Laplace Transforms

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 5.

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Convergence can be an issue depending on $\phi(t)$.

We will use the one-sided Laplace Transform. Some fields use the two sided Laplace Transform which poses more convergence issues.

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If $s = i2\pi f,\ \sigma = 0$

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$$L[a(t) * b(t)] = A(s)B(s)$$

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Now recall our generic differential equation,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1 \frac{d^1 y}{dt^1} + a_0 y$$

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Laplace transform both sides to find,

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→ The transfer function of our system of equations in terms of poles and zeros located on the s-plane

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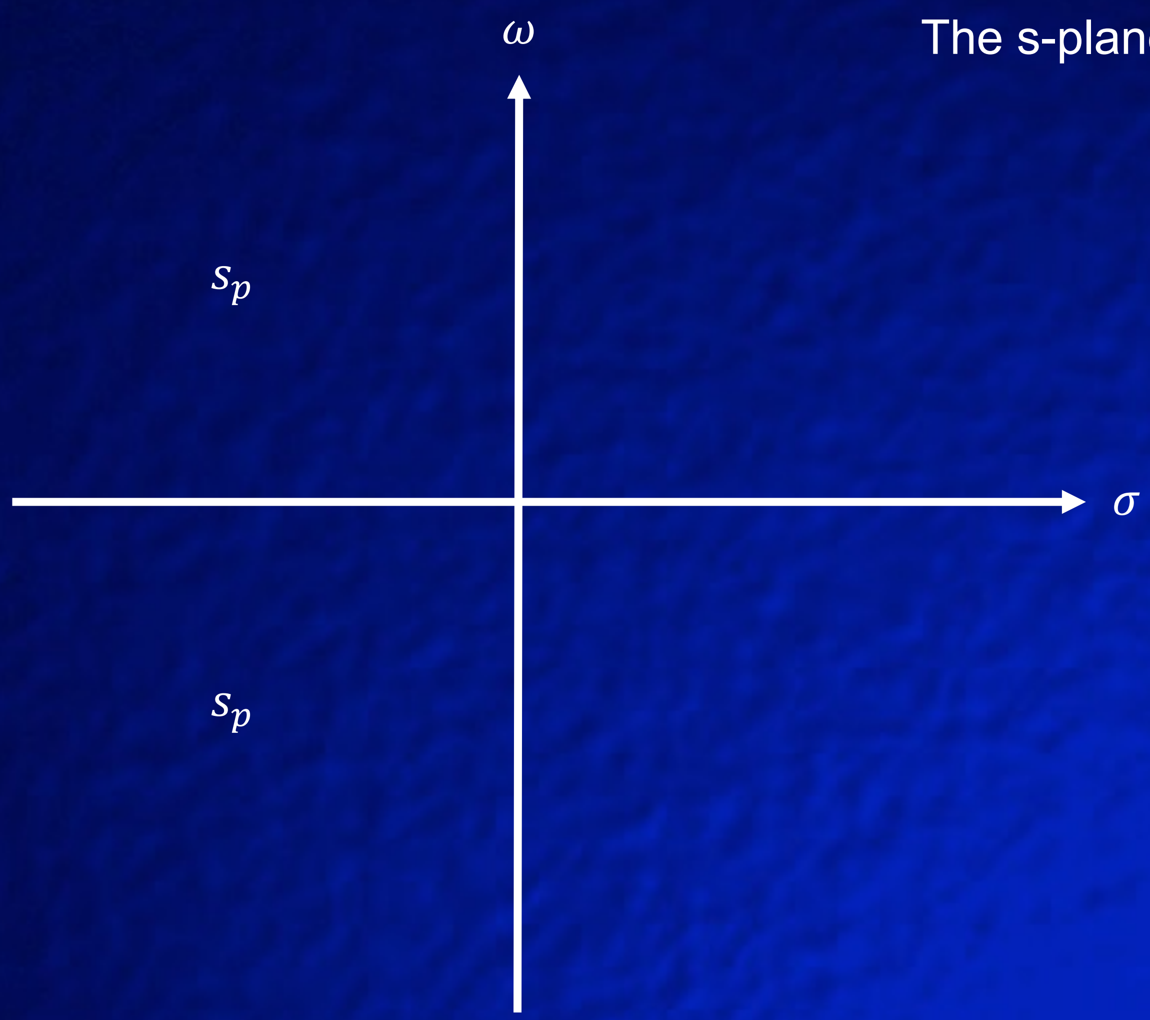
$s = \sigma + i\omega$

The s-plane



Let $\phi(t) = 1$ $\Phi(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big/_{0}^{\infty}$

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Forward transforms are relatively easy. Not so the inverse.

For $t > 0$, the inverse Laplace transform is $\phi(t) = L^{-1}[\Phi(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s)e^{st}ds$

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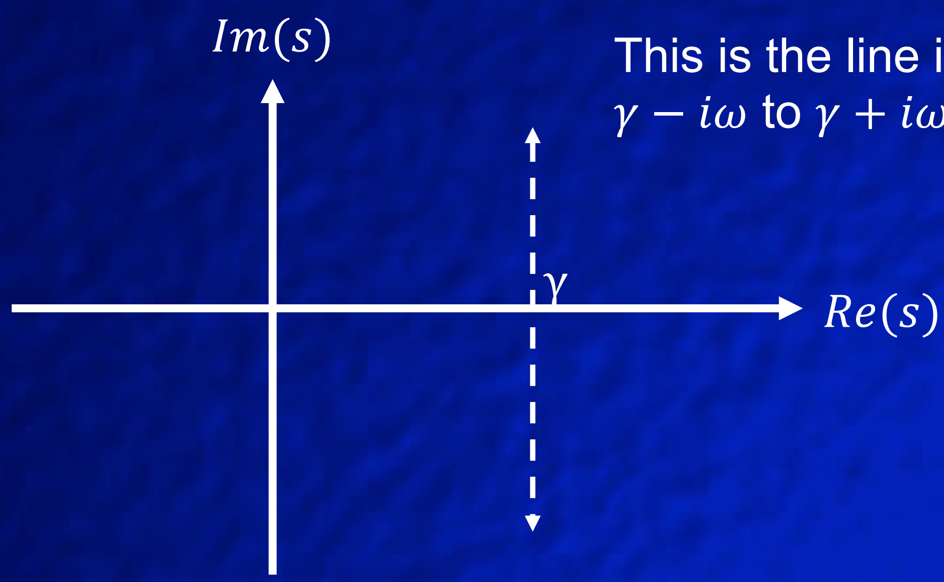
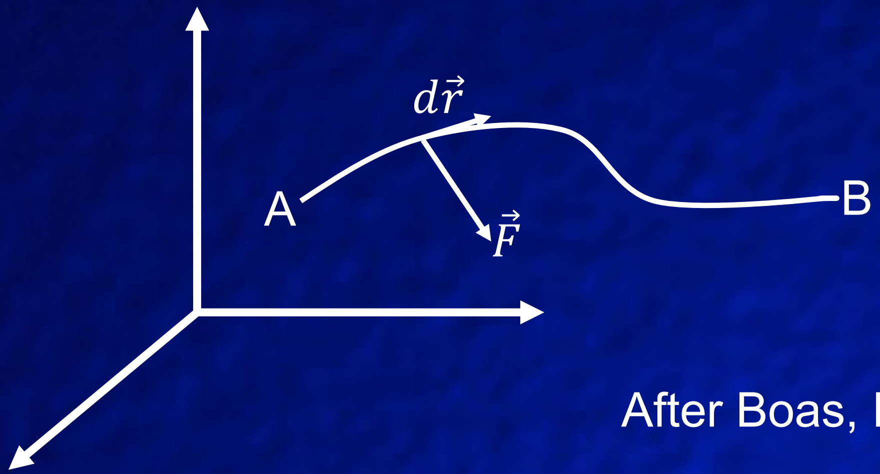


This is the line integral from $\gamma - i\omega$ to $\gamma + i\omega$

Line Integrals (e.g. see Mary Boas, Mathematical Methods in the Physical Sciences (page 257ff, in my ancient edition).



After Boas, Figure 8.1

Work done on an object by a force $\vec{F}$ which undergoes an infinitesimal displacement $d\vec{r}$ is

$$dW = \vec{F} \cdot d\vec{r}$$

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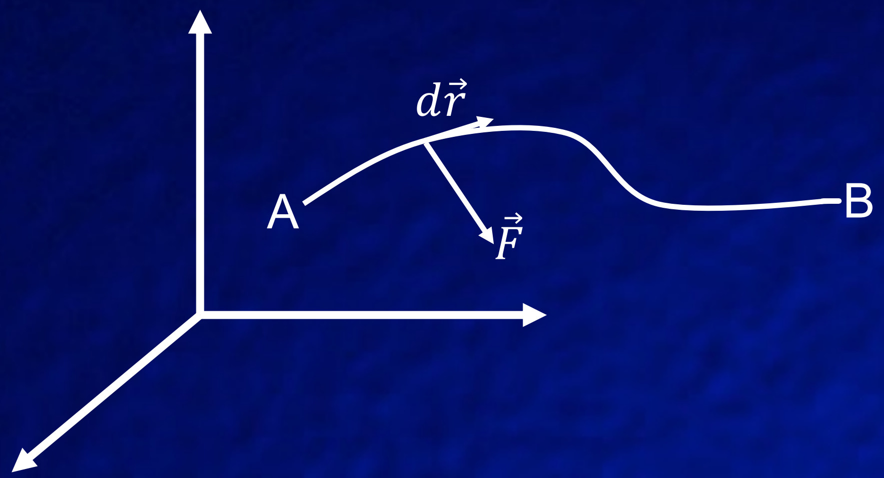
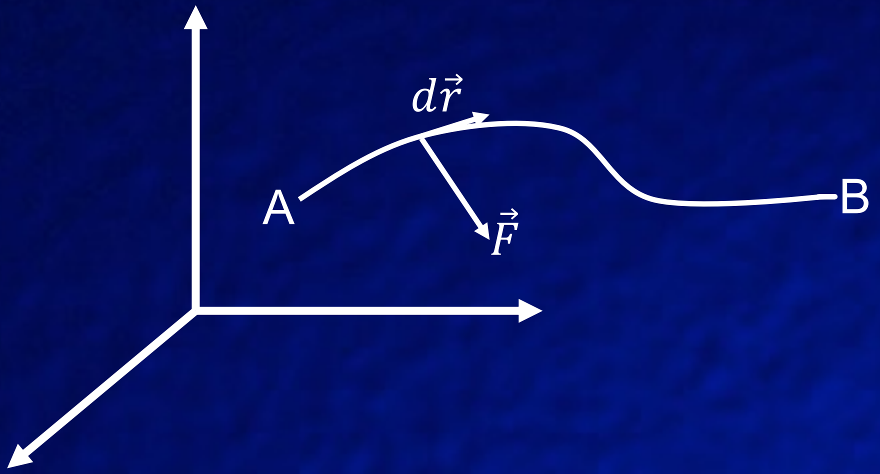


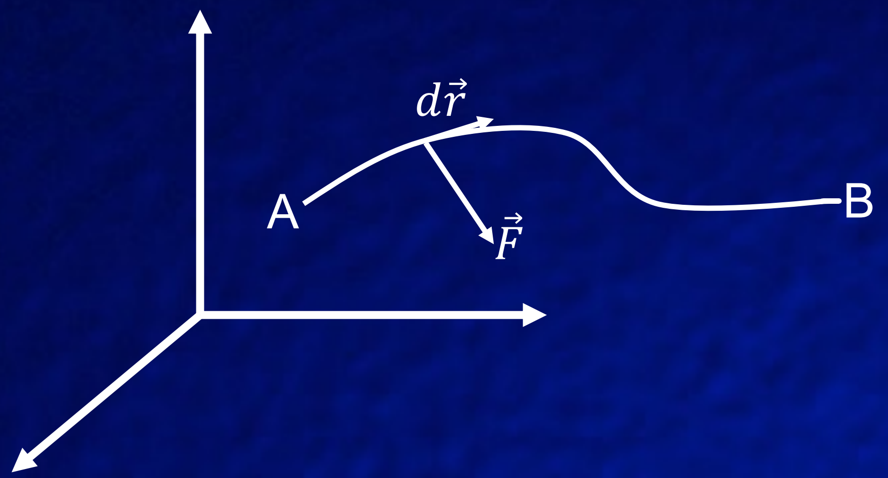
Suppose an object moves along the path from A to B and $\vec{F}$ varies along the path.



Suppose an object moves along the path from A to B and $\vec{F}$ varies along the path.

Then along curve, there is only 1 independent variable that is a function of position in the 3-d space.

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$



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The integral of dW then becomes and ordinary integral of 1 variable, r, along the line from A to B. This is the line integral.

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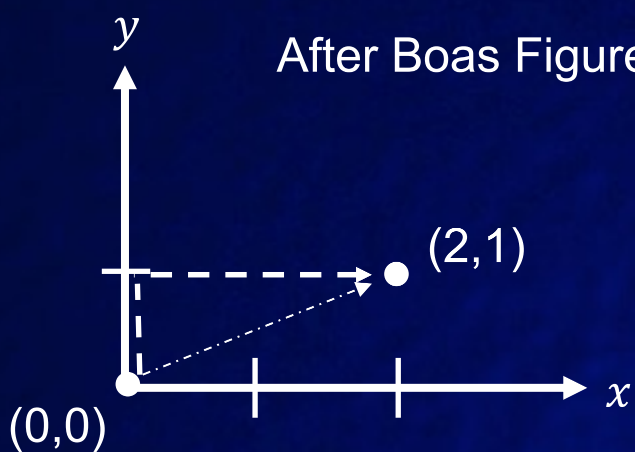
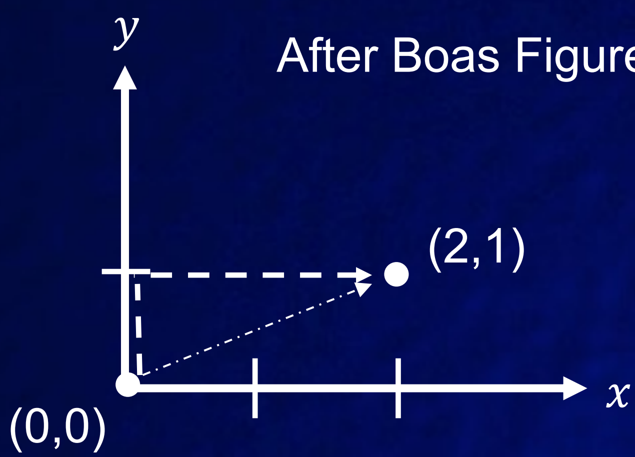


After Boas Figure 8.2

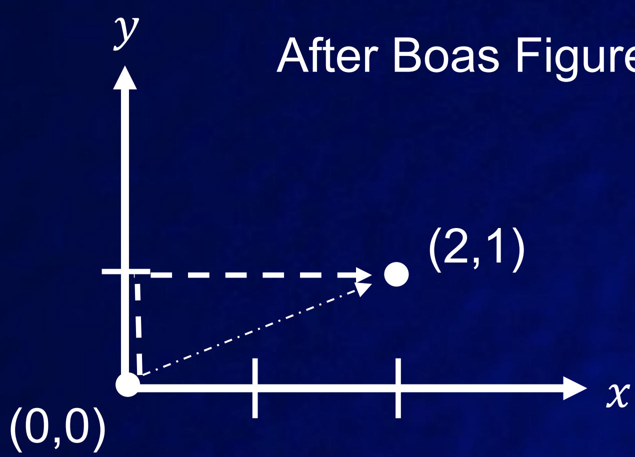
Line integral example.



After Boas Figure 8.2

Line integral example.

Let $\vec{F} = xy\hat{x} - y^2\hat{y}$ $\quad A = (0,0), B = (2,1)$



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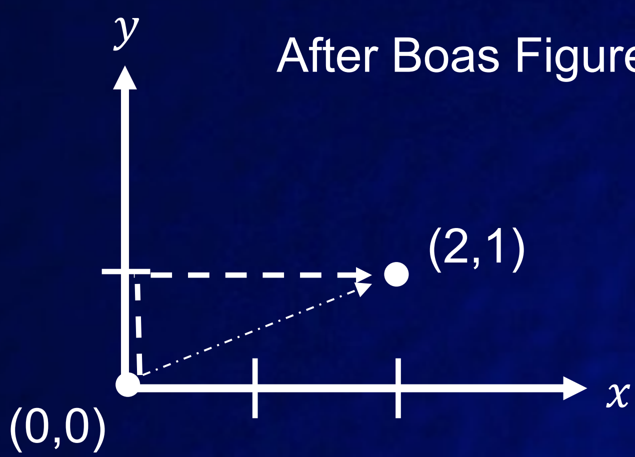

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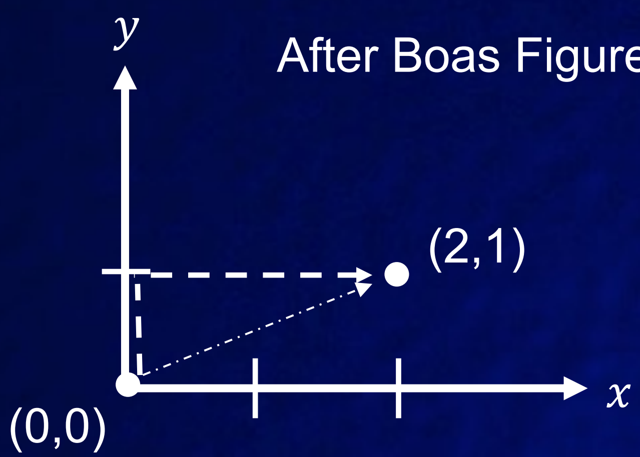

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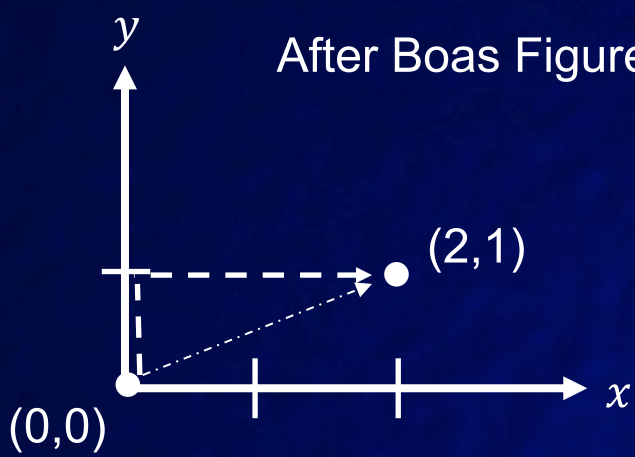
If the path from A to B is a straight line (recall $y = mx + b$)

$$y = \frac{1}{2}x \qquad dy = \frac{1}{2}dx$$

After Boas Figure 8.2



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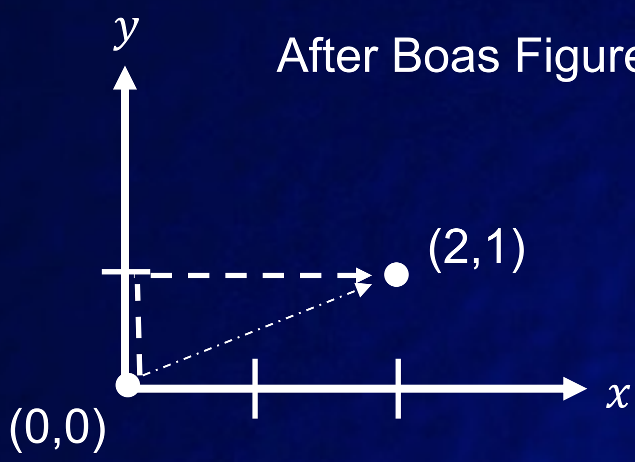

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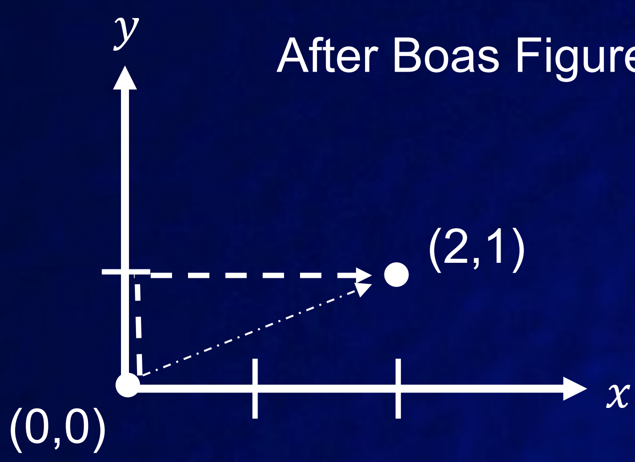

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We could have just as easily set $x = 2y$ and integrated over y from 0 to 1.

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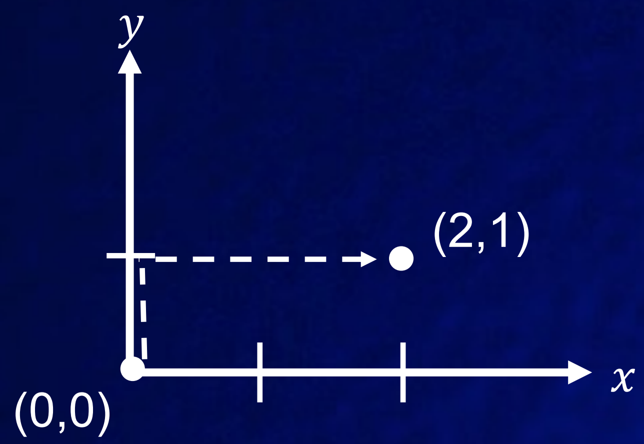
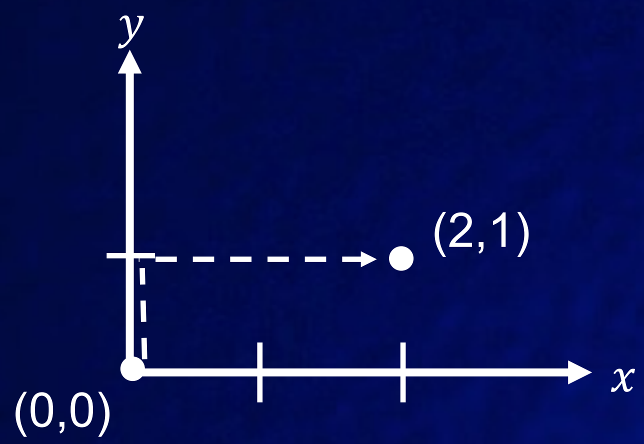
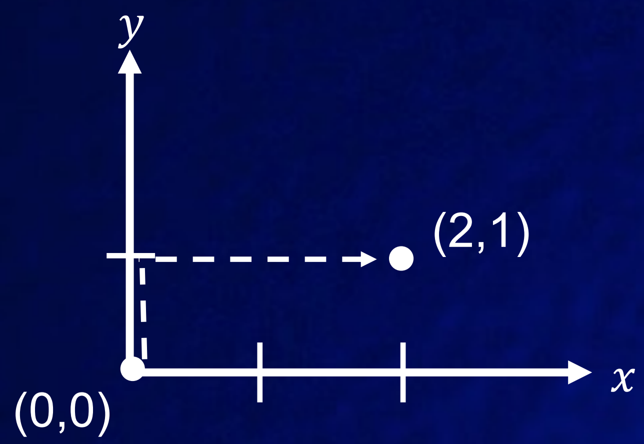


Now let's find the work needed to move the object from A to B along the L-shaped path.



Now let's find the work needed to move the object from A to B along the L-shaped path.

That requires to integrals. First keeping x $= 0$ along the vertical path from (0,0) to (0,1). Then, keeping y=1, horizontally from (0,1) to (2,1).



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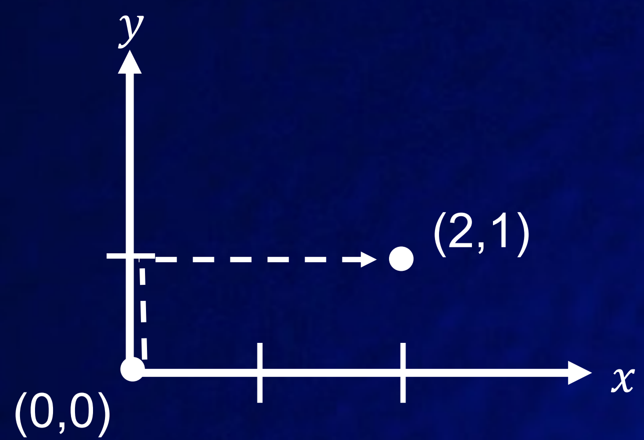

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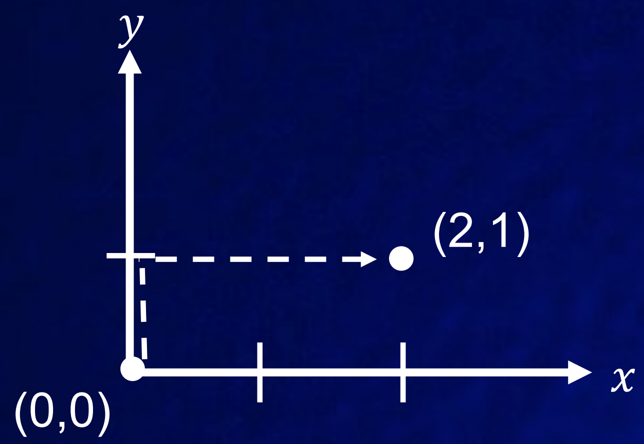

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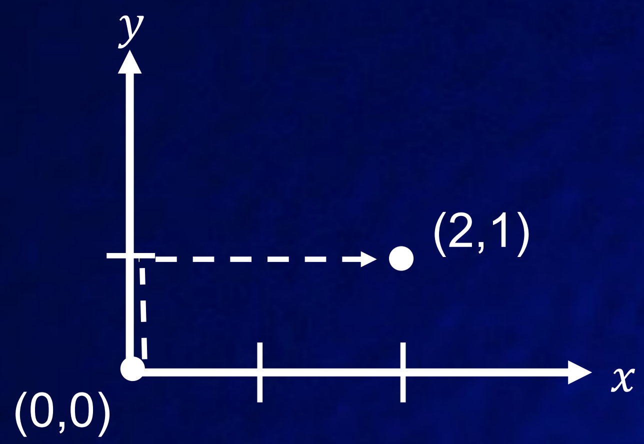

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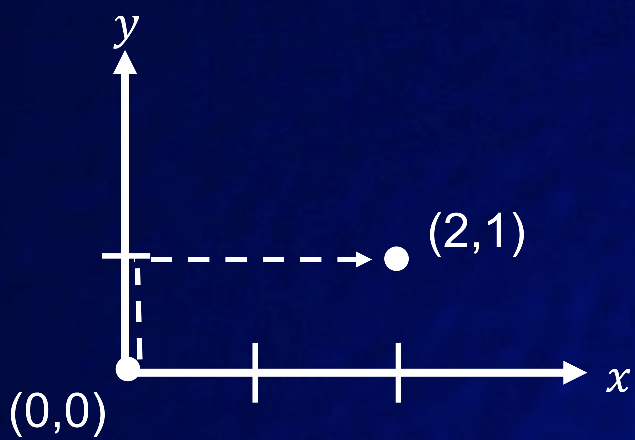

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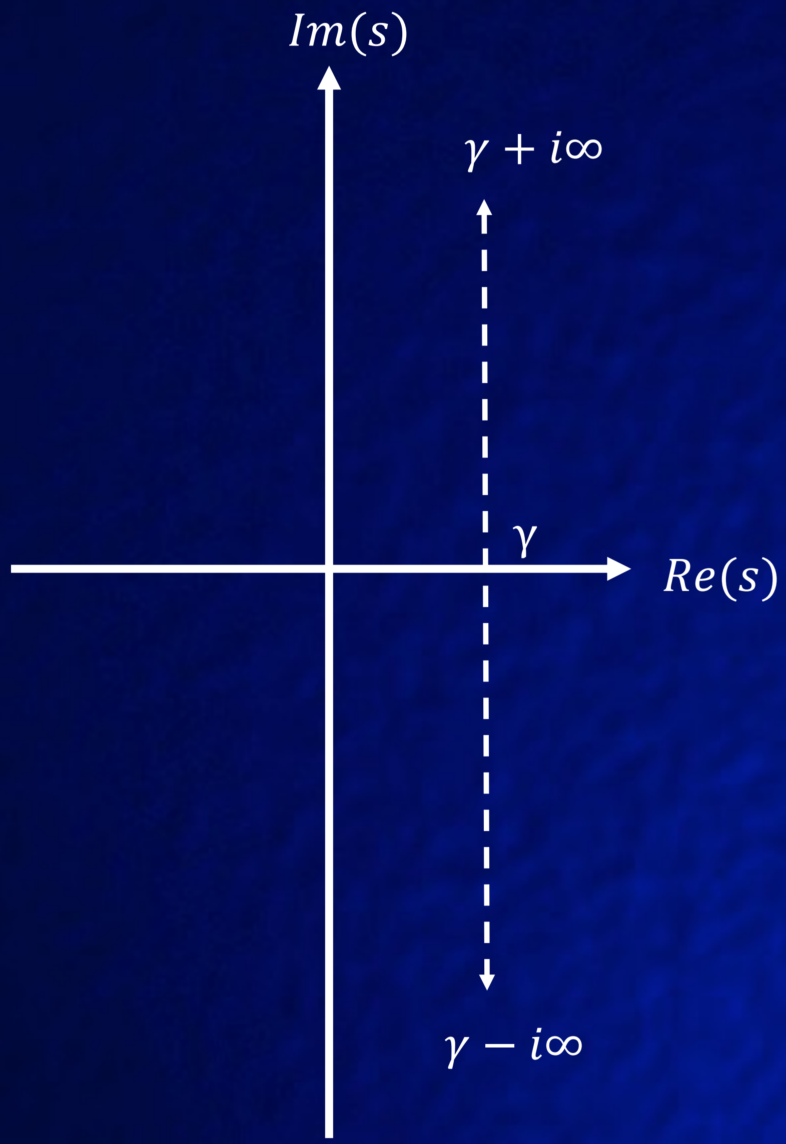
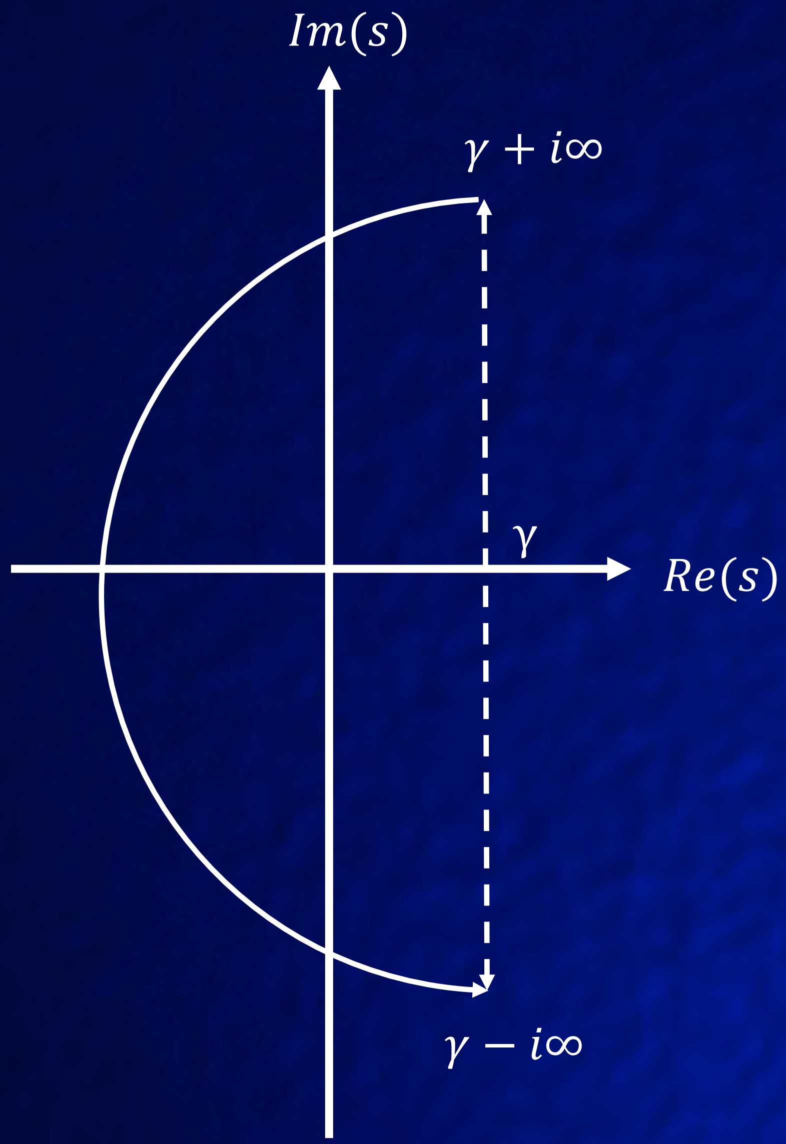
The work done in this case depends on the path.



$$\phi(t) = L^{-1}[\Phi(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \Phi(s) e^{st} ds$$



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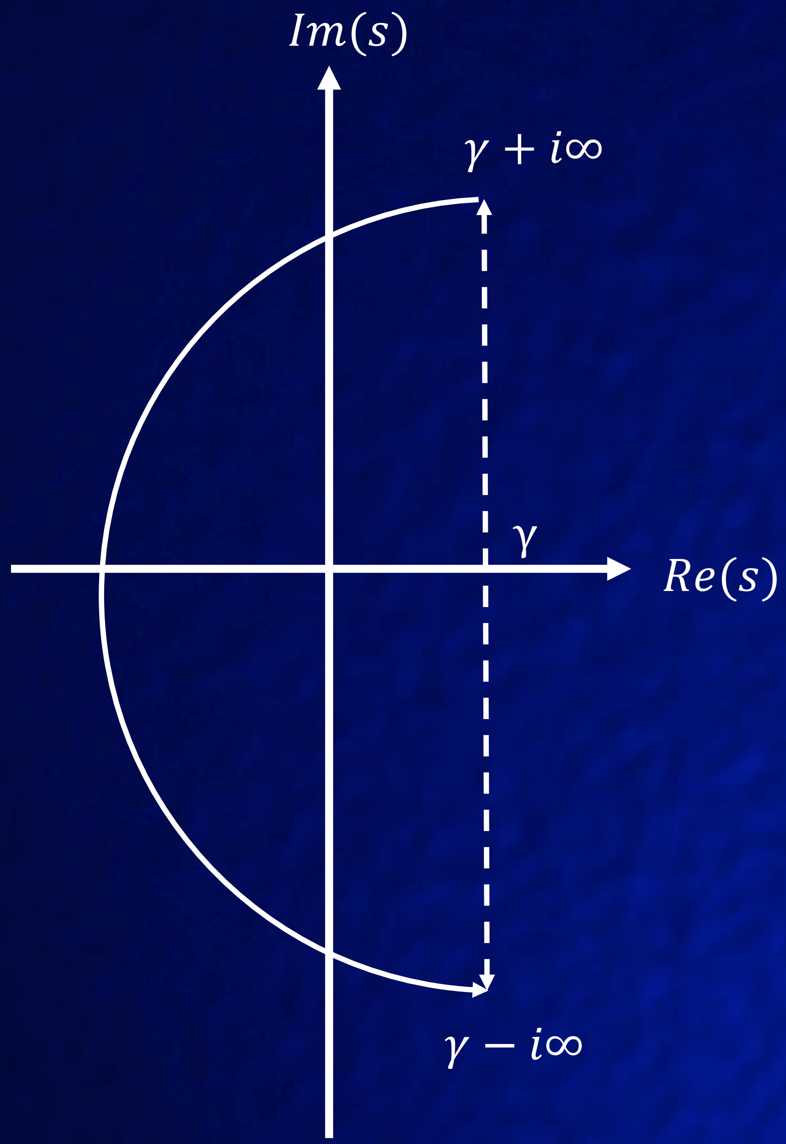

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$$\phi(t) = \frac{1}{2\pi i}\oint \Phi(s)e^{st}ds$$

And we can take advantage of the residue theorem from complex analysis.

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Any contour that surrounds the same set of poles has the same value for the contour integral. And we can then use the residue theorem to help solve L^{-1}.

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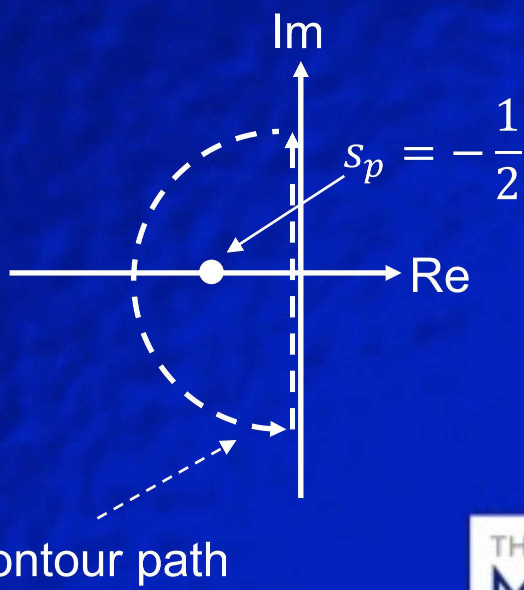

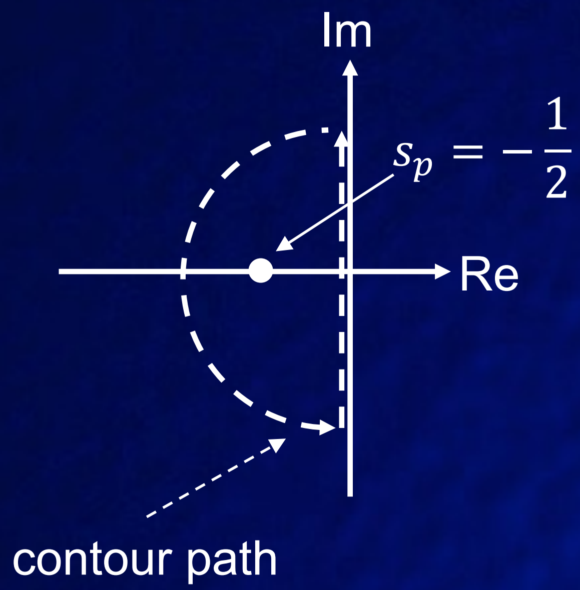

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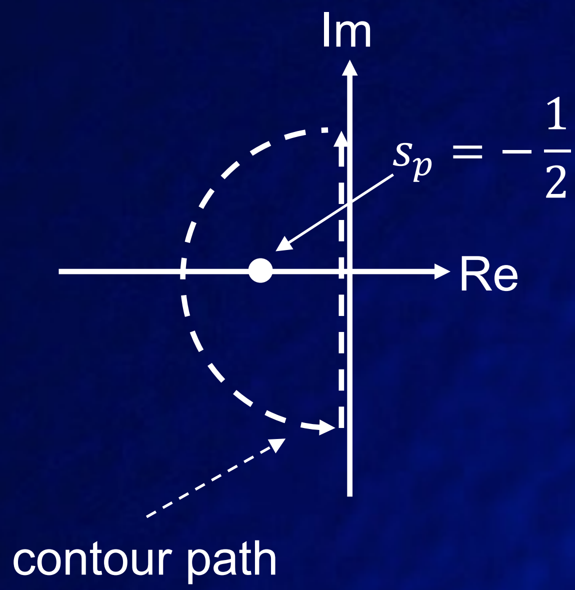

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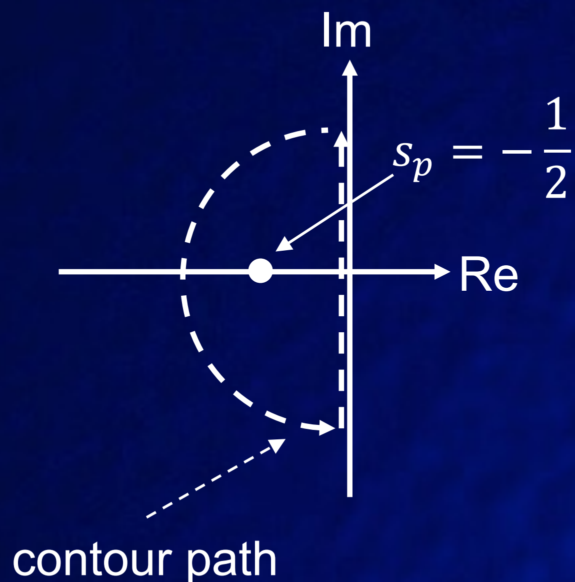

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The line integral over the semi-circular arc is then,

$$\lim_{r\to\infty}\int_{\pi/2}^{3\pi/2}\frac{1}{1+2re^{i\theta}}e^{(re^{i\theta})t}\left(rie^{i\theta}d\theta\right)$$

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Observe that, $\displaystyle \lim_{r\to\infty}\frac{re^{i\theta}}{1+2re^{i\theta}}=\lim_{r\to\infty}\frac{e^{i\theta}}{\frac{1}{r}+2e^{i\theta}}=\frac{1}{2}$

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amp Phase

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For $\frac{\pi}{2}<\theta<\frac{3\pi}{2},\sigma<0\ \ \therefore\lim_{r\to\infty}e^{\sigma t}e^{i\omega t}=e^{-\infty}e^{i\omega t}=0$

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amp Phase

For $\frac{\pi}{2}<\theta<\frac{3\pi}{2},\sigma<0$ $\therefore \lim_{r\to\infty}e^{\sigma t}e^{i\omega t}=e^{-\infty}e^{i\omega t}=0$

1/2 0

Finally, $\int_{\pi/2}^{3\pi/2}\lim_{r\to\infty}\frac{re^{i\theta}}{1+2re^{i\theta}}ie^{rte^{i\theta}}d\theta=0$

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amp Phase

For $\frac{\pi}{2}<\theta<\frac{3\pi}{2},\sigma<0$ $\therefore\lim_{r\to\infty}e^{\sigma t}e^{i\omega t}=e^{-\infty}e^{i\omega t}=0$

Finally, $\int_{\pi/2}^{3\pi/2}\lim_{r\to\infty}\frac{re^{i\theta}}{1+2re^{i\theta}}ie^{rte^{i\theta}}d\theta=0$ (1/2, 0)

Line integral around the semi-circular arc is 0 so it's okay to use contour integration to find L^{-1}.

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Whew! Inverse Laplace Transforms can be difficult to determine analytically.

It is therefore common to use tables.

http://www.ceri.memphis.edu/people/mwithers/CERI7106/other/Laplace_Table.pdf

The Chandler Wobble.

The Chandler wobble is the free nutation of the earth rotational axis due to changes in mass distribution.

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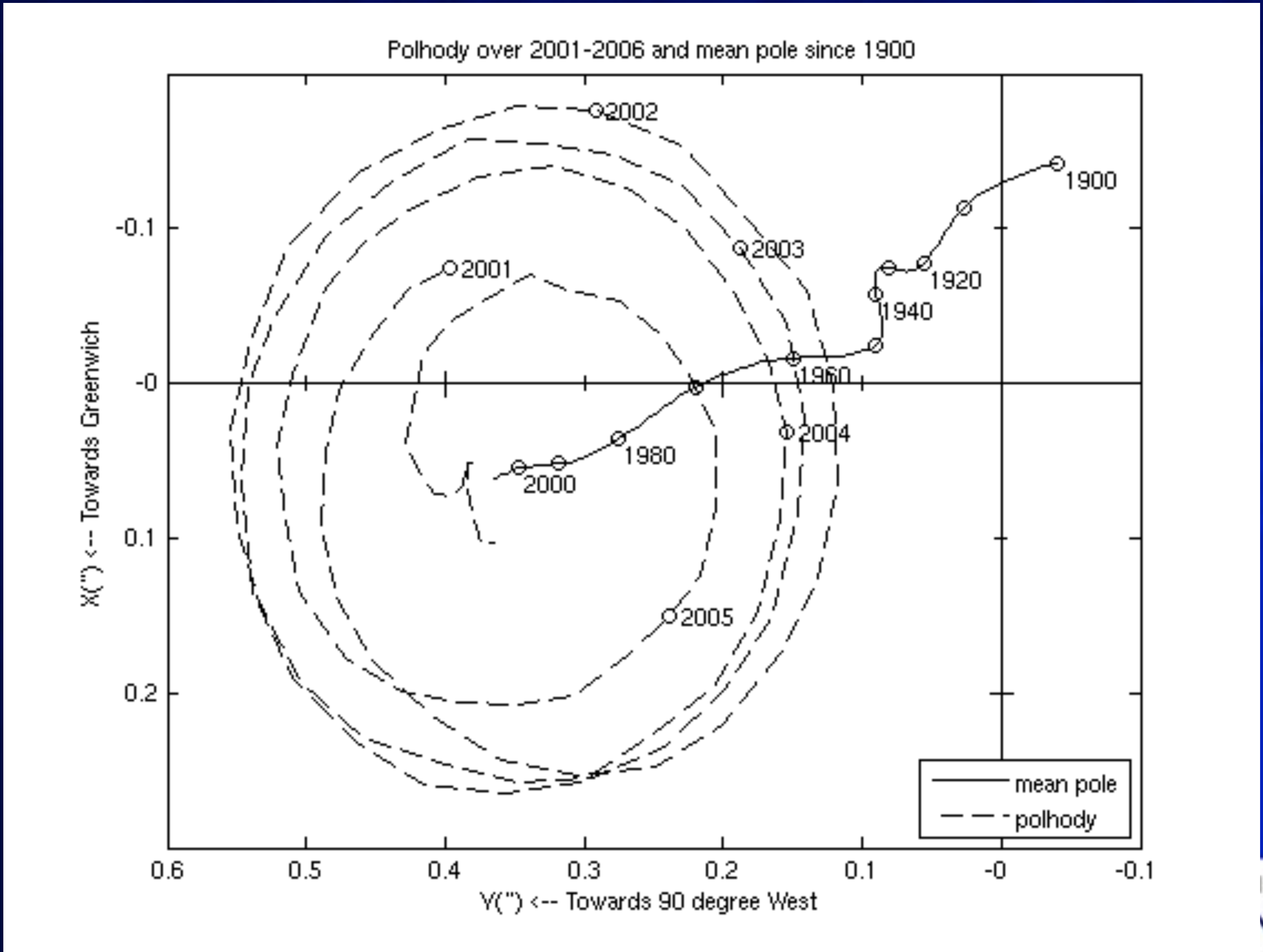
One arcsecond is about 27m. The 2004 Sumatra earthquake caused the rotation pole to move about 2.5 cm.

https://www.iers.org/IERS/EN/Science/EarthRotation/PolarMotion.html



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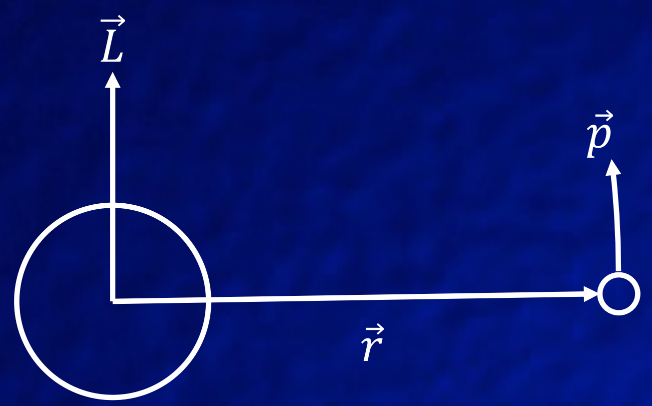

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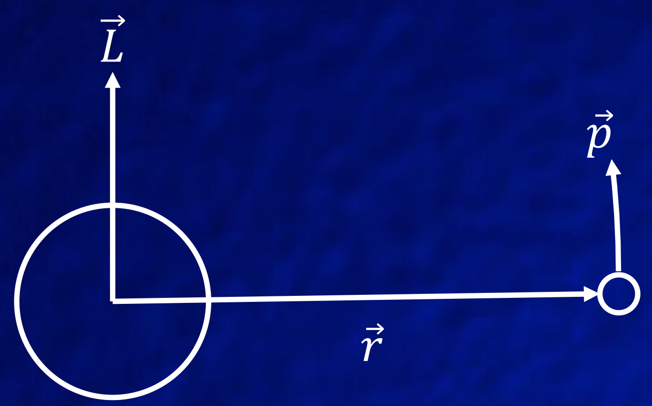

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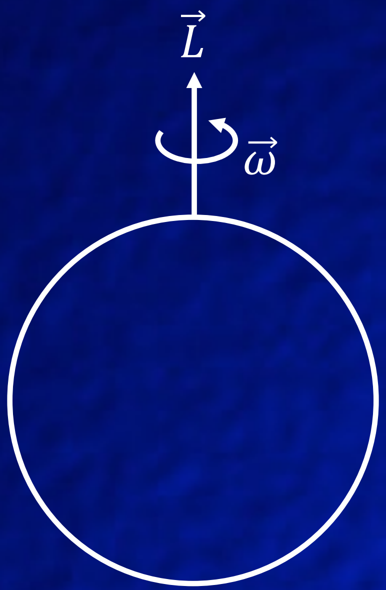
Consider a spinning figure skater. If she pulls in her arms closer to her body, she spins faster because she changed her mass distribution (reduced $\vec{r}$ for the mass of her hands and arms). This makes her spin faster to conserve angular momentum

For a rotating rigid body, angular momentum $\vec{L} = I\vec{\omega}$ where I is rotational inertia (the mass distribution) and $\vec{\omega}$ is angular frequency (e.g. rotations per time).



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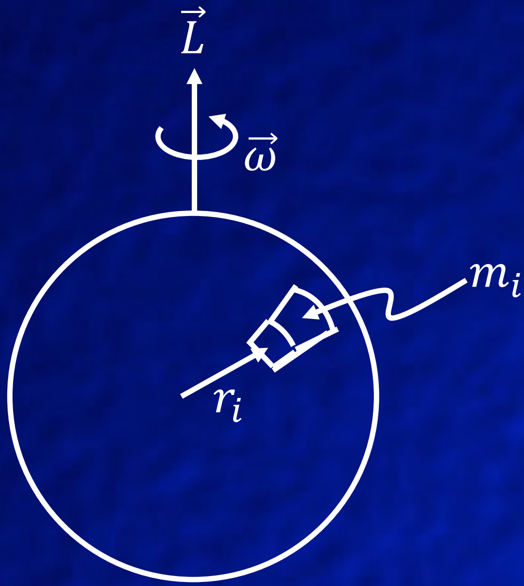

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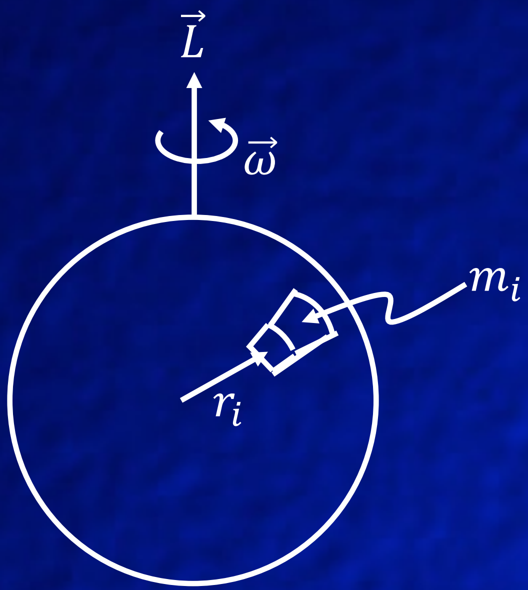

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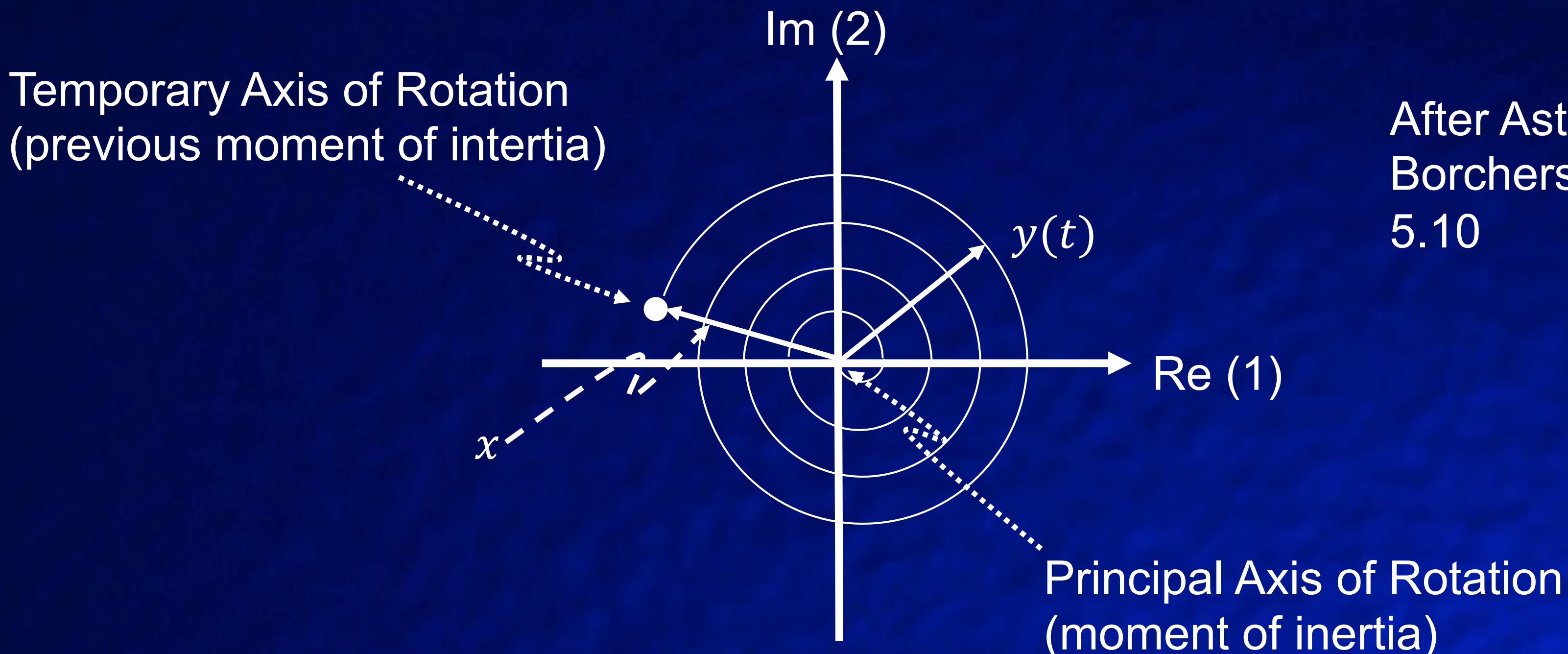
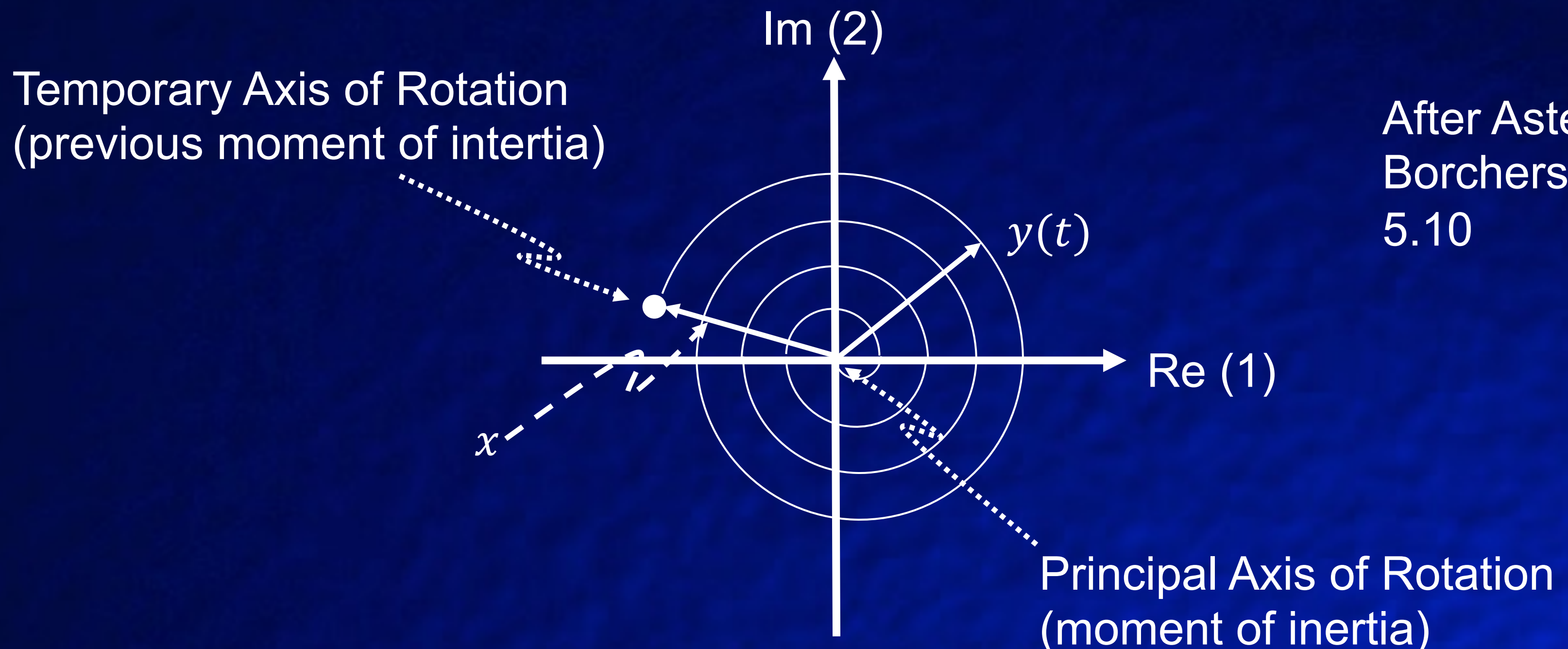
Changes in I = changes in ω with nutation depending on the distribution of the changes in I.

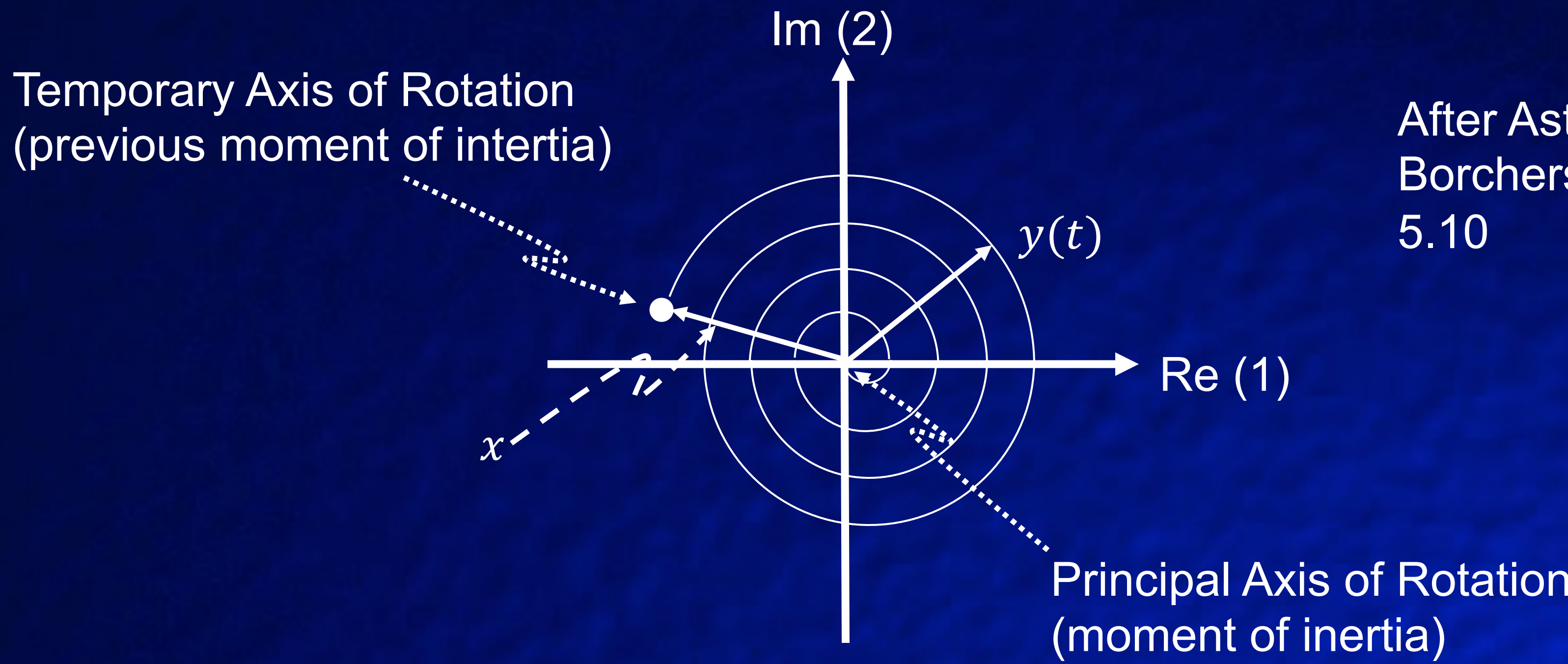


After Aster and Borchers Figure 5.10



After Aster and Borchers Figure 5.10

The equilibrium state is at (0,0), the principal axis of rotation.



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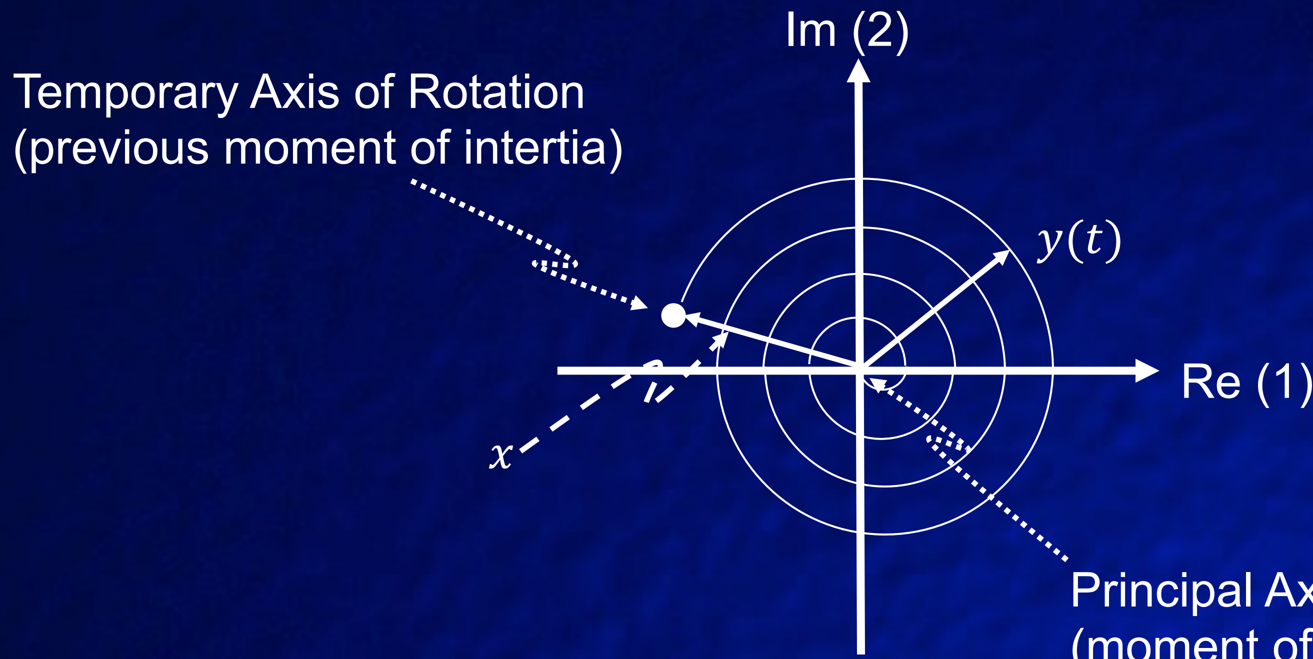

After Aster and Borchers Figure 5.10

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The system will then nutate (or wobble) about the new principal axis of rotation and the nutation will decay as $y(t)$, until the temporary axis and principal axis are aligned.

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We can think of x as the input and y as the output, then ϕ is the impulse response of the system.

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The governing differential equations for such a system are (Aster and Borchers 5.35 and 5.36):

$$\frac{\dot{y}_1}{\omega_c} + y_2 = x_2 \qquad -\frac{\dot{y}_2}{\omega_c} + y_1 = x_1$$

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ω_c is the characteristic frequency which, for a rigid body, is: $\omega_c = \frac{C-A}{C}\Omega$

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C = polar moment of inertia

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For convenience, we can combine our two axis system (1,2) of equations into a system of complex equations by setting,

$$x = x_1 + ix_2 \qquad y = y_1 + iy_2$$

$$x_1 = -\frac{\dot{y}_2}{\omega_c} + y_1$$

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$$L[x] = L\left[\frac{i\dot{y}}{\omega_c} + y\right]$$

Recall, $L[\dot{y}] = sL[y] - y(0)$

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$$x_1 = -\frac{\dot{y}_2}{\omega_c} + y_1 \qquad x = x_1 + i x_2 = \left(-\frac{\dot{y}_2}{\omega_c} + y_1\right) + i\left(\frac{\dot{y}_1}{\omega_c} + y_2\right)$$

$$x_2 = \frac{\dot{y}_1}{\omega_c} + y_2 \qquad = \frac{1}{\omega_c}(i\dot{y}_1 - \dot{y}_2) + y_1 + i y_2 \qquad y = y_1 + i y_2$$

$$= \frac{i}{\omega_c}\left(\dot{y}_1 - \frac{\dot{y}_2}{i}\right) + y = \frac{i}{\omega_c}(\dot{y}_1 + i\dot{y}_2) + y$$

$$= \frac{i\dot{y}}{\omega_c} + y$$

$$L[x] = L\left[\frac{i\dot{y}}{\omega_c} + y\right]$$

Recall, $L[\dot{y}] = sL[y] - y(0)$

Assume 0

$$X(s) = \frac{i}{\omega_c} sY(s) - y(0) + Y(s) = Y(s)\left(\frac{is}{\omega_c} + 1\right) = Y(s)\left(\frac{\omega_c + is}{\omega_c}\right)$$

$$\frac{Y(s)}{X(s)} = \Phi(s) = \frac{\omega_c}{\omega_c + is}$$

How many poles?

$$\frac{Y(s)}{X(s)} = \Phi(s) = \frac{\omega_c}{\omega_c + is}$$

Set $\omega_c + is_p = 0, \;\; \omega_c = -is_p, \;\; s_p = i\omega_c$

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Is there a problem with this transfer function?

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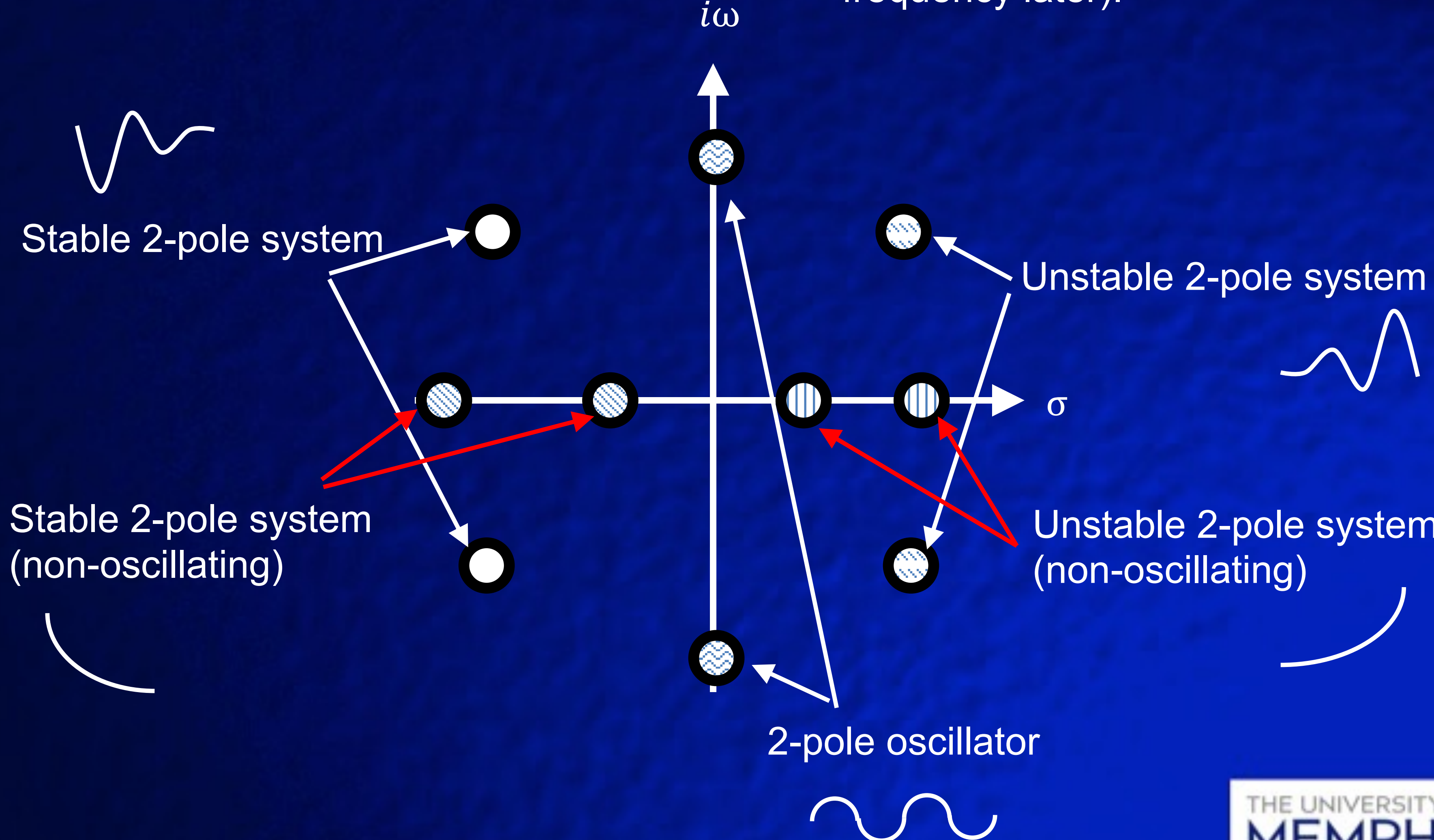
On which half of the s-plane is it located?

Aster Pole Zero Notes.

S-plane (more on complex frequency later).



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$Re(s_p) < 0, Im(s_p) > 0$ ⟶ Decay, stable.

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From tables

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This is the impulse response of our system so that for a given input, x, we can find the predicted output, $y(t) = \phi(t) * x(t)$.

Run matlab program chandler.m

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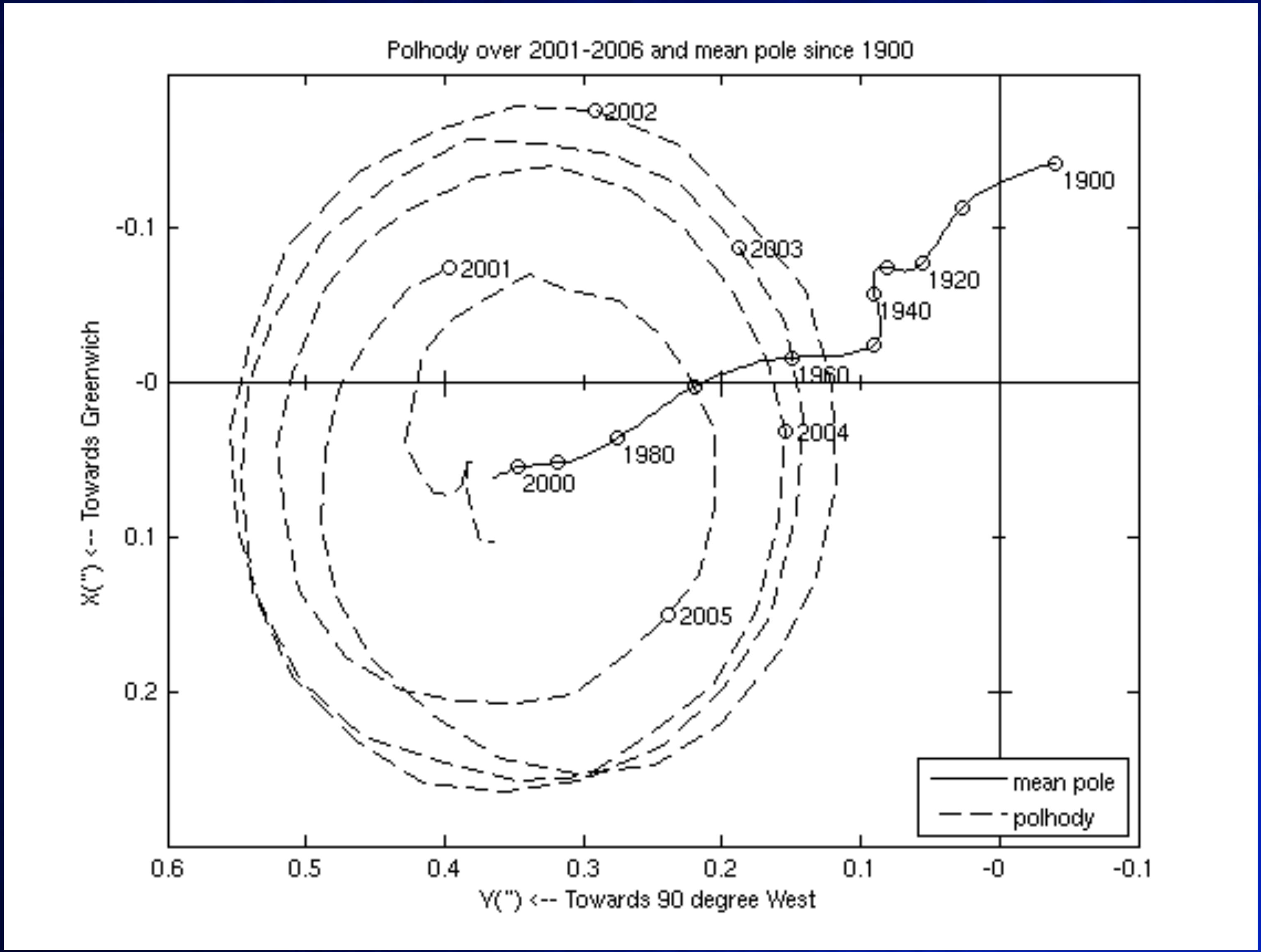
So why doesn't $q(t) = -i\omega_c e^{i\omega_c t}$ look more like what we see in the data?



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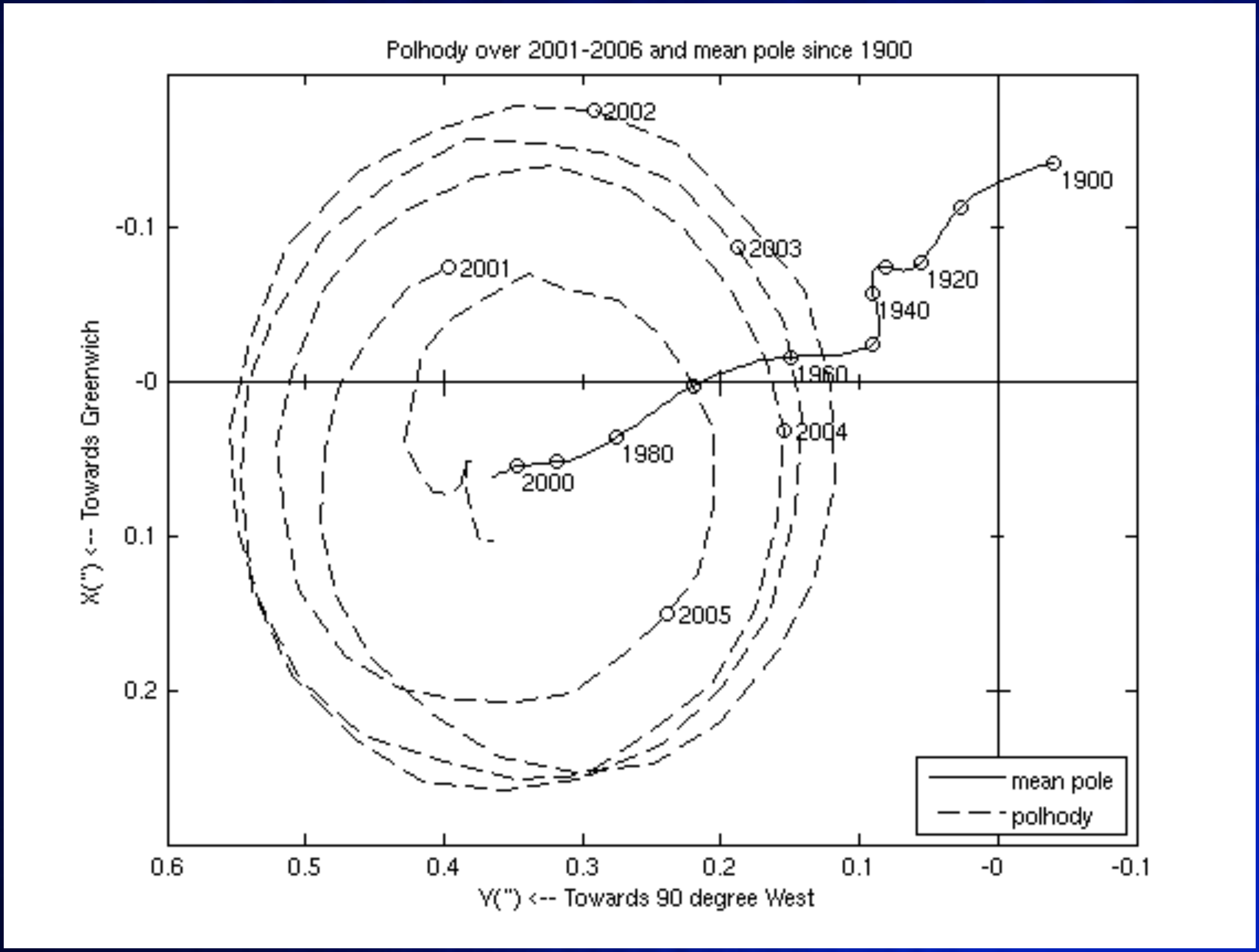
Multiple successive inputs from numerous different sources are superimposed.



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