Z - Transforms

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 5

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Recall, $\Phi(s) = L[\phi(t)] = \int_0^\infty \phi(t) e^{-st} dt$

Recall, $\Phi(s) = L[\phi(t)] = \int_0^\infty \phi(t) e^{-st} dt$

Now let $z = e^s$ & $t \to n$ (discrete)

Recall, $\Phi(s) = L[\phi(t)] = \int_0^{\infty} \phi(t) e^{-st} dt$

Now let $z = e^s$ & $t \rightarrow n$ (discrete) then, $z^{-n} = e^{-sn}$ similar to e^{-st}

Recall, $\Phi(s) = L[\phi(t)] = \int_0^\infty \phi(t) e^{-st} dt$

Now let $z = e^s$ & $t \to n$ (discrete) then, $z^{-n} = e^{-sn}$ similar to e^{-st}

We can then define, $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

Recall, $\Phi(s) = L[\phi(t)] = \int_0^\infty \phi(t)e^{-st}dt$

Now let $z = e^s$ & $t \to n$ (discrete) then, $z^{-n} = e^{-sn}$ similar to e^{-st}

We can then define, $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

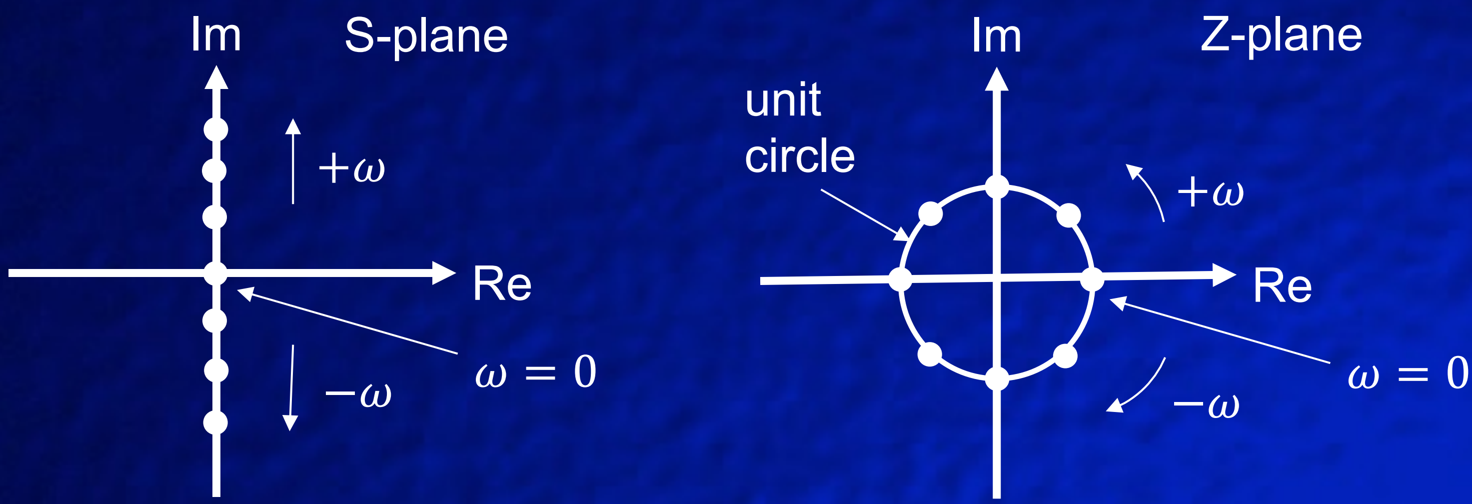
This is the Z transform, the discrete analog of the Laplace transform.

As with the FT, some versions of the ZT may be one-sided $n = (0, \infty)$ and some conventions may use $+n$ instead of $-n$ in the exponent.

Note that $z = e^{s} = e^{\sigma + i\omega} = e^{\sigma} e^{i\omega} = \underbrace{e^{\sigma}}_{\text{amp.}} \underbrace{[\cos(\omega) + i\sin(\omega)]}_{\text{phase}}$

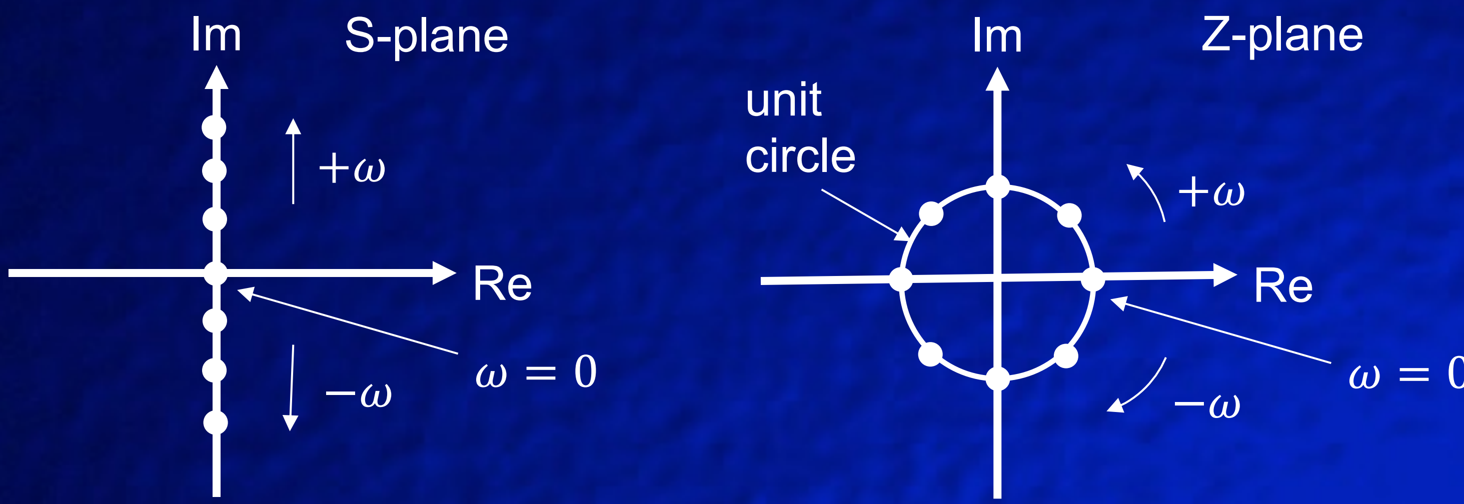
Note that $z = e^s = e^{\sigma+i\omega} = e^{\sigma}e^{i\omega} = \underbrace{e^{\sigma}}_{\text{amp.}}\underbrace{[\cos(\omega) + i\sin(\omega)]}_{\text{phase}}$

If $\sigma = 0$, amplitude = 1 which tells us the FT lies on the z-plane unit circle.



Note that $z = e^{s} = e^{\sigma + i\omega} = e^{\sigma}e^{i\omega} = \underbrace{e^{\sigma}}_{\text{amp.}}\underbrace{[\cos(\omega) + i\sin(\omega)]}_{\text{phase}}$

If $\sigma = 0$, amplitude = 1 which tells us the FT lies on the z-plane unit circle.



ω in the z-plane is, in practice, the ω in the FT normalized to a circle $(0,2\pi)$ or $(-\pi,\pi)$. Some authors thus use Ω instead of ω to distinguish the two. It is normalized by the sample rate so that one revolution around the circle in the z-plane represents the periodicity we saw in the DFT.

$$Z[x_{n-l}] = \sum_{n=-\infty}^{\infty} x_{n-l} z^{-n}$$

$$Z[x_{n-l}] = \sum_{n=-\infty}^{\infty} x_{n-l} z^{-n} = z^{-l} \sum_{n=-\infty}^{\infty} x_{n-l} z^{-(n-l)}$$

$$Z[x_{n-l}] = \sum_{n=-\infty}^{\infty} x_{n-l} z^{-n} = z^{-l} \sum_{n=-\infty}^{\infty} x_{n-l} z^{-(n-l)}$$

$$= z^{-l} \sum_{m=-\infty}^{\infty} x_m z^{-m}$$

$$Z[x_{n-l}] = \sum_{n=-\infty}^{\infty} x_{n-l} z^{-n} = z^{-l} \sum_{n=-\infty}^{\infty} x_{n-l} z^{-(n-l)}$$

$$= z^{-l} \sum_{m=-\infty}^{\infty} x_m z^{-m} = z^{-l} X(z)$$

time shift ⟷ "*phase shift*"

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

$$Z[x_{n-l}] = \sum_{n=-\infty}^{\infty} x_{n-l} z^{-n} = z^{-l} \sum_{n=-\infty}^{\infty} x_{n-l} z^{-(n-l)}$$

$$= z^{-l} \sum_{m=-\infty}^{\infty} x_m z^{-m} = z^{-l} X(z)$$

time shift ⟷ "*phase shift*"

The inverse Z transform is found using the residue theorem or tables.

$$x_n = \frac{1}{2\pi i} \oint X(z) z^{n-1} dz$$

THE UNIVERSITY OF MEMPHIS. Center for Earthquake Research and Information

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-n} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-k} z^{-(n-k)}$$

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-n} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-k} z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{n=-\infty}^{\infty} y_{(n-k)} z^{-(n-k)}$$

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-n} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-k} z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{n=-\infty}^{\infty} y_{(n-k)} z^{-(n-k)} = \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{m=-\infty}^{\infty} y_m z^{-m}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Let $w_n = x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$

$$W(z) = \sum_{n=-\infty}^{\infty} w_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k y_{n-k} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-n} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_k y_{n-k} z^{-k} z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{n=-\infty}^{\infty} y_{(n-k)} z^{-(n-k)} = \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{m=-\infty}^{\infty} y_m z^{-m}$$

$$= X(z)Y(z)$$

Convolution ⟷ Multiplication

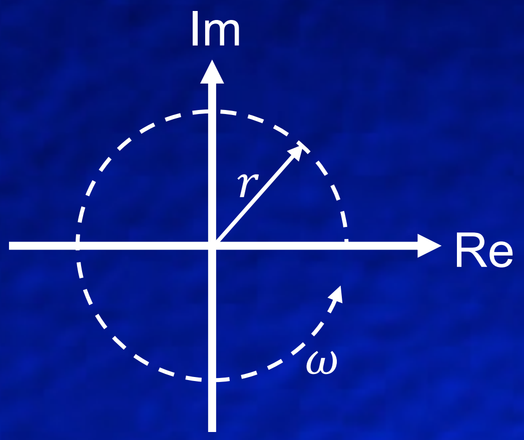
$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n}$$

$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n} = \sum_{n=-\infty}^{\infty} \phi_n \left(re^{i\omega}\right)^{-n} \qquad r = e^{\sigma}$$

THE UNIVERSITY OF MEMPHIS. Center for Earthquake Research and Information

$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n} = \sum_{n=-\infty}^{\infty} \phi_n \left(re^{i\omega}\right)^{-n} \qquad r = e^{\sigma}$$

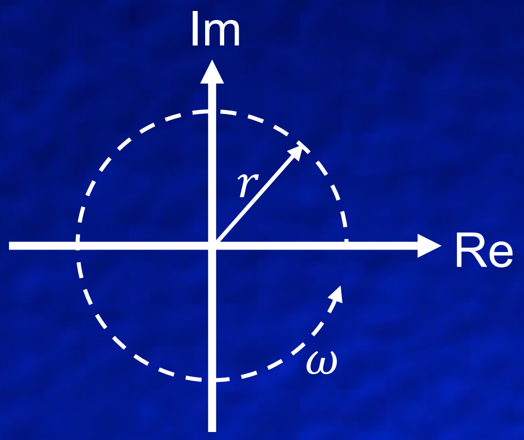
$$= \sum_{n=-\infty}^{\infty} \phi_n r^{-n} e^{-i\omega n}$$



If $r = 1$ (σ in $s = \sigma + i\omega$ is 0) then the Z transform reduces to the DFT.

$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n} = \sum_{n=-\infty}^{\infty} \phi_n \left(re^{i\omega}\right)^{-n} \qquad r = e^{\sigma}$$

$$= \sum_{n=-\infty}^{\infty} \phi_n r^{-n} e^{-i\omega n}$$

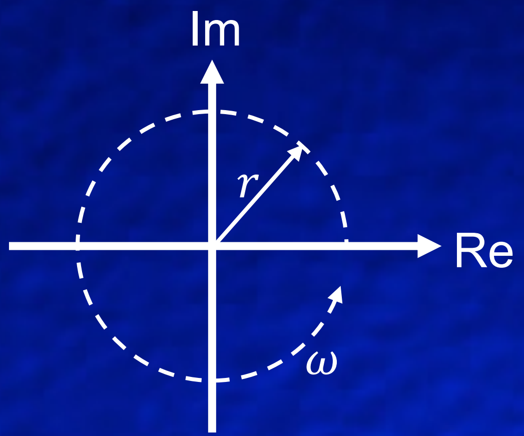


If $r = 1$ (σ in $s = \sigma + i\omega$ is 0) then the Z transform reduces to the DFT.

Where does $\Phi(z)$ exist? That is, where is $|\Phi(z)| < \infty$?

$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n} = \sum_{n=-\infty}^{\infty} \phi_n \left(re^{i\omega}\right)^{-n} \qquad r = e^{\sigma}$$

$$= \sum_{n=-\infty}^{\infty} \phi_n r^{-n} e^{-i\omega n}$$



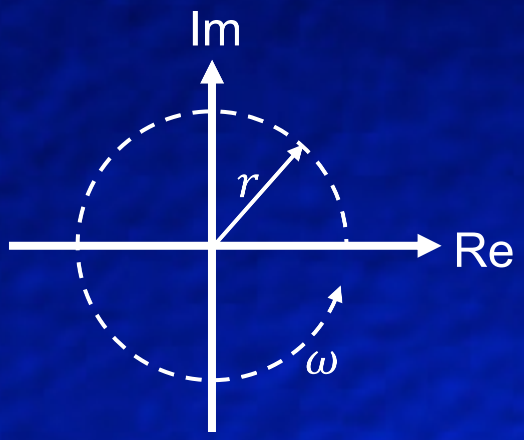
If $r = 1$ (σ in $s = \sigma + i\omega$ is 0) then the Z transform reduces to the DFT.

Where does $\Phi(z)$ exist? That is, where is $|\Phi(z)| < \infty$?

$e^{-i\omega n}$ is always $< \infty$ so we require $\displaystyle\sum_{n=-\infty}^{\infty} |\phi_n r^{-n}| < \infty$

$$\Phi(z) = Z[\phi_n] = \sum_{n=-\infty}^{\infty} \phi_n z^{-n} = \sum_{n=-\infty}^{\infty} \phi_n \left(re^{i\omega}\right)^{-n} \qquad r = e^{\sigma}$$

$$= \sum_{n=-\infty}^{\infty} \phi_n r^{-n} e^{-i\omega n}$$



If $r = 1$ (σ in $s = \sigma + i\omega$ is 0) then the Z transform reduces to the DFT.

Where does $\Phi(z)$ exist? That is, where is $|\Phi(z)| < \infty$?

$e^{-i\omega n}$ is always $< \infty$ so we require $\displaystyle\sum_{n=-\infty}^{\infty} |\phi_n r^{-n}| < \infty$

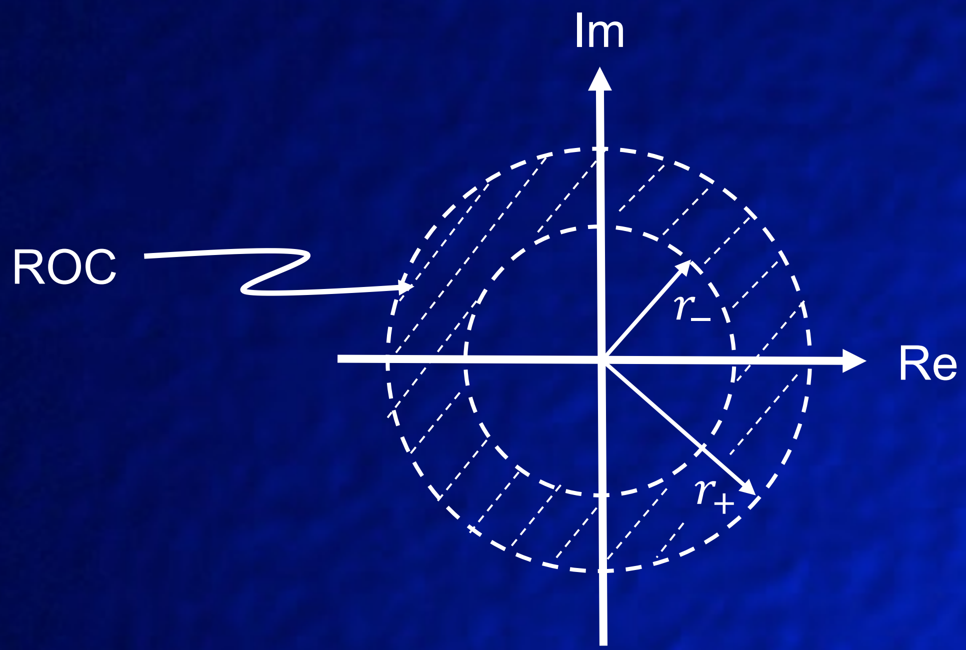
Depending on ϕ_n this is not true for any r.

$$\sum_{n=-\infty}^{\infty} |\phi_n r^{-n}| < \infty$$

In general, there is a range of r where the sum converges. This is called the Region of Convergence (ROC).

$$\sum_{n=-\infty}^{\infty} |\phi_n r^{-n}| < \infty$$

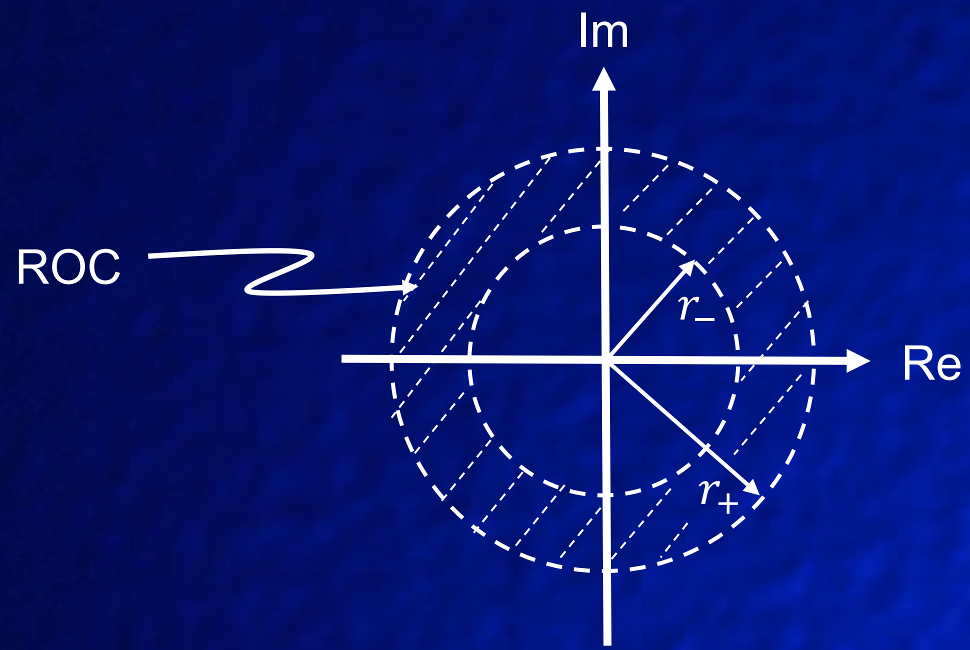
In general, there is a range of r where the sum converges. This is called the Region of Convergence (ROC).



Though the ROC could instead be outside r_+ or inside r_-

$$\sum_{n=-\infty}^{\infty} |\phi_n r^{-n}| < \infty$$

In general, there is a range of r where the sum converges. This is called the Region of Convergence (ROC).



Though the ROC could instead be outside r_+ or inside r_-

And of course if $r = 1$ is within the ROC, then the DFT exists too.

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1 - r)$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1 - r)$$

$$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1 - r)$$

$$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

$$S_n(1 - r) = 1 - r^{n+1}$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1 - r)$$

$$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

$$S_n(1 - r) = 1 - r^{n+1}$$

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1 - r)$$

$$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

$$S_n(1 - r) = 1 - r^{n+1}$$

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

For $-1 < r < 1$ $\quad \lim_{n\to\infty} S_n = \frac{1-0}{1-r} = \frac{1}{1-r}$

We'll use a geometric series as an example of finding a Z transform. Let S_n be a series unrelated to $s = \sigma + i\omega$.

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$S_n - rS_n = S_n(1-r)$$

$$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$$

$$S_n(1-r) = 1 - r^{n+1}$$

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

For $-1 < r < 1$ $\quad \lim_{n\to\infty} S_n = \frac{1-0}{1-r} = \frac{1}{1-r}$

If $r = cz^{-1}$ $\quad S_{n\to\infty} = \frac{1}{1 - \frac{c}{z}} = \frac{z}{z-c}$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}}$ $\quad |r| < 1$

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}}$ $\quad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$

How many zeros and where are they?

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

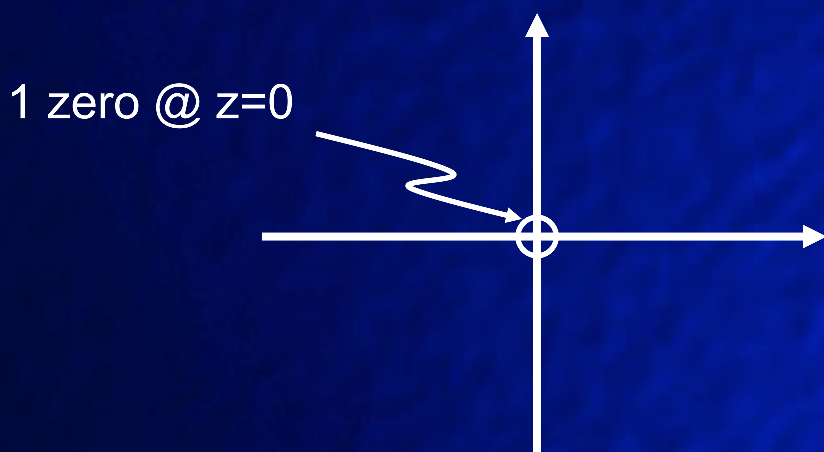
$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$



THE UNIVERSITY OF MEMPHIS Center for Earthquake Research and Information

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

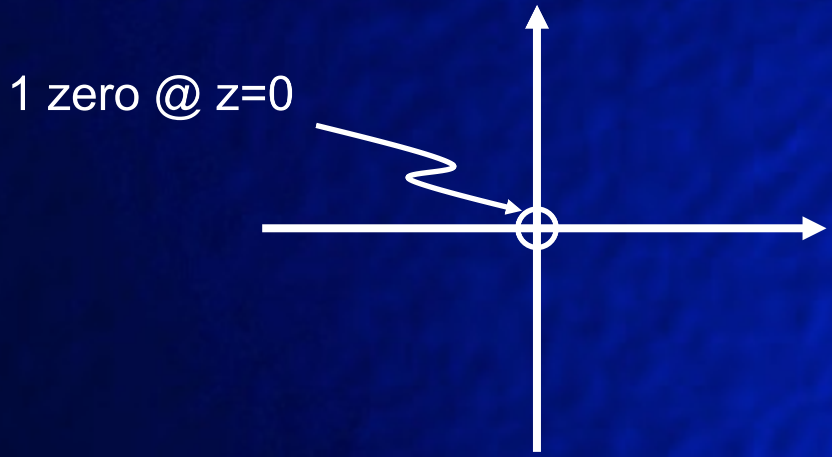
$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$



How many poles and where are they?

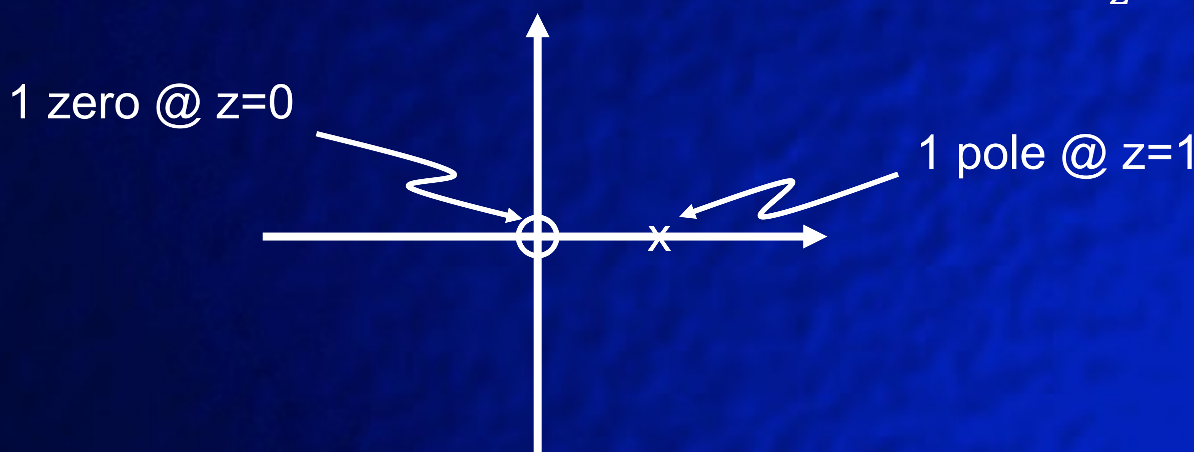
Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$ A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}}$ $\quad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$



Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$

Where is the ROC?

Now let $x_n = \begin{cases} c^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$Z[x_n] = \sum_{n=0}^{\infty} c^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{c}{z}\right)^n$$

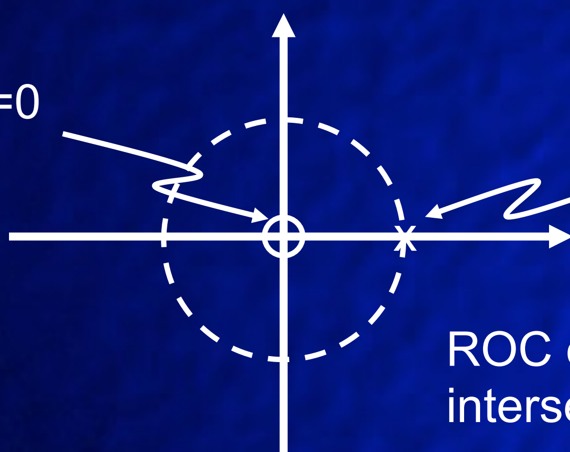
A geometric series where $r = \frac{c}{z}$

From before, $S_n = \sum_{k=0}^{n} r^k = \frac{1}{1 - \frac{c}{z}} \qquad |r| < 1$

$$\therefore X(z) = \frac{1}{1 - cz^{-1}} = \frac{z}{z - c} \qquad \left|\frac{c}{z}\right| < 1, \qquad |z| > |c|$$

If $c = 1$, then x_n is a step function and $X(z) = Z[H_n] = \frac{z}{z-1}$

1 zero @ z=0



1 pole @ z=1

ROC outside circle that intersects pole

Recall the Z transform $z = e^{s}$ $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Recall the Z transform $z = e^{s}$ $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

Examples from Discrete-Time Signal Processing, Alan Oppenheim and Ronald Schaffer, Prentice Hall, 1989.

Let $x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$ where h_n is a step function.

Recall the Z transform $z = e^{s}$ $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

Examples from Discrete-Time Signal Processing, Alan Oppenheim and Ronald Schaffer, Prentice Hall, 1989.

Let $x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$ where h_n is a step function.

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n\right] z^{-n}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Recall the Z transform $z = e^{z}$ $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

Examples from Discrete-Time Signal Processing, Alan Oppenheim and Ronald Schaffer, Prentice Hall, 1989.

Let $x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$ where h_n is a step function.

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n\right] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n}$$

Recall the Z transform $z = e^{z}$ $X(z) = Z[x_n] = \sum_{n=-\infty}^{\infty} x_n z^{-n}$

Examples from Discrete-Time Signal Processing, Alan Oppenheim and Ronald Schaffer, Prentice Hall, 1989.

Let $x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$ where h_n is a step function.

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n\right] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Recall $\lim_{n\to\infty} S_n = \sum_{k=0}^{n} r^k = \frac{1}{1-r} \qquad |r| < 1$

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$$

Recall $\lim_{n\to\infty} S_n = \sum_{k=0}^{n} r^k = \frac{1}{1-r} \qquad |r| < 1$

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$$

$$X(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\left(-\frac{1}{3}z^{-1}\right)}$$

THE UNIVERSITY OF MEMPHIS. Center for Earthquake Research and Information

Recall $\lim_{n\to\infty} S_n = \sum_{k=0}^{n} r^k = \frac{1}{1-r} \qquad |r|<1$

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$$

$$X(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\left(-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{z\left(z+\frac{1}{3}\right)+z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Recall $\lim_{n\to\infty} S_n = \sum_{k=0}^{n} r^k = \frac{1}{1-r}$ $\quad |r| < 1$

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - \left(-\frac{1}{3} z^{-1}\right)}$$

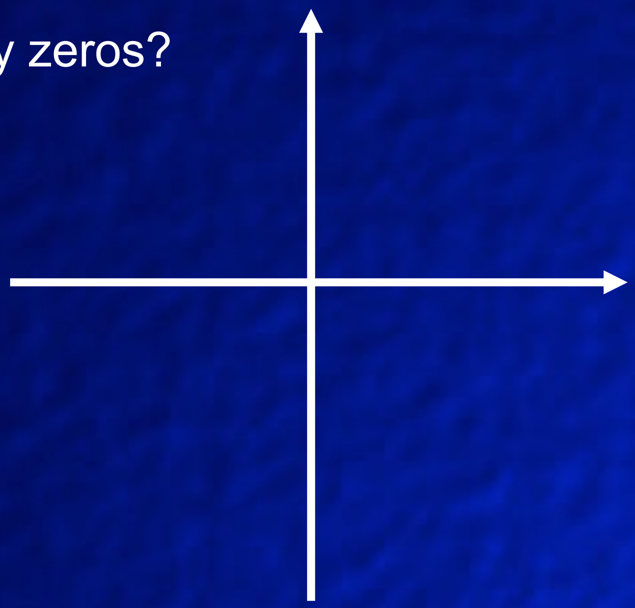
$$= \frac{z}{z - \frac{1}{2}} + \frac{z}{z + \frac{1}{3}} = \frac{z\left(z + \frac{1}{3}\right) + z\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

$$\left|\frac{1}{2} z^{-1}\right| < 1, \left|-\frac{1}{3} z^{-1}\right| < 1 \longrightarrow |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

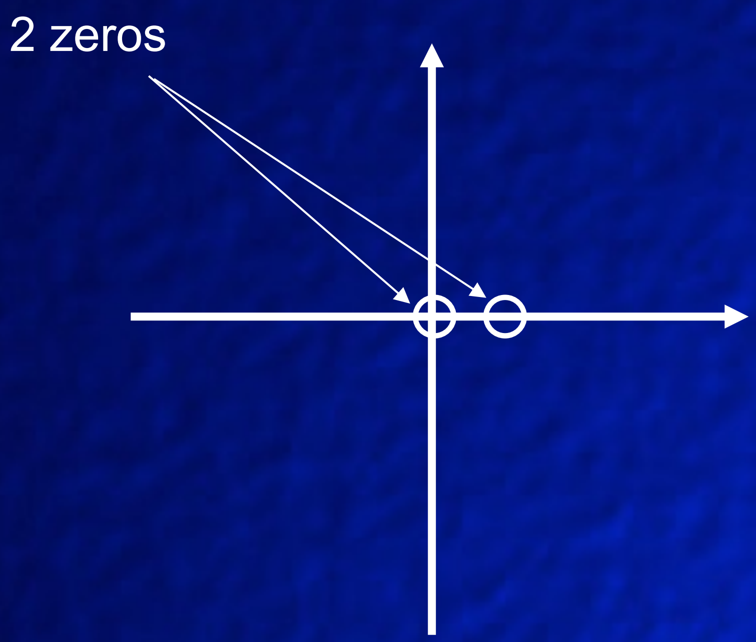
THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

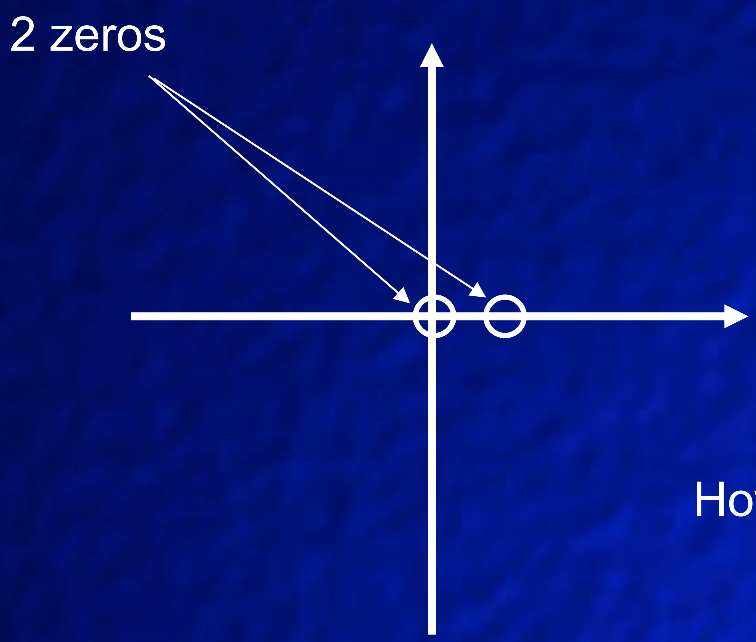
How many zeros?



$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

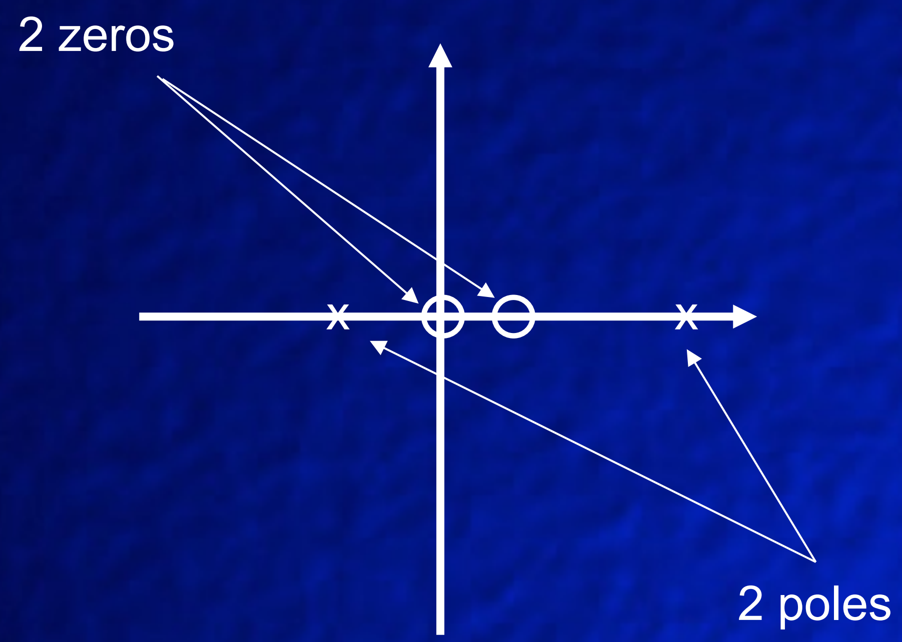


$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

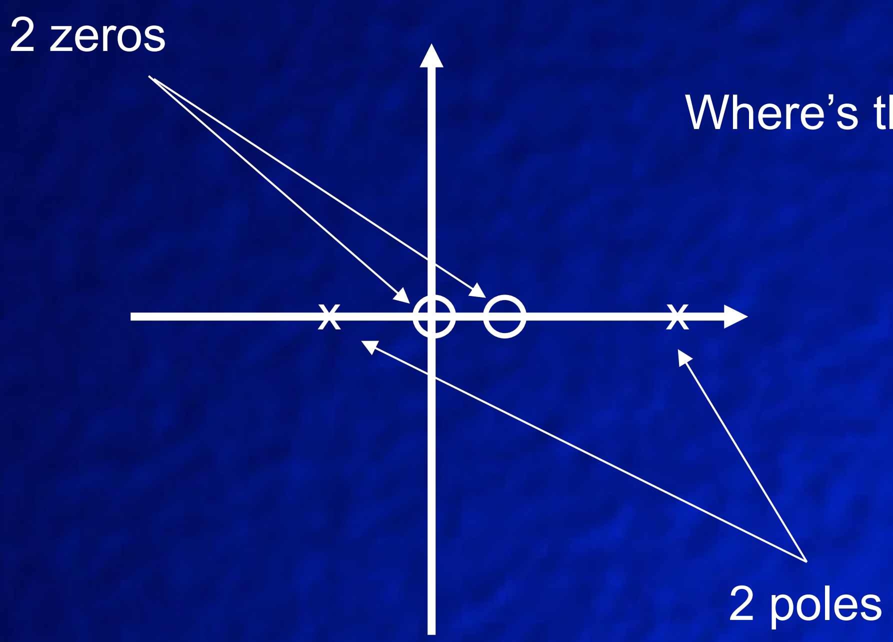


How many poles?

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

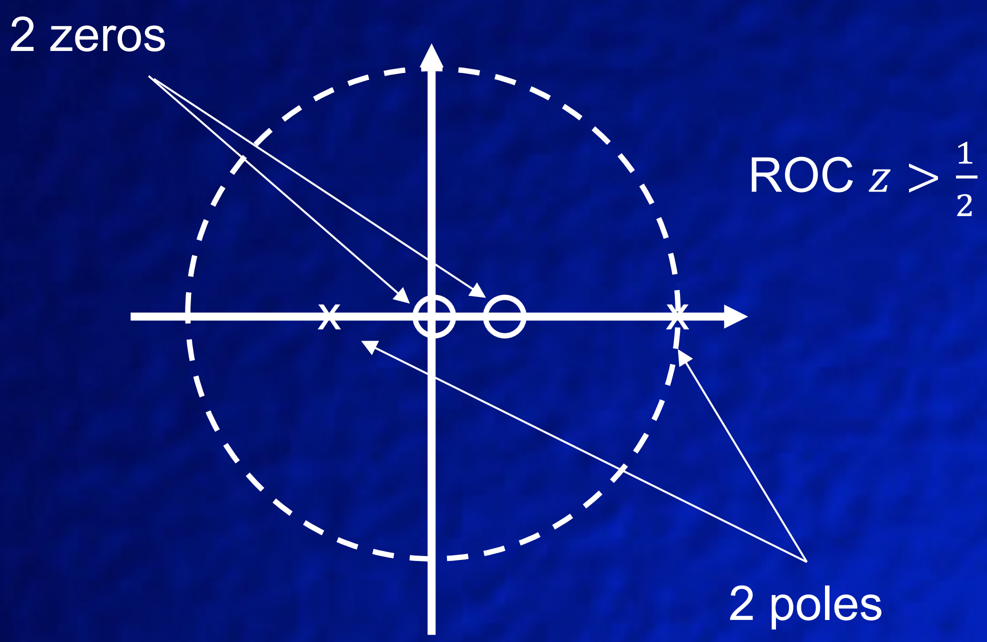


$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

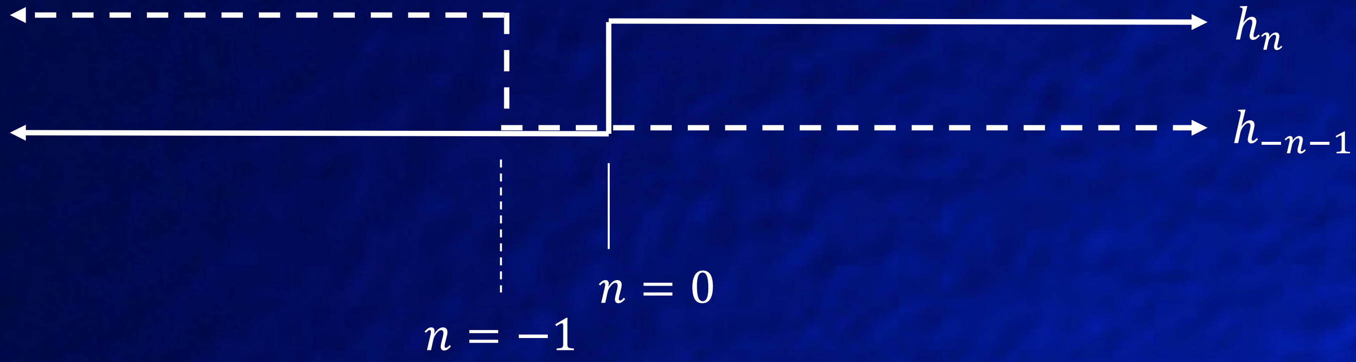


Where's the ROC?

$$x_n = \left(\frac{1}{2}\right)^n h_n + \left(-\frac{1}{3}\right)^n h_n \qquad X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad |z| > \frac{1}{2}, |z| > \frac{1}{3}$$

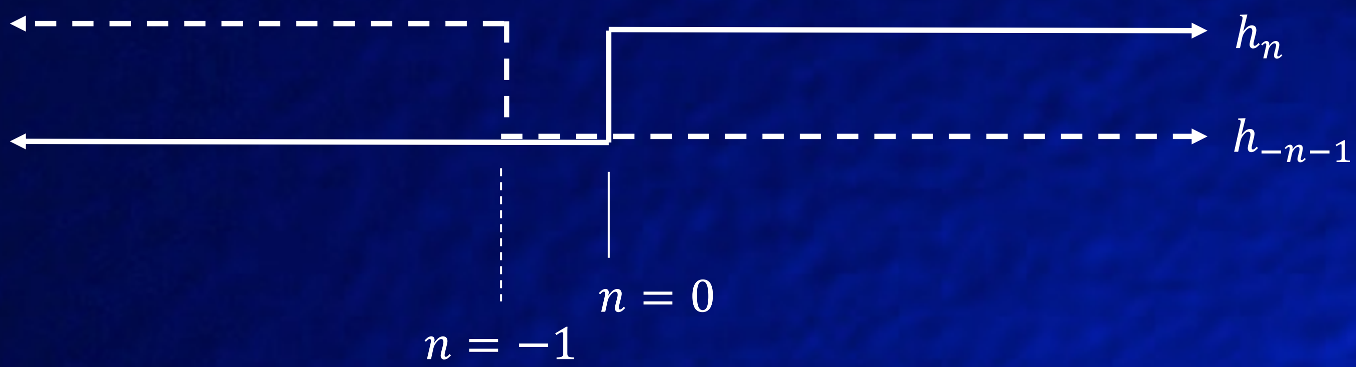


Now let $x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$ $\qquad$ h_n is again a step function.



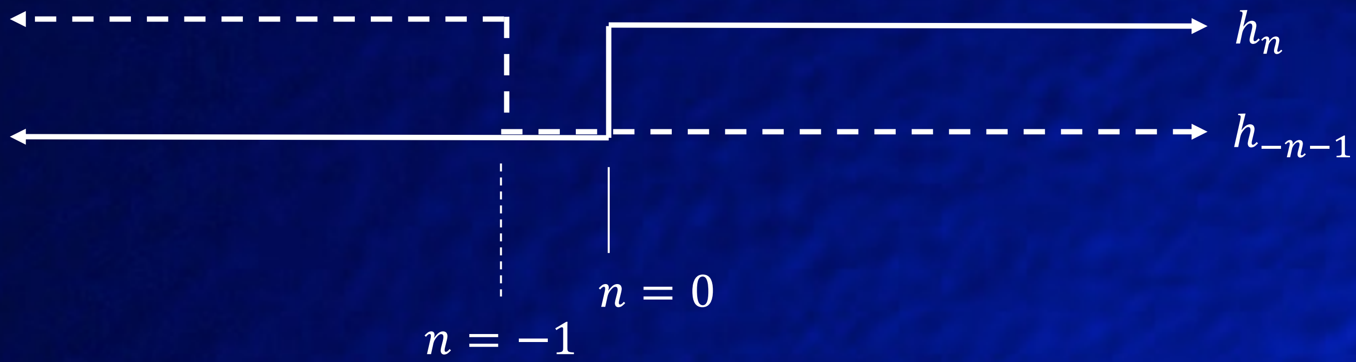
THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Now let $x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$ h_n is again a step function.



$$X(z) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n h_n z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n h_{-n-1} z^{-n}$$

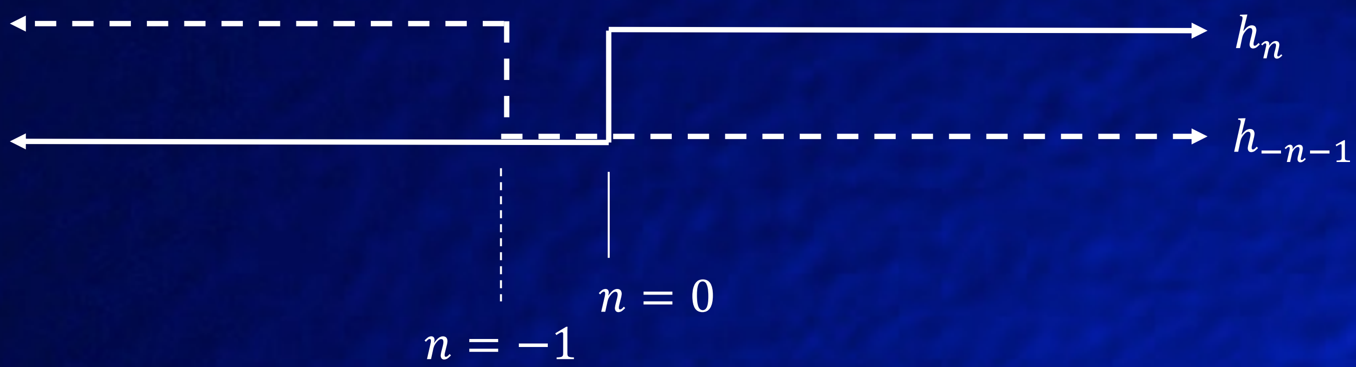
Now let $x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$ h_n is again a step function.



$$X(z) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n h_n z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n h_{-n-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n + \sum_{n=-\infty}^{0} \left(\frac{1}{2} z^{-1}\right)^n$$

Now let $x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$ $\quad$ h_n is again a step function.



$$X(z) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n h_n z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n h_{-n-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n + \sum_{n=-\infty}^{0} \left(\frac{1}{2} z^{-1}\right)^n$$

We can sum from $-\infty$ to 0 instead of -1 because summing to 0 just adds 0 in this case.

We saw the geometric series with a positive exponent,

$$S_n = \sum_{n=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r| < 1, n \to \infty$$

We saw the geometric series with a positive exponent,

$$S_n = \sum_{n=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r| < 1, n \to \infty$$

$$S_n = \sum_{n=-\infty}^{0} r^k = \frac{1}{1-r}, \qquad |r| > 1, n \to -\infty$$

It works for a negative exponent too.

We saw the geometric series with a positive exponent,

$$S_n = \sum_{n=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r| < 1, n \to \infty$$

$$S_n = \sum_{n=-\infty}^{0} r^k = \frac{1}{1-r}, \qquad |r| > 1, n \to -\infty$$

It works for a negative exponent too.

$$x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$$

$$\therefore X(z) = \frac{1}{1-\left(-\frac{1}{3}z^{-1}\right)} + \frac{1}{1-\left(\frac{1}{2}z^{-1}\right)} \qquad \left|-\frac{1}{3}z^{-1}\right| < 1 \qquad \left|\frac{1}{2}z^{-1}\right| > 1$$

We saw the geometric series with a positive exponent,

$$S_n = \sum_{n=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r| < 1, n \to \infty$$

It works for a negative exponent too.

$$S_n = \sum_{n=-\infty}^{0} r^k = \frac{1}{1-r}, \qquad |r| > 1, n \to -\infty$$

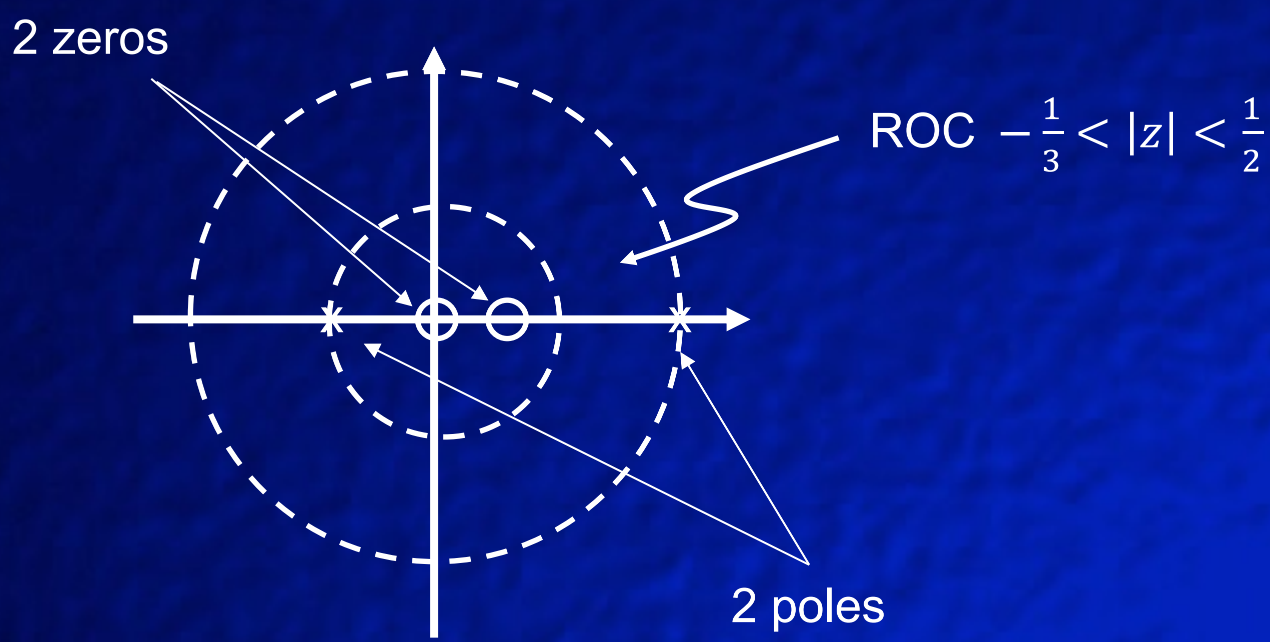
$$x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1}$$

$$\therefore X(z) = \frac{1}{1-\left(-\frac{1}{3}z^{-1}\right)} + \frac{1}{1-\left(\frac{1}{2}z^{-1}\right)} \qquad \left|-\frac{1}{3}z^{-1}\right| < 1 \qquad \left|\frac{1}{2}z^{-1}\right| > 1$$

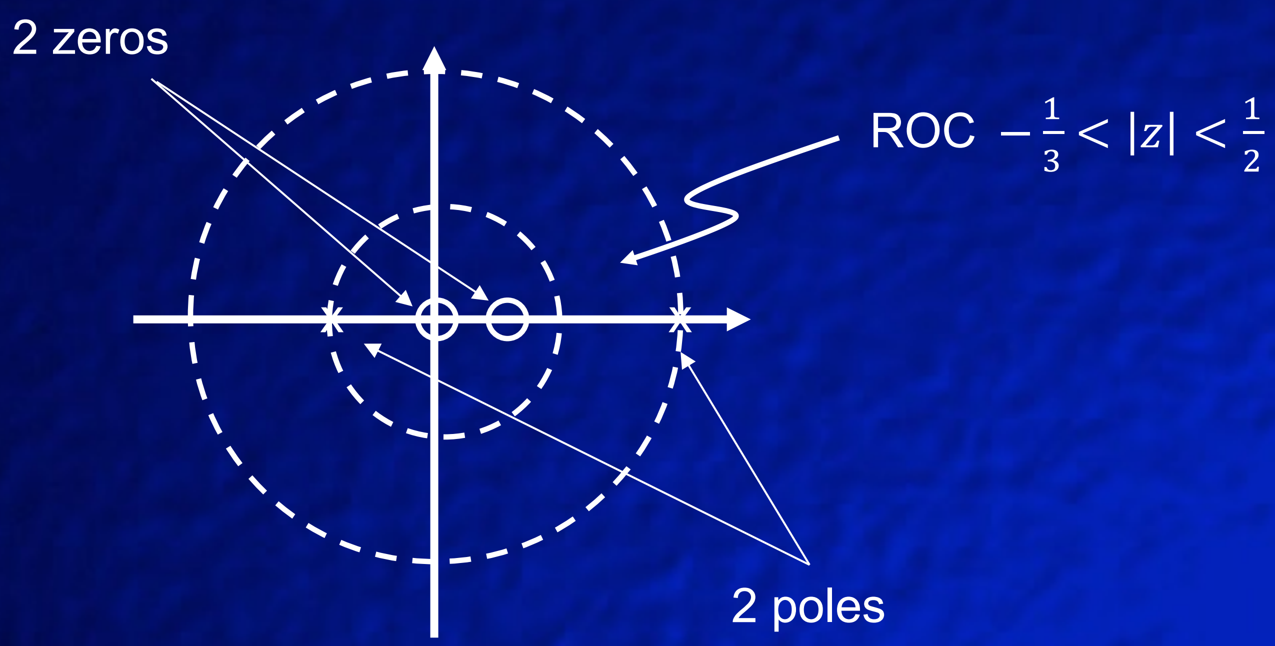
$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad -\frac{1}{3} < |z| < \frac{1}{2}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

$$x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1} \qquad X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)} \qquad -\frac{1}{3} < |z| < \frac{1}{2}$$



$$x_n = \left(-\frac{1}{3}\right)^n h_n + \left(\frac{1}{2}\right)^n h_{-n-1} \qquad X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad -\frac{1}{3} < |z| < \frac{1}{2}$$



We have a different x_n with the same $X(z)$ but with a different ROC. That tells us that $X(z)$ alone may be non-unique unless we include the ROC.

5.3.2 – Properties of ROC

- *A ring or disk in the z-plane centered at the origin.*
- *The Fourier Transform of x(n) is converge absolutely iff the ROC includes the unit circle.*
- *The ROC cannot include any poles*
- *Finite Duration Sequences: The ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.*
- *Right sided sequences (causal seq.): The ROC extends outward from the outermost finite pole in $X(z)$ to $z=\infty$.*
- *Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to $z=0$.*
- *Two-sided sequence: The ROC is a ring bounded by two circles passing through two pole with no poles inside the ring*

Source: slidesharecdn.com

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Source: chegg.com (though this table is reproduced in multiple texts and websites)

u is the unit step function