Deconvolution

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See Aster and Borchers, Time Series Analysis, chapter 6.

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For example, if we obtain a seismogram, that's our observable, $d(t)$. And we have the impulse response of our seismometer, $g(t)$. We then wish to find the ground motion input, $m(t)$.

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$$d(t) = g(t) * m(t) = \int_{-\infty}^{\infty} g(\tau) m(t - \tau) d\tau$$

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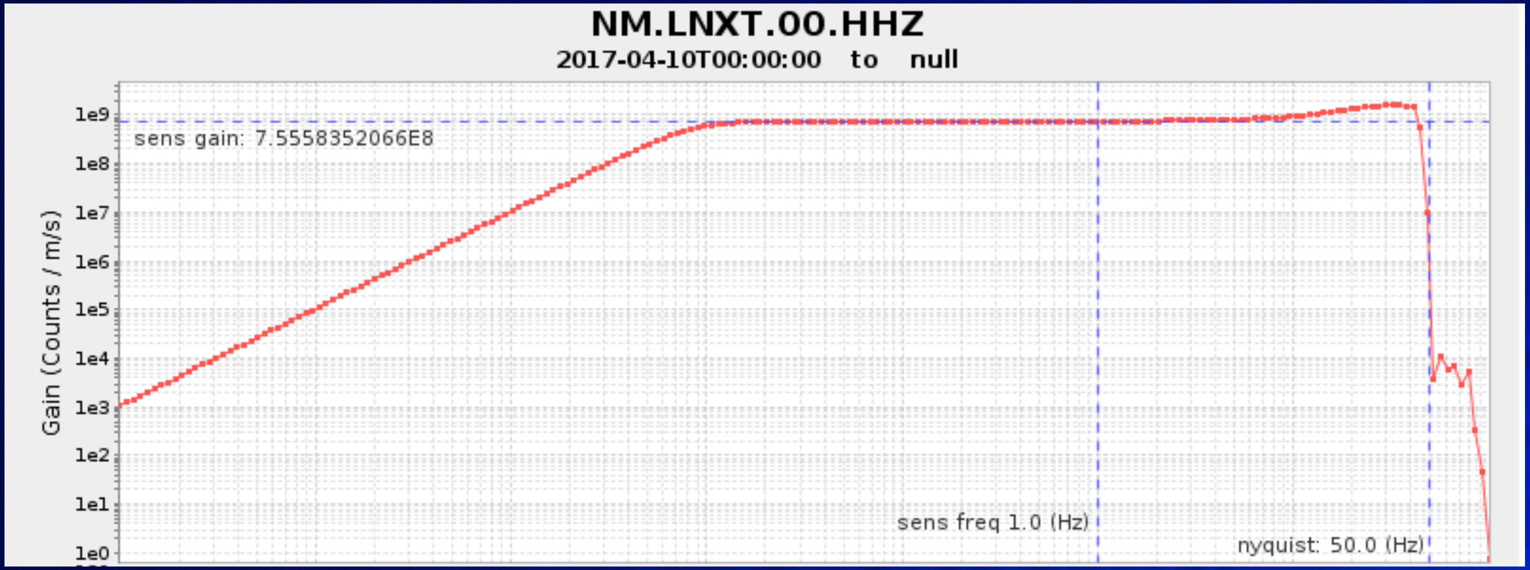
$M_k = \dfrac{D_k}{G_k \Delta t}$ → Spectral Division

Spectral division is not stable.

Recall the amplitude response for an example broadband seismometer



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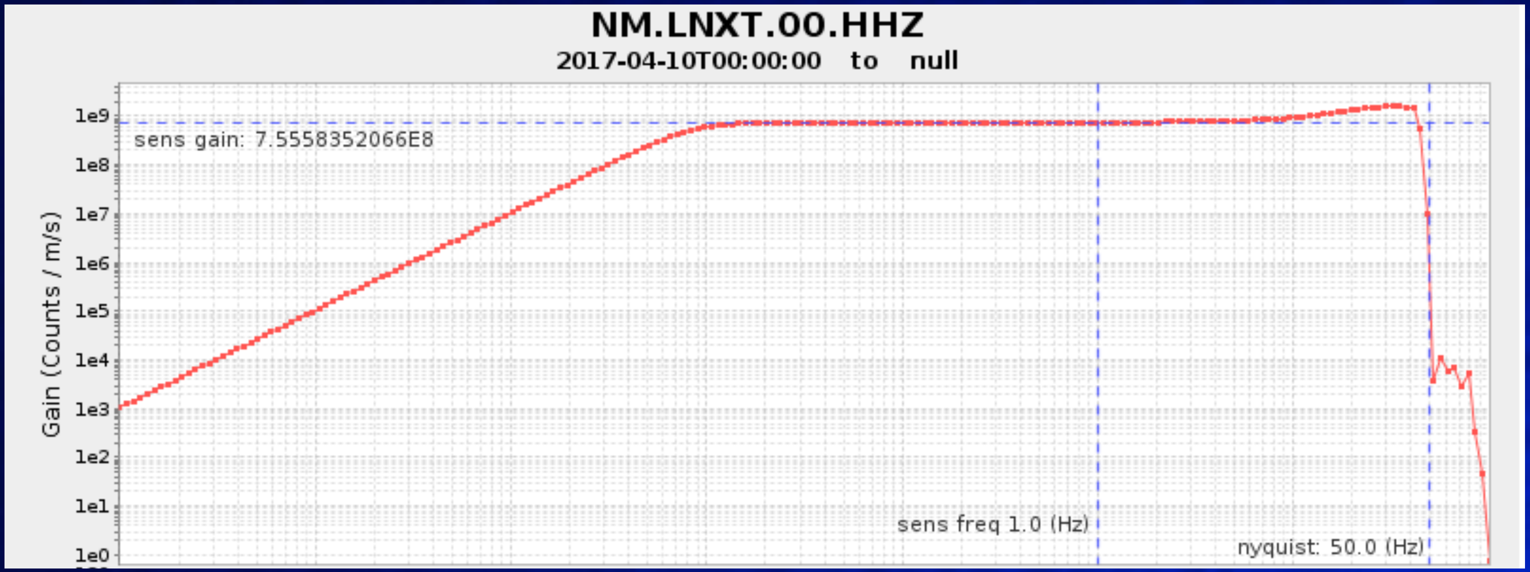

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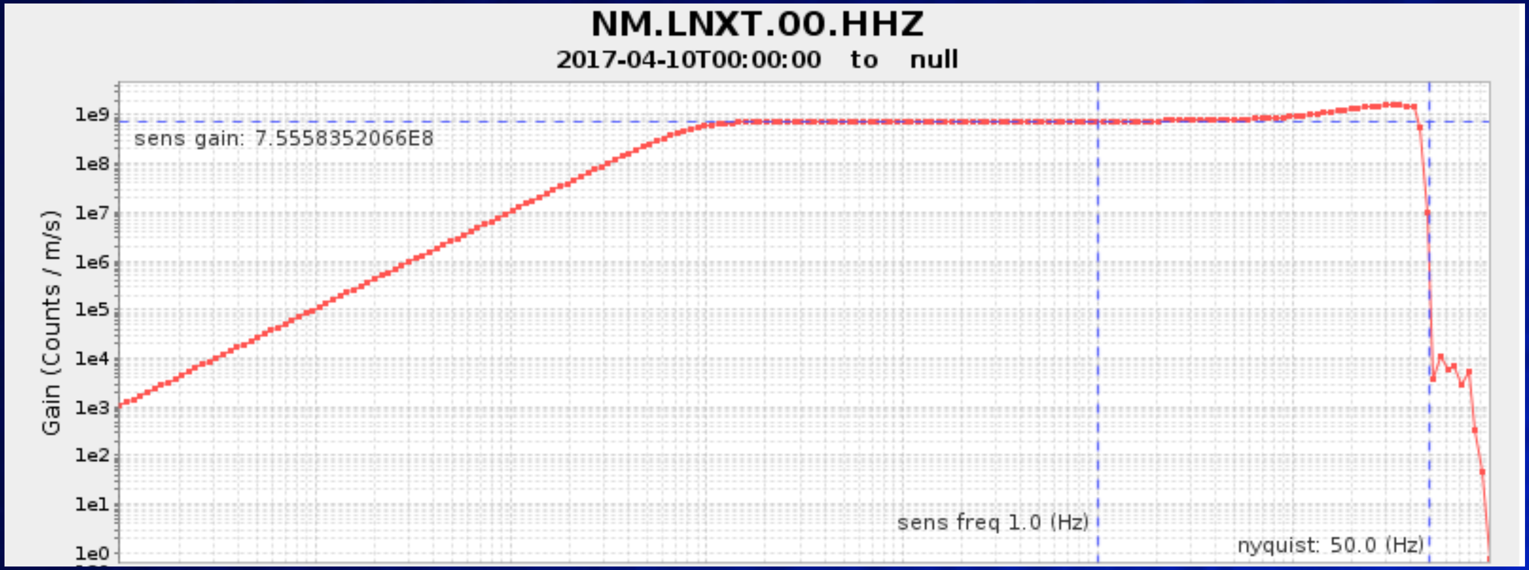

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And using spectral division, we divide by small numbers at those frequencies.

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What if the noise is added after the convolution (e.g. instrument noise)?

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Remember we’re trying to deconvolve the instrument response (G) from the data (D) to get the input ground motion (M). Where G is small, we amplify the noise.

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We wish to keep the denominator from getting small so we add a small constant to it.

$$M_k = \frac{D_k}{(G_k + \lambda)\Delta t}$$

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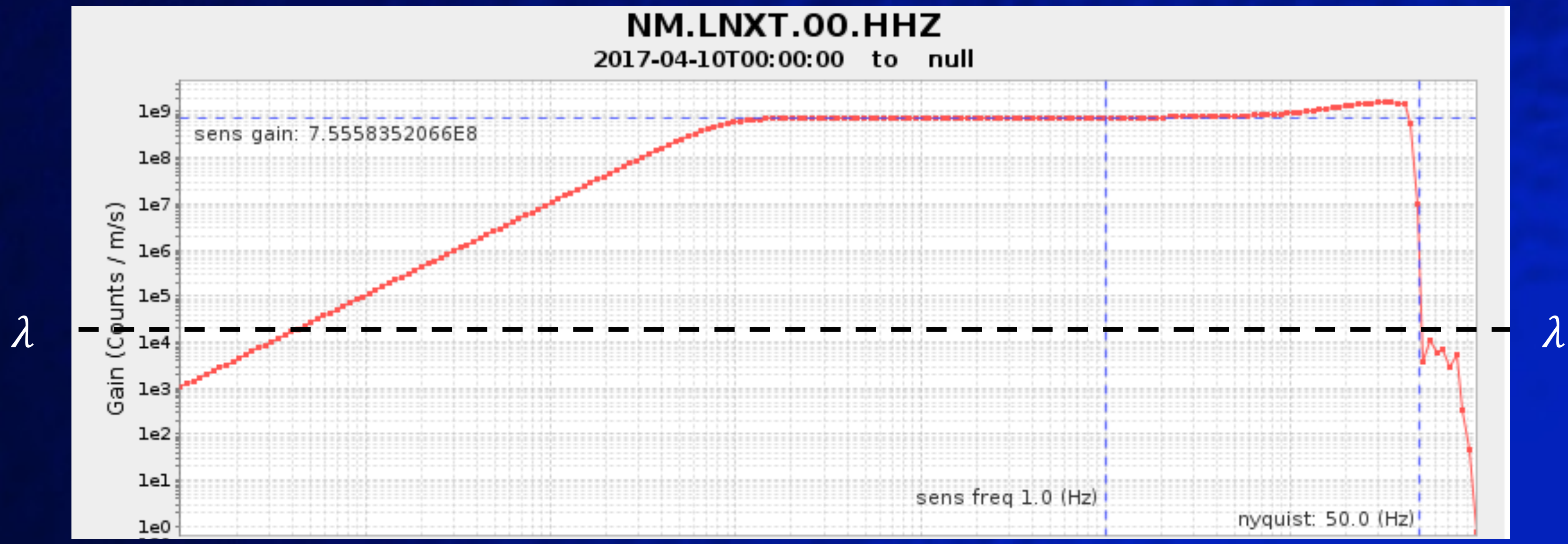
When G is large, λ will have a small effect.

When G is small, λ will have a larger effect.

When $G = -\lambda$, we have a problem.



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Preserves the phase of G, with constant amplitude w.

But be careful of 0.

$$\hat{G}(f) = \begin{cases} G(f), & |G(f)| > w \\ \dfrac{wG(f)}{|G(f)|}, & 0 < |G(f)| \leq w \\ w, & G(f) = 0 \end{cases}$$

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If you know $N(f)$ (e.g. the average instrument noise say from an ASL study for your system) then that can define w.

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As an example, let the input be, $m(t) = te^{-t}$

And the linear system be, $g(t) = e^{-5t}\sin(10t)$

Run matlab program deconvdemoA to demonstrate recovering $m(t)$ in the presence of noise using water level regularization.

Another more elegant method is Tikhonov regularization.

$$\hat{M}_k = \frac{G_k^* D_k}{(G_k^* G_k + \lambda)\Delta t}$$

When $|G_k| \gg \lambda,\ M_k = \frac{D_k}{G_k}$

When $|G_k| \ll \lambda,\ M_k$ is reduced rather than amplified.

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One could also use average pre-event noise and average event amplitude or some similar scheme to choose λ

Run matlab program deconvdemoB