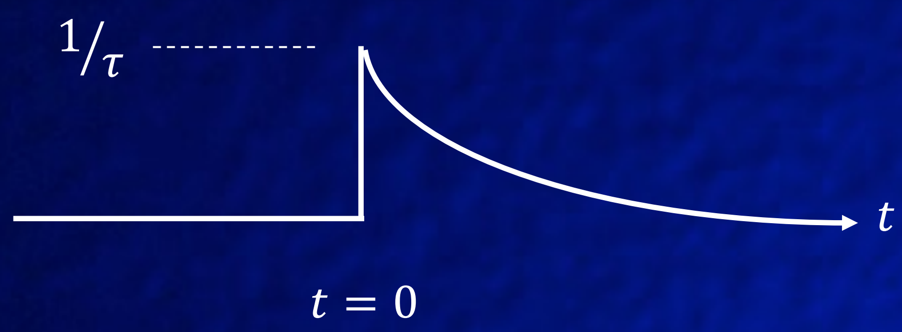

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In the impulse invariance method of IIR filter design we select the discrete recursive filter with impulse response that best matches the desired continuous response.

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This decays exponentially like we want, in order to model $\phi(t)$.

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We also want $\phi(0) = \frac{1}{\tau} = y_0$.

Set $\frac{1}{\tau} = 1 - \alpha \rightarrow \alpha = 1 - \frac{1}{\tau}$

See A&B figure 5.12

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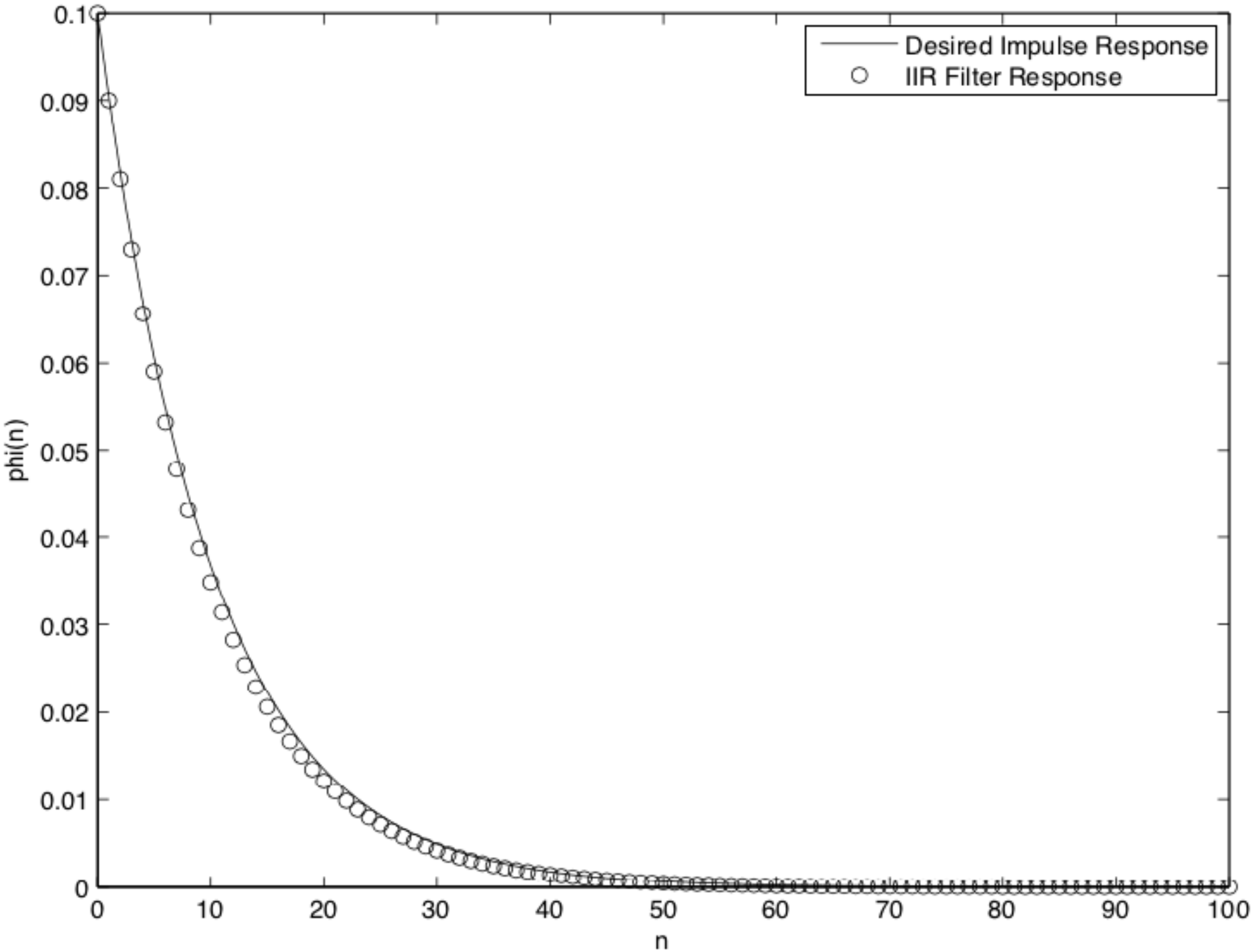

Figure 5.12: Impulse invariance discrete realization compared to a target continuous response in the time domain.

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τ is the characteristic decay time.

See A&B figure 5.13

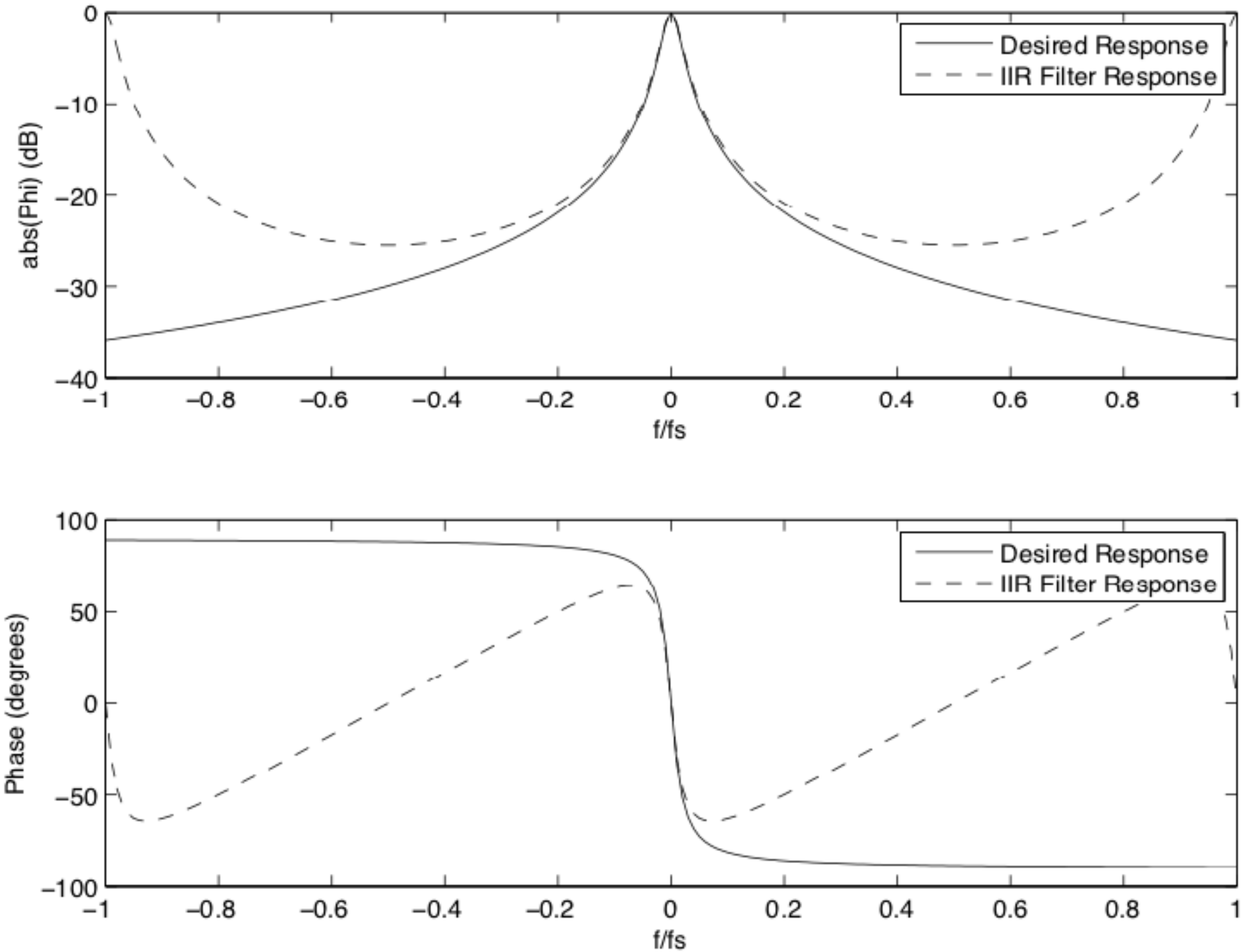

Figure 5.13: Impulse invariance discrete realization compared to a target continuous response in the frequency domain.

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Not a particularly formulaic design method.

Let’s try the Bilinear Transform.

The bilinear transform ”pre-warps” the s-domain into the z-domain (i.e. from infinite frequency to periodic) using the tangent.

Our low pass filter from before, $\tau\dot{y} + y = x$ $\quad \Phi(s) = \dfrac{1}{1+\tau s}$ $\quad \phi(t) = \dfrac{1}{\tau} H(t) e^{-t/\tau}$

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So that,

$$\Phi(z) = \frac{1}{1+\tau \frac{2}{\Delta} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{1+\left(\frac{2\tau}{\Delta}\right) i \tan\left(\frac{\pi f}{f_s}\right)}$$

See A&B Figures 5.14 and 5.15

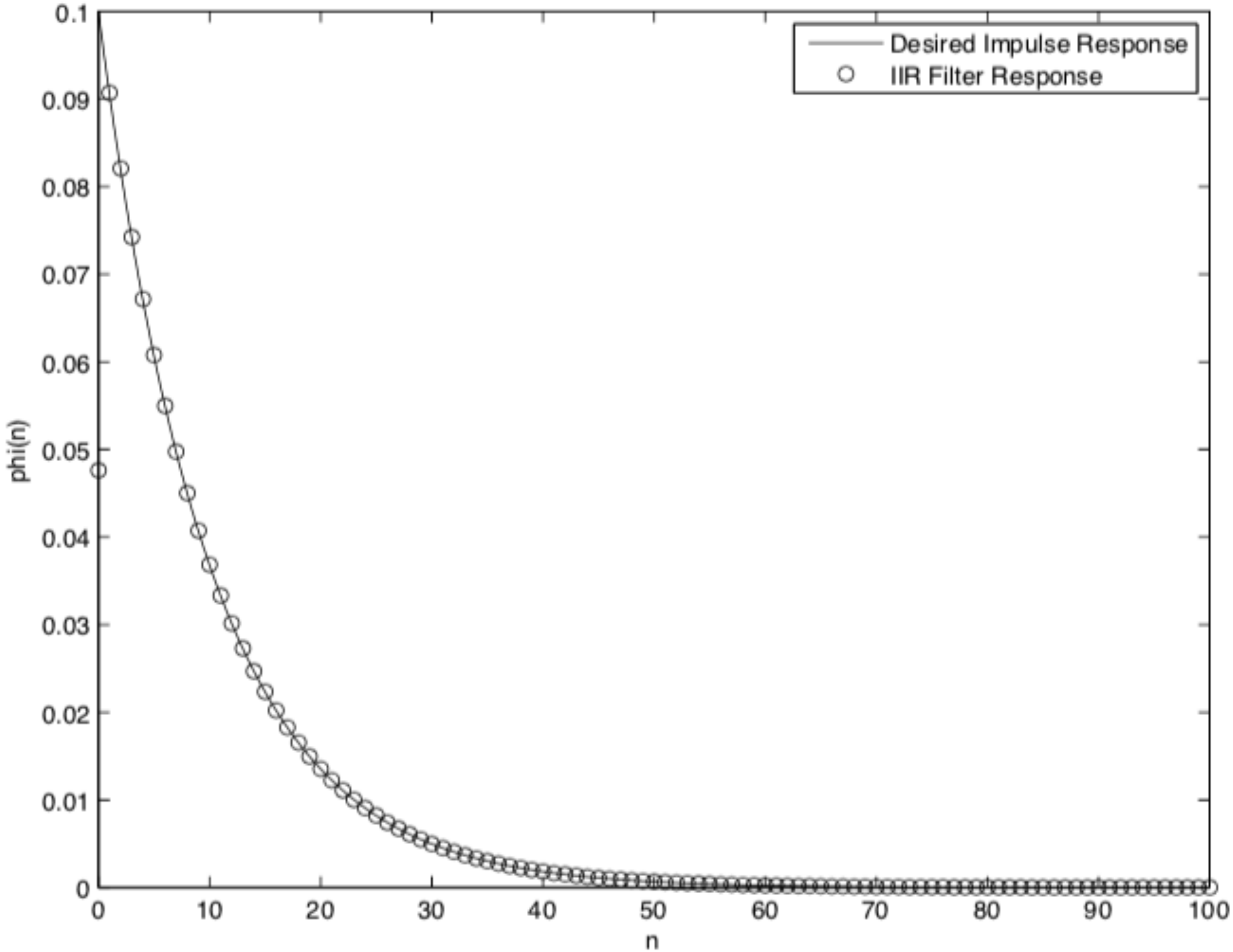

Figure 5.14: Bilinear z transform discrete realization response compared to a target continuous response in the time domain.

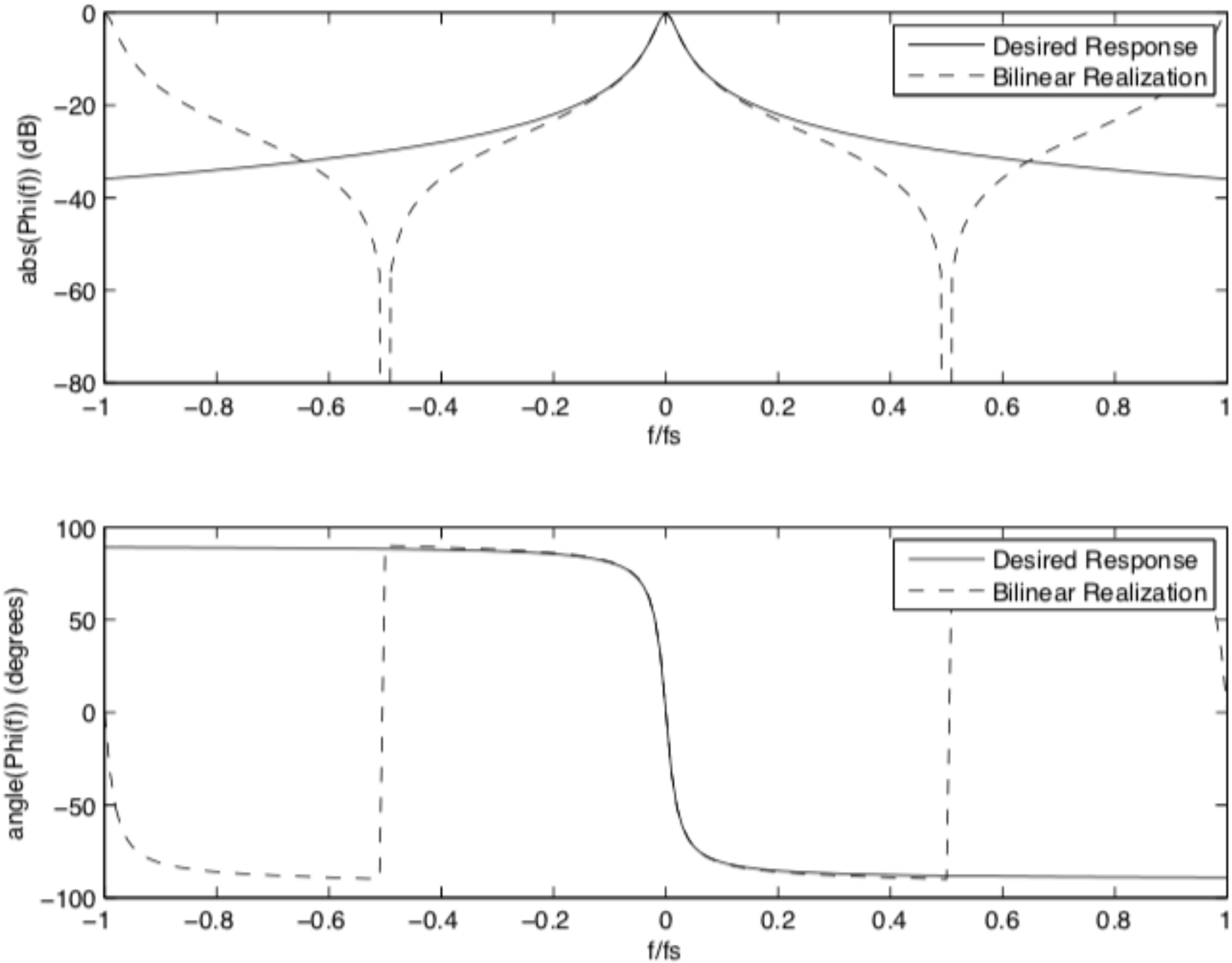

Figure 5.15: Bilinear z transform discrete realization response compared to a target continuous response in the frequency domain.

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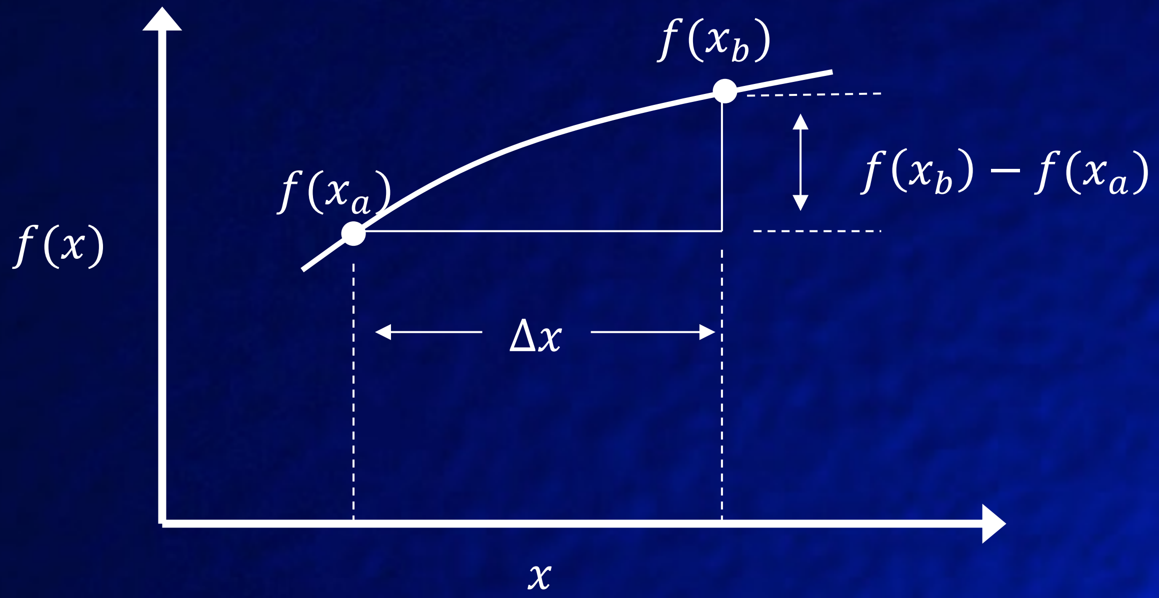
So that the n^{th} y is

$$y(n\Delta) = \int_{(n-1)\Delta}^{n\Delta} \dot{y}(u)du + y[(n-1)\Delta]$$

The trapezoid rule



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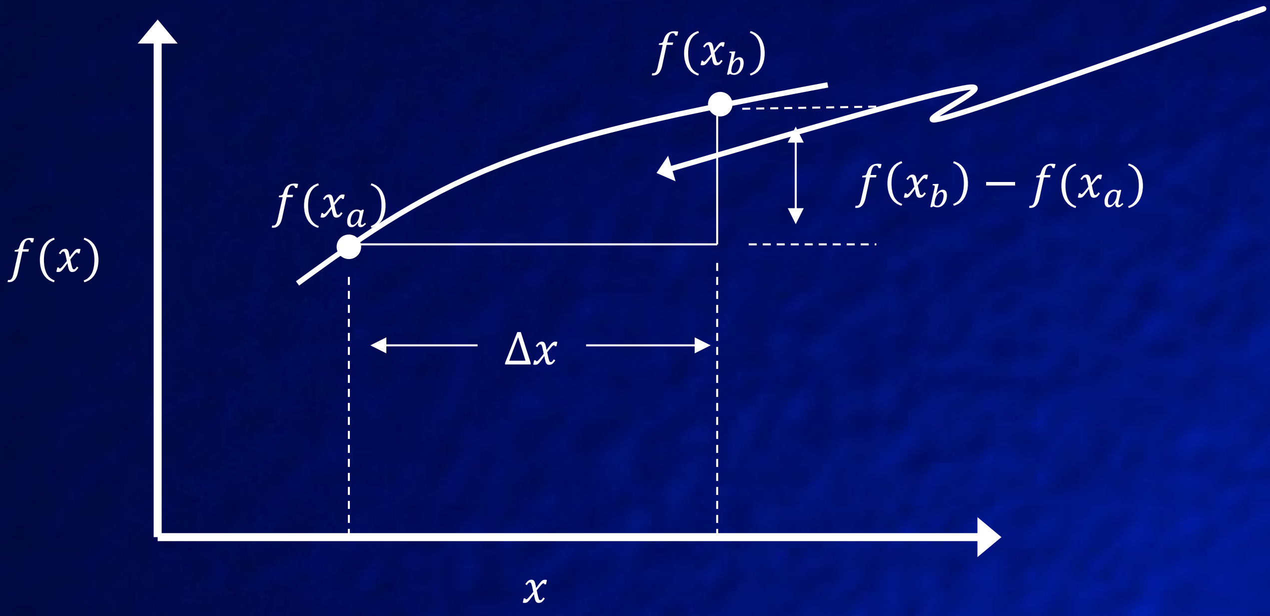
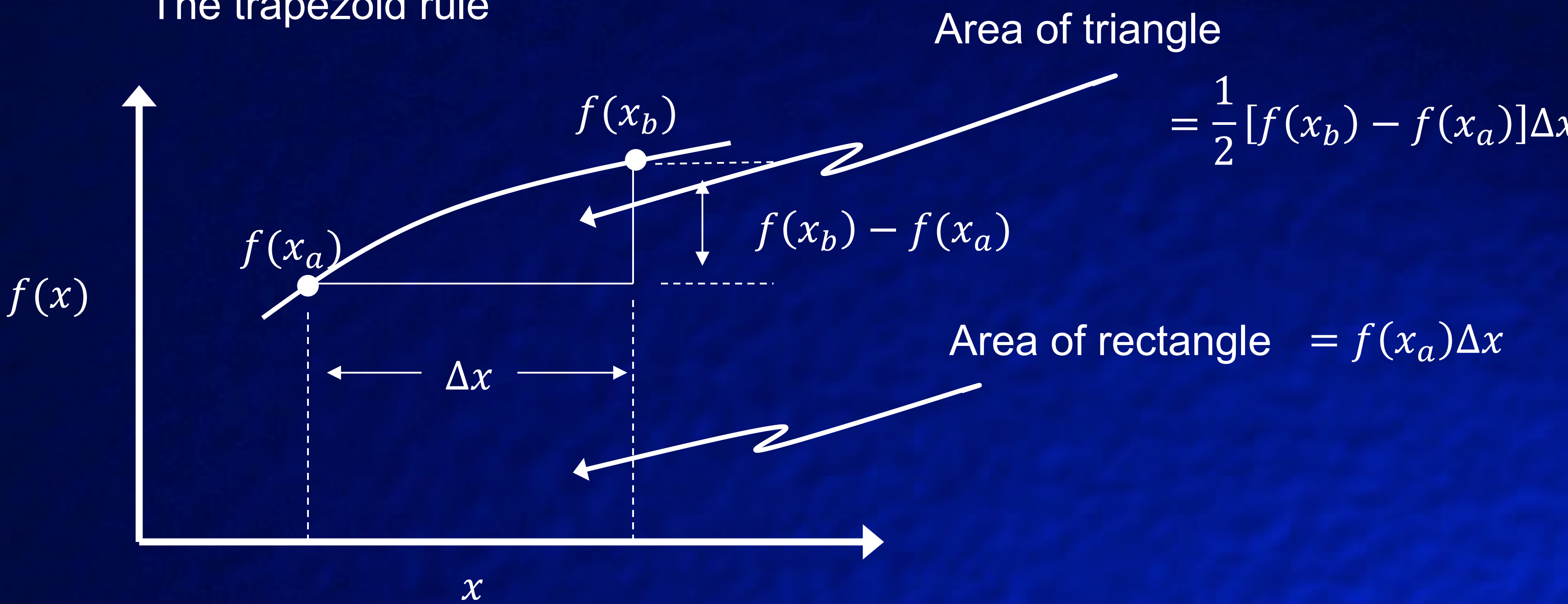


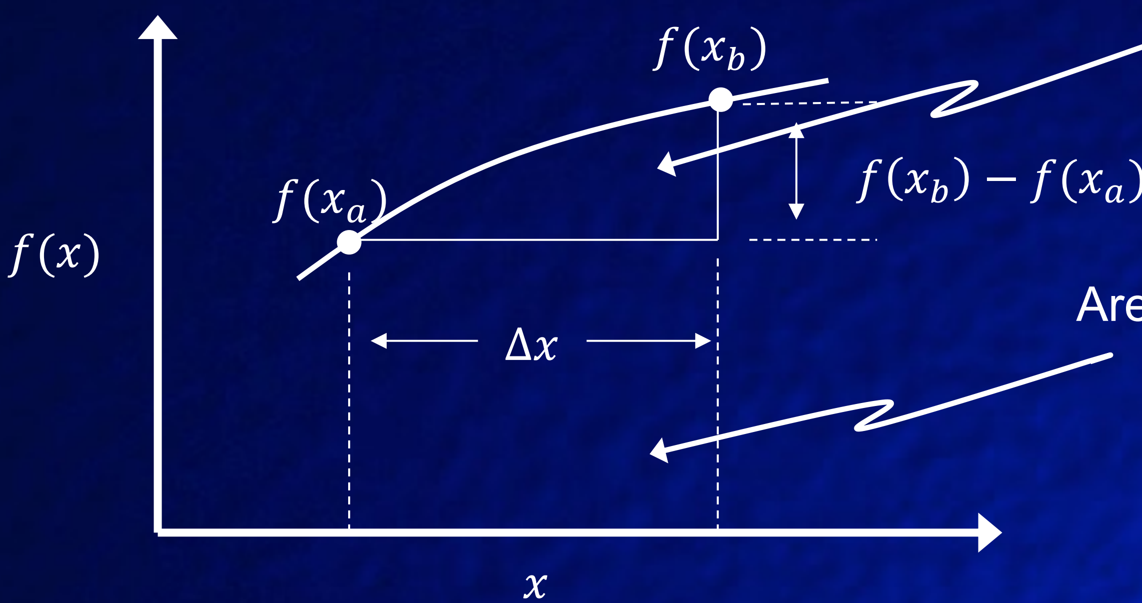
Area of triangle

$$= \frac{1}{2}[f(x_b) - f(x_a)]\Delta x$$

The trapezoid rule



The trapezoid rule



Area of triangle $= \frac{1}{2}[f(x_b) - f(x_a)]\Delta x$

Area of rectangle $= f(x_a)\Delta x$

Total area $= \frac{1}{2}[f(x_b) - f(x_a)]\Delta x + f(x_a)\Delta x \quad = \frac{\Delta x}{2}[f(x_a) + f(x_b)]$

The trapezoid rule

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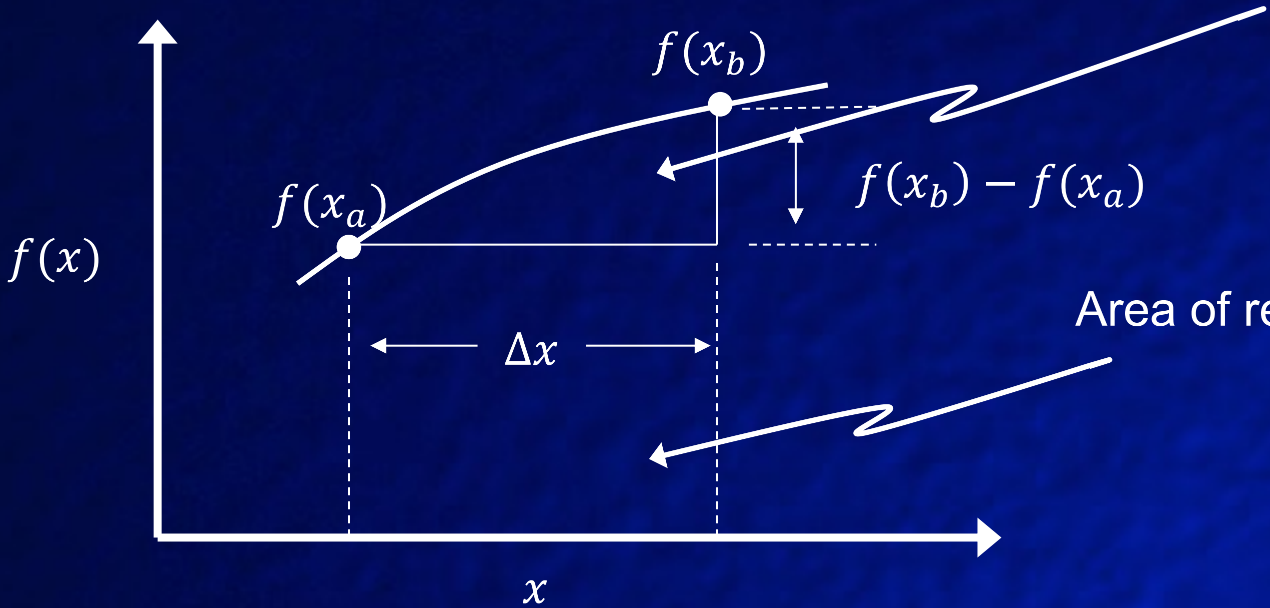

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Total area $= \frac{1}{2}[f(x_b) - f(x_a)]\Delta x + f(x_a)\Delta x \; = \frac{\Delta x}{2}[f(x_a) + f(x_b)]$

So that

$$y(n\Delta) \cong \int_{(n-1)\Delta}^{n\Delta} \dot{y}(u)du + y[(n-1)\Delta] = \frac{\Delta}{2}\{\dot{y}[\Delta(n-1) + \dot{y}(\Delta n)] + y[\Delta(n-1)]\}$$

Our original differential equation is

$$\dot{y}_n = \frac{1}{\tau}(x_n - y_n)$$

$$\dot{y}_{n-1} = \frac{1}{\tau}(x_{n-1} - y_{n-1})$$

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Rewriting the trapezoid rule,

$$\frac{\Delta}{2}\{\dot{y}[\Delta(n-1) + \dot{y}(\Delta n)] + y[\Delta(n-1)]\} \longrightarrow y_n \cong \frac{\Delta}{2}(\dot{y}_{n-1} + \dot{y}_n) + y_{n-1}$$

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Substituting $\quad y_n \cong \frac{\Delta}{2}\left[\frac{1}{\tau}(x_n + x_{n-1} - y_n - y_{n-1})\right] + y_{n-1}$

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Rewriting the trapezoid rule,

$$\frac{\Delta}{2}\{\dot{y}[\Delta(n-1) + \dot{y}(\Delta n)] + y[\Delta(n-1)]\} \Rightarrow y_n \cong \frac{\Delta}{2}(\dot{y}_{n-1} + \dot{y}_n) + y_{n-1}$$

Substituting $y_n \cong \frac{\Delta}{2}\left[\frac{1}{\tau}(x_n + x_{n-1} - y_n - y_{n-1})\right] + y_{n-1}$

$$= \frac{\Delta}{2\tau}x_n + \frac{\Delta}{2\tau}x_{n-1} - \frac{\Delta}{2\tau}y_n - \frac{\Delta}{2\tau}y_{n-1} + y_{n-1}$$

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From previous slide

$$y_n \cong \frac{\Delta}{2\tau} x_n + \frac{\Delta}{2\tau} x_{n-1} - \frac{\Delta}{2\tau} y_n - \frac{\Delta}{2\tau} y_{n-1} + y_{n-1}$$

Collect terms

$$y_n + \frac{\Delta}{2\tau} y_n + \frac{\Delta}{2\tau} y_{n-1} - y_{n-1} = \frac{\Delta}{2\tau}(x_n + x_{n-1})$$

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Clean up

$$y_n\left(1 + \frac{\Delta}{2\tau}\right) - y_{n-1}\left(1 - \frac{\Delta}{2\tau}\right) = \frac{\Delta}{2\tau}(x_n + x_{n-1})$$

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Now find the z transform of both sides

$$\left(1 + \frac{\Delta}{2\tau}\right)\sum_{n=-\infty}^{\infty} y_n z^{-n} - \left(1 - \frac{\Delta}{2\tau}\right)\sum_{n=-\infty}^{\infty} y_{n-1} z^{-n} = \frac{\Delta}{2\tau}\left(\sum_{n=-\infty}^{\infty} x_n z^{-n} + \sum_{n=-\infty}^{\infty} x_{n-1} z^{-n}\right)$$

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$$\left(1 + \frac{\Delta}{2\tau}\right)Y(z) - \left(1 - \frac{\Delta}{2\tau}\right)z^{-1}Y(z) = \frac{\Delta}{2\tau}X(z)(1 + z^{-1})$$

From previous slide, $\left(1+\frac{\Delta}{2\tau}\right)Y(z)-\left(1-\frac{\Delta}{2\tau}\right)z^{-1}Y(z)=\frac{\Delta}{2\tau}X(z)(1+z^{-1})$

$$\frac{Y(z)}{X(z)}=\frac{\frac{\Delta}{2\tau}(1+z^{-1})}{\left(1+\frac{\Delta}{2\tau}\right)-\left(1-\frac{\Delta}{2\tau}\right)z^{-1}}$$

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$$\Phi(z)=\frac{1}{1+\frac{2\tau}{\Delta}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$ → The discrete approximation of $\frac{1}{1+\tau s}$

Where, we used the bilinear transform method of replacing, $s=\frac{2}{\Delta}\cdot\frac{1-z^{-1}}{1+z^{-1}}$

$$\Phi(z) = \frac{1}{1 + \frac{2\tau}{\Delta}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

We can approximate the frequency response in f by substituting

$$z = e^{i2\pi f/f_s}$$

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Recall,

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$$

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Let $\theta = {}^{\pi f}/_{f_s}$

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$$i\tan(\pi f/f_s) = \frac{1 - e^{-i2\pi f/f_s}}{1 + e^{-i2\pi f/f_s}}$$

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$$i\tan(\pi f/f_s) = \frac{1 - e^{-i2\pi f/f_s}}{1 + e^{-i2\pi f/f_s}}$$

So that,

$$\Phi\left(z = e^{i2\pi f/f_s}\right) = \frac{1}{1 + \frac{2\tau}{\Delta} i\tan\left({}^{\pi f}/_{f_s}\right)}$$

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$$\Phi\left(z = e^{i2\pi f/f_s}\right) = \frac{1}{1 + \frac{2\tau}{\Delta} i \tan\left({}^{\pi f}/_{f_s}\right)}$$

This is the discrete approximation of the continuous system,

$$\Phi(s) = \frac{1}{1 + \tau s} \qquad \text{Where} \qquad s \cong \frac{2i}{\Delta} \tan\left({}^{\pi f}/_{f_s}\right)$$

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This maps (or warps) the continuous frequency response into $\left(-\frac{f_s}{2}, \frac{f_s}{2}\right)$ so that for some continuous $\Phi_c(s)$ we can obtain the discrete version $\Phi_d(z)$ by setting,

$$s = \frac{2}{\Delta} \frac{1 - z^{-1}}{1 + z^{-1}}$$

Where Δ is the sample interval (100 sps has 0.01s interval).

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See A&B Figures 5.14 and 5.15

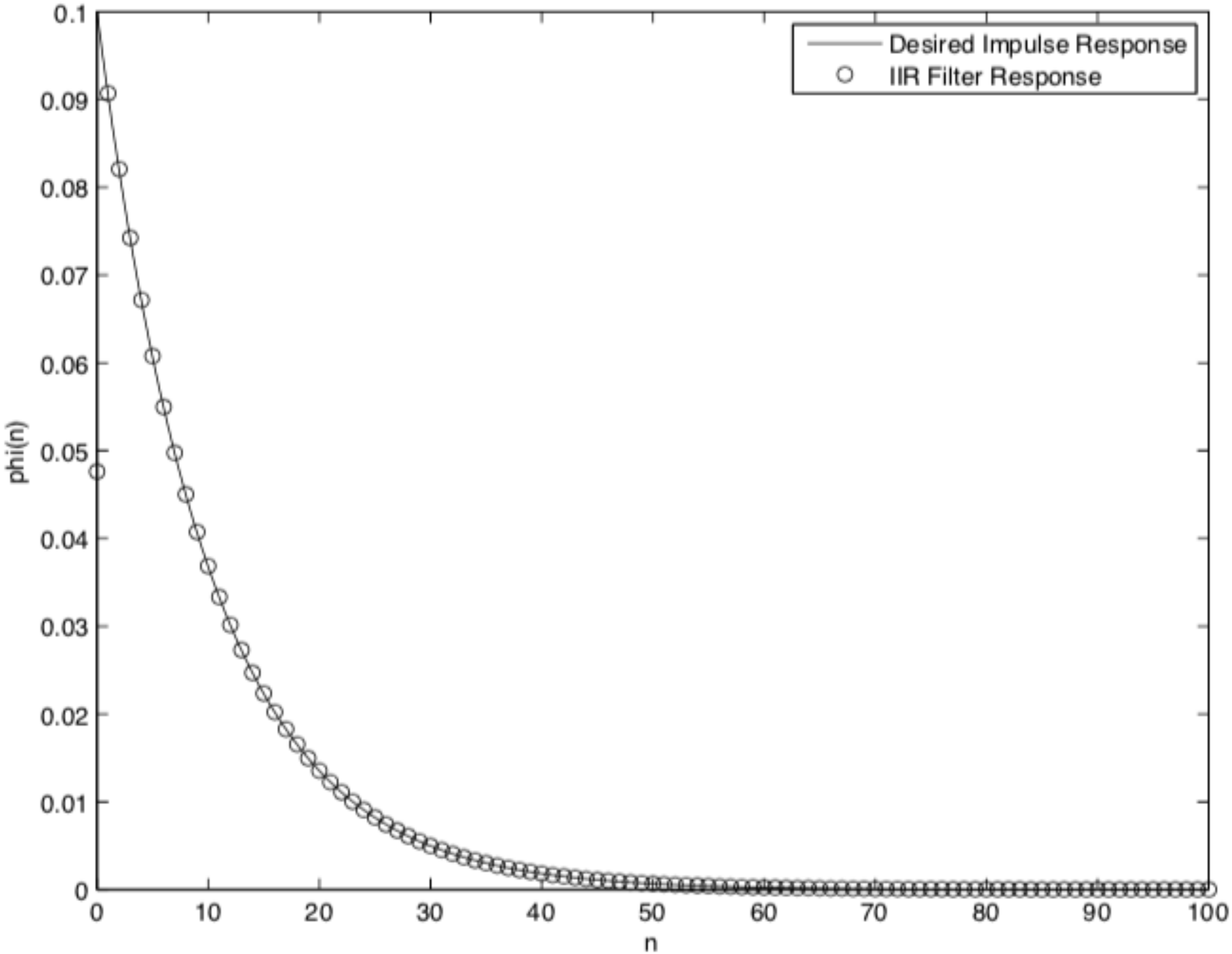

Figure 5.14: Bilinear z transform discrete realization response compared to a target continuous response in the time domain.

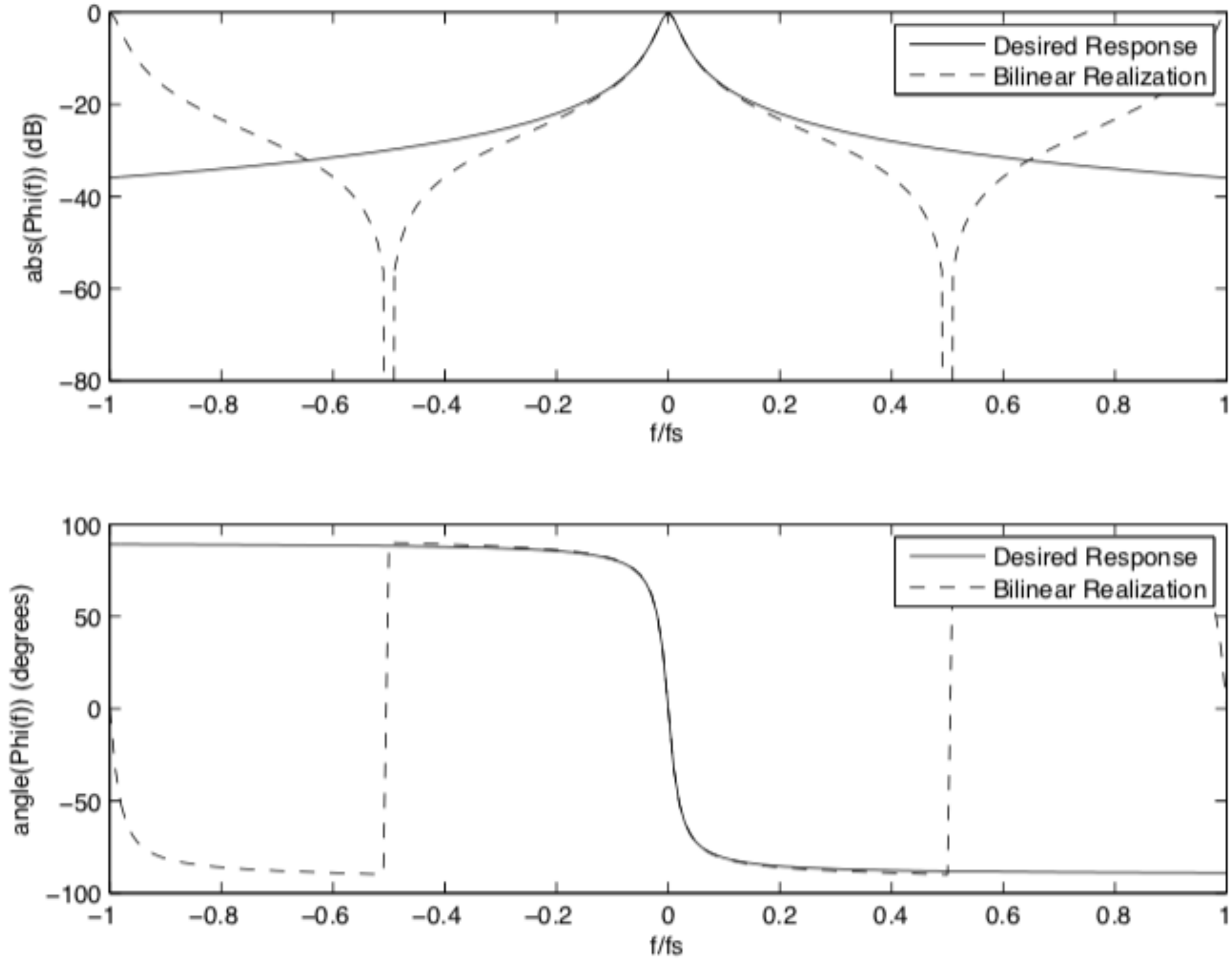

Figure 5.15: Bilinear z transform discrete realization response compared to a target continuous response in the frequency domain.

Matlab has very useful tools for IIR filter design. There are several standard filter types that attempt to approximate the ideal continuous response with the band limited discrete response. Typically the choice of filter type depends on whether you wish to minimize ripple in the passband, the stopband, or some balance in between.

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We first obtain the filter coefficients (the a's and b's in our differential equation that characterizes the filter) using the specific filter type (e.g. *butter()*) and then obtain the filter response with the command *freqz()*.

We can also eliminate phase distortion by filtering the data twice. The first time as normal, then flip the resulting filtered time series end to end from start to finish (as one does for the second time series in convolution) then filter again and restore the correct time order. So that any frequency dependent time shifts in the first application of the filter, are subtracted out in the second. Matlab has a command to do this for us called *filtfilt()*.

Run matlab program filter_examps.m

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