Sampled Time Series

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 3.

THE UNIVERSITY OF
MEMPHIS.
Center for Earthquake Research
and Information

A normal distribution, or bell curve, is a gaussian distribution of the form,

$$f(x) = e^{-(x-\mu)^2 / 2\sigma^2}$$

A normal distribution, or bell curve, is a gaussian distribution of the form,

$$f(x) = e^{-(x-\mu)^2 / 2\sigma^2}$$

where, $\mu = mean$
$\sigma = standard\ deviation\ (\sigma^2 = variance)$

This distribution is very common (hence the name normal distribution).

The Fourier Series is conceptually similar to the FT but mathematically quite different.

The Fourier Series is conceptually similar to the FT but mathematically quite different.

We can expand any periodic function into a weighted sum of sines and cosines,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The Fourier Series is conceptually similar to the FT but mathematically quite different.

We can expand any periodic function into a weighted sum of sines and cosines,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Where,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\,dx$$

The Fourier Series in complex form,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \int_{-\pi}^{\pi} f(x) e^{inx} dx$

The Fourier Series in complex form,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$ where $c_n = \int_{-\pi}^{\pi} f(x) e^{inx} dx$

We’ll see that the discrete Fourier Transform is

$$\phi_n = \frac{1}{N} \sum_{k=0}^{N-1} \Phi_k e^{i2\pi kn/N}$$

$$\Phi_k = \sum_{n=0}^{N-1} \phi_n e^{-i2\pi kn/N}$$

The Fourier Series in complex form,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{where } c_n = \int_{-\pi}^{\pi} f(x) e^{inx} dx$$

We’ll see that the discrete Fourier Transform is

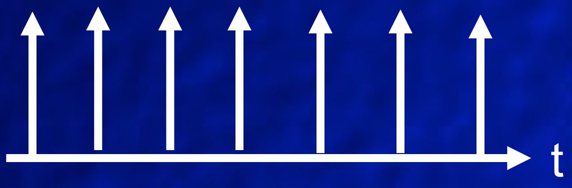
$$\left.\begin{aligned} \phi_n &= \frac{1}{N} \sum_{k=0}^{N-1} \Phi_k e^{{i2\pi kn}/{N}} \\ \Phi_k &= \sum_{n=0}^{N-1} \phi_n e^{{-i2\pi kn}/{N}} \end{aligned}\right\}$$

Discrete, band limited, finite length series.

We can use the sampling function, or replicating function also referred to as the Shah function due to its resemblance to the Cyrillic letter shah, Ш

We can use the sampling function, or replicating function also referred to as the Shah function due to its resemblance to the Cyrillic letter shah, Ш

The Shah function, $Ш(t)$, is a series of delta functions,



It is also sometimes called a dirac comb (after the dirac delta).

We’ll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a Guassian.

We'll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a Guassian.

Let $\phi(t) = e^{-\pi t^2}$

We’ll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a Guassian.

Let $\phi(t) = e^{-\pi t^2}$

$$F[\phi(t)] = \Phi(f) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi f t} dt$$

We'll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a Guassian.

Let $\phi(t) = e^{-\pi t^2}$

$$F[\phi(t)] = \Phi(f) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi f t} dt$$

$$= e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t^2+2ift-f^2)} dt$$

(Remember this trick, it will be useful later)

We'll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a Guassian.

Let $\phi(t) = e^{-\pi t^2}$

$$F[\phi(t)] = \Phi(f) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi f t} dt$$

$$= e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t^2+2ift-f^2)} dt$$

(Remember this trick, it will be useful later)

$$(t+if)^2 = t^2 + 2ift - f^2$$

We'll use the dirac comb to help mathematically develop the Discrete Fourier Transform from the continuous but first we need to find the FT of a gaussian.

Let $\phi(t) = e^{-\pi t^2}$

$$F[\phi(t)] = \Phi(f) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi f t} dt$$

$$= e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t^2+2ift-f^2)} dt$$

(Remember this trick, it will be useful later)

$$(t + if)^2 = t^2 + 2ift - f^2$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t+if)^2} dt$$

THE UNIVERSITY OF MEMPHIS. Center for Earthquake Research and Information

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t+if)^2} dt$$

Let $\tau = t + if,\ d\tau = dt$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t+if)^2} dt$$

Let $\tau = t + if, \; d\tau = dt$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau$$

From tables, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\therefore \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau = 1$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t+if)^2} dt$$

Let $\tau = t + if,\; d\tau = dt$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau$$

From tables, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\therefore \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau = 1$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau = e^{-\pi f^2}$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi(t+if)^2} dt$$

Let $\tau = t + if,\ d\tau = dt$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi \tau^2} d\tau$$

From tables, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\therefore \int_{-\infty}^{\infty} e^{-\pi\tau^2} d\tau = 1$$

$$\Phi(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi \tau^2} d\tau = e^{-\pi f^2}$$

The Fourier Transform of a gaussian is another gaussian!

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Now consider the function,

$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

Now consider the function,

$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

$e^{-\pi\tau^2 t^2}$ is a broad gaussian of width ${}^{1}/_{\tau}$ (it’s standard deviation).

Now consider the function,

$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

$e^{-\pi\tau^2 t^2}$ is a broad gaussian of width $1/\tau$ (it's standard deviation).

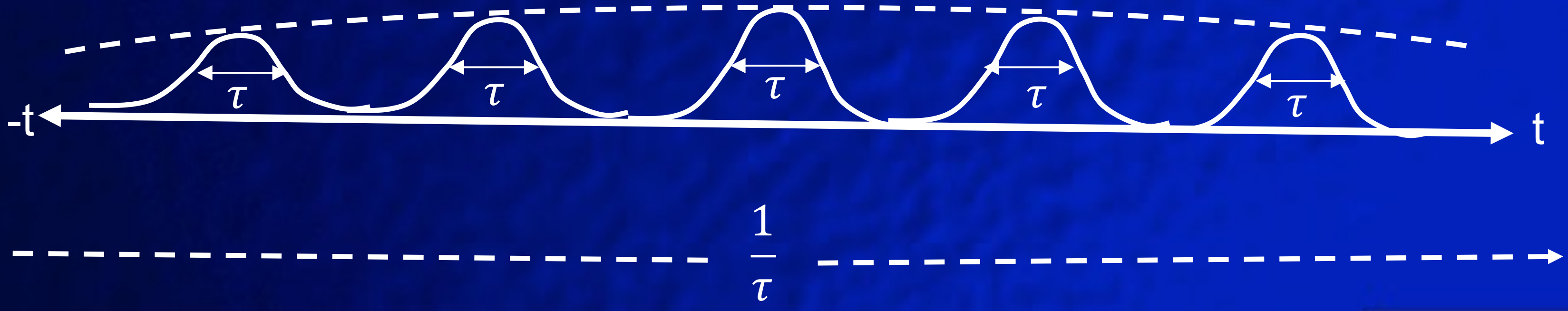
The sum is a series of gaussian “spikes” each of width τ.

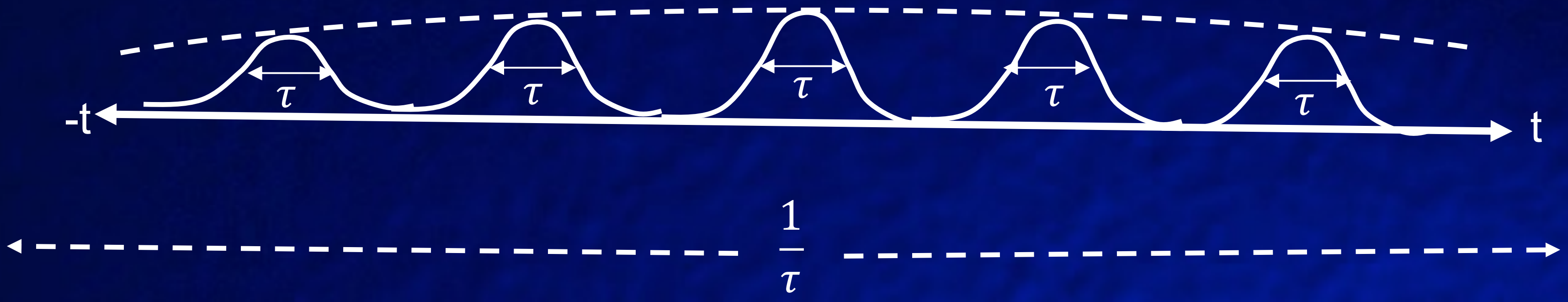
Now consider the function,

$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

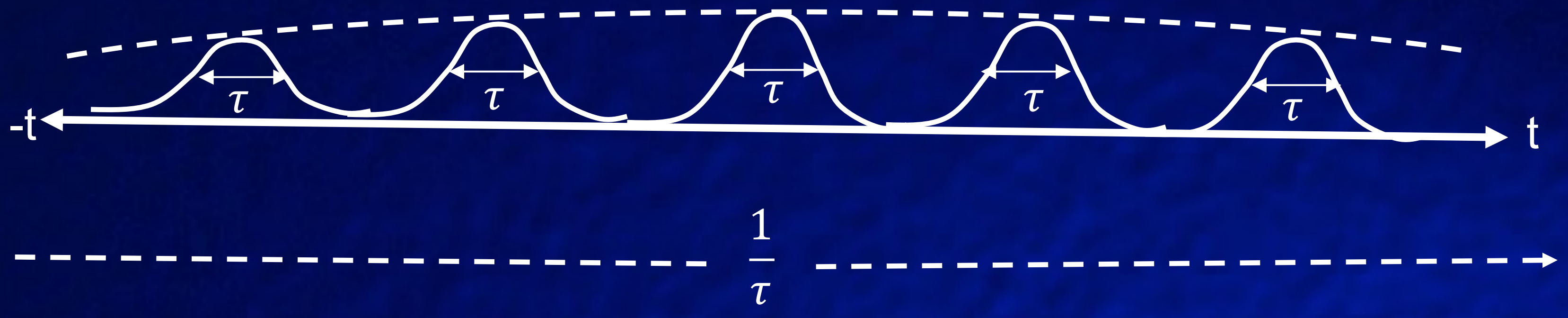
$e^{-\pi\tau^2 t^2}$ is a broad gaussian of width ${}^{1}/_{\tau}$ (it's standard deviation).

The sum is a series of gaussian “spikes” each of width τ.





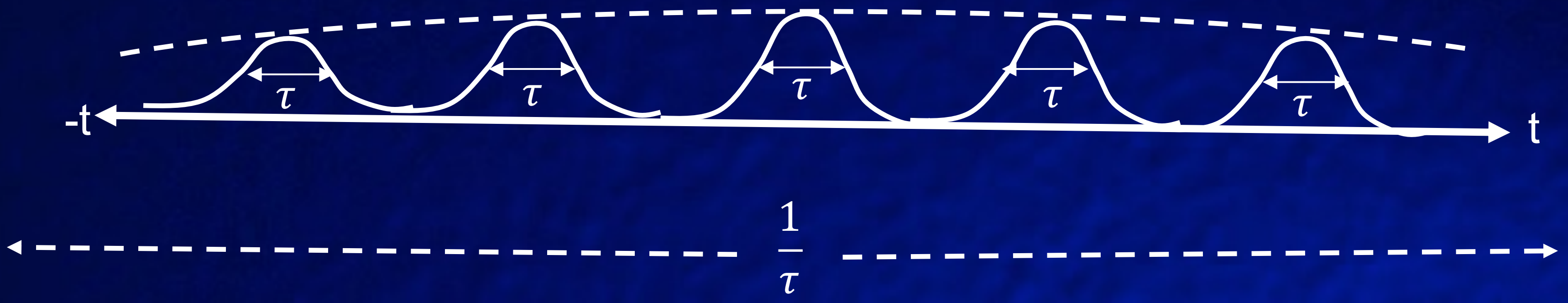
$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$



$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

In the limit, $\lim_{\tau \to 0} f(t)$, the broad envelope converges to 1 and the “spikes” become infinitely narrow and tall so that,

THE UNIVERSITY OF MEMPHIS. Center for Earthquake Research and Information



$$f(t) = \frac{1}{\tau} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2}$$

In the limit, $\lim_{\tau \to 0} f(t)$, the broad envelope converges to 1 and the “spikes” become infinitely narrow and tall so that,

$$\text{Ш}(t) = \lim_{\tau \to 0} f(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

We leave as an exercise for the student to show that the Fourier Series of our gaussian spikes is

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2/\tau^2} = \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} e^{i2\pi n t}$$

We leave as an exercise for the student to show that the Fourier Series of our gaussian spikes is

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2 / \tau^2} = \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} e^{i2\pi nt}$$ so that,

$$\text{Ш}(t) = \lim_{\tau\to 0} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} e^{i2\pi nt}$$

We leave as an exercise for the student to show that the Fourier Series of our gaussian spikes is

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi(t-n)^2/\tau^2} = \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} e^{i2\pi nt}$$

so that,

$$Ш(t) = \lim_{\tau\to 0} e^{-\pi\tau^2 t^2} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} e^{i2\pi nt}$$

$$F[\,Ш(t)] = \lim_{\tau\to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} F\left[e^{-\pi\tau^2 t^2} e^{i2\pi nt}\right]$$

Recall the shifting property,

$$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$$

Recall the sifting property,

$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$ similarly, $F^{-1}[\Phi(f-n)] = e^{i\pi nt}\phi(t)$

Recall the sifting property,

$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$ similarly, $F^{-1}[\Phi(f-n)] = e^{i\pi nt}\phi(t)$

A phase shift of nt in time is equivalent to a shift of f-n in frequency. Then,

$$F\left[e^{-\pi\tau^2t^2}e^{i2\pi nt}\right] = F\left[e^{-\pi\tau^2t^2}\right]_{f=f-n}$$

Recall the sifting property,

$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$ similarly, $F^{-1}[\Phi(f-n)] = e^{i\pi nt}\phi(t)$

A phase shift of nt in time is equivalent to a shift of f-n in frequency. Then,

$$F\left[e^{-\pi\tau^2t^2}e^{i2\pi nt}\right] = F\left[e^{-\pi\tau^2t^2}\right]_{f=f-n}$$

$$F[\amalg(t)] = \lim_{\tau\to 0}\sum_{n=-\infty}^{\infty} e^{-\pi\tau^2n^2}F\left[e^{-\pi\tau^2t^2}\right]_{f=f-n}$$

Recall the sifting property,

$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$ similarly, $F^{-1}[\Phi(f-n)] = e^{i\pi nt}\phi(t)$

A phase shift of nt in time is equivalent to a shift of f-n in frequency. Then,

$$F\left[e^{-\pi\tau^2t^2}e^{i2\pi nt}\right] = F\left[e^{-\pi\tau^2t^2}\right]_{f=f-n}$$

$$F[\text{Ш}(t)] = \lim_{\tau\to 0}\sum_{n=-\infty}^{\infty} e^{-\pi\tau^2n^2}F\left[e^{-\pi\tau^2t^2}\right]_{f=f-n}$$

Recall the scaling property, $F[\phi(at)] = \frac{1}{a}\Phi\left(\frac{f}{a}\right)$ and knowing the FT of a gaussian is another gaussian,

Recall the sifting property,

$F[\phi(t-a)] = e^{i2\pi af}\Phi(f)$ similarly, $F^{-1}[\Phi(f-n)] = e^{i\pi nt}\phi(t)$

A phase shift of nt in time is equivalent to a shift of f-n in frequency. Then,

$$F\left[e^{-\pi\tau^2 t^2} e^{i2\pi nt}\right] = F\left[e^{-\pi\tau^2 t^2}\right]_{f=f-n}$$

$$F[\,\text{Ш}(t)] = \lim_{\tau\to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} F\left[e^{-\pi\tau^2 t^2}\right]_{f=f-n}$$

Recall the scaling property, $F[\phi(at)] = \frac{1}{a}\Phi\left(\frac{f}{a}\right)$ and knowing the FT of a gaussian is another gaussian,

$$F\left[e^{-\pi\tau^2 t^2}\right]_{f=f-n} = \frac{1}{\tau} e^{-\pi(f-n)^2/\tau^2}$$

$$F[\text{Ш}(t)] = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} \frac{1}{\tau} e^{-\pi(f-n)^2/\tau^2}$$

$$F[\text{Ш}(t)] = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} \frac{1}{\tau} e^{-\pi(f-n)^2/\tau^2}$$

$$= \text{Ш}[f]$$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

$$F[\text{Ш}(t)] = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} \frac{1}{\tau} e^{-\pi(f-n)^2/\tau^2}$$

$$= \text{Ш}[f]$$

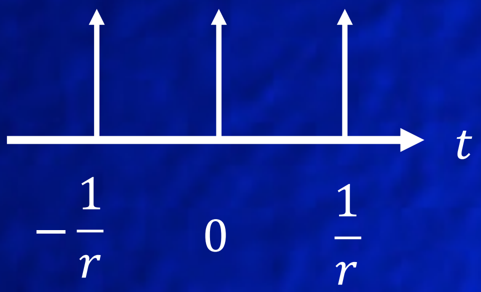
Now let $\psi(t) = \phi(t) \cdot r \cdot \text{Ш}(rt)$ where r is the sampling rate.

$$F[\text{Ш}(t)] = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} e^{-\pi\tau^2 n^2} \frac{1}{\tau} e^{-\pi(f-n)^2/\tau^2}$$

$$= \text{Ш}[f]$$

Now let $\psi(t) = \phi(t) \cdot r \cdot \text{Ш}(rt)$ where r is the sampling rate.

If the rate is in samples per second, then samples are separated in time by $\frac{1}{r}$ seconds.



$\psi(t) = \phi(t) \cdot r \cdot \Pi(rt)$

$F[\psi(t)] = F[\phi(t)] * F[\mathrm{r}\Pi(rt)]$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

$\psi(t) = \phi(t) \cdot r \cdot \Pi(rt)$ $F[\psi(t)] = F[\phi(t)] * F[\mathrm{r}\Pi(rt)]$

$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(rt)e^{-i2\pi ft}dt$ Let $\tau = rt$ $d\tau = rdt$

$\psi(t) = \phi(t) \cdot r \cdot \Pi(rt)$ $F[\psi(t)] = F[\phi(t)] * F[\mathrm{r}\Pi(rt)]$

$$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(rt)e^{-i2\pi ft}dt$$

Let $\tau = rt$ $d\tau = rdt$

$$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(\tau)e^{-i2\pi f^{\tau}/_{r}}\frac{d\tau}{r}$$

$\psi(t) = \phi(t) \cdot r \cdot \Pi(rt)$ $\quad F[\psi(t)] = F[\phi(t)] * F[\mathrm{r}\Pi(rt)]$

$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(rt)e^{-i2\pi ft}dt$ Let $\tau = rt$ $\quad d\tau = rdt$

$$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(\tau)e^{-i2\pi f^{\tau}/_{r}} \frac{d\tau}{r}$$

$$= \Phi(f) * \Pi\left(\frac{f}{r}\right)$$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

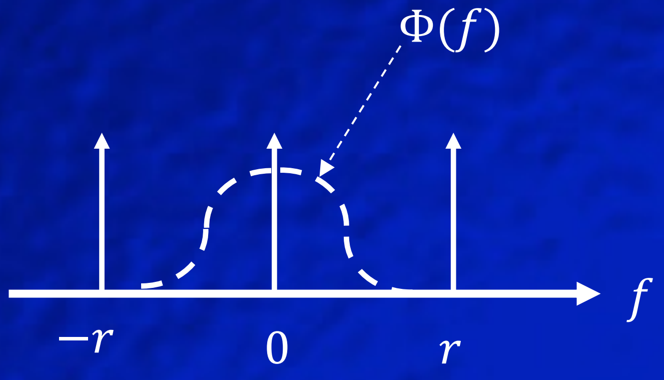
$$\psi(t) = \phi(t) \cdot r \cdot \Pi(rt) \qquad F[\psi(t)] = F[\phi(t)] * F[\mathrm{r}\Pi(rt)]$$

$$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(rt)e^{-i2\pi ft}dt$$

Let $\tau = rt$ $\quad d\tau = rdt$

$$\Psi(f) = \Phi(f) * \int_{-\infty}^{\infty} r\Pi(\tau)e^{-i2\pi f^{\tau}/_{r}}\frac{d\tau}{r}$$

$$= \Phi(f) * \Pi\left(\frac{f}{r}\right)$$



$\Pi\left(\frac{f}{r}\right)$ replicates $\Phi(f)$ at intervals of r.

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \Pi\left(\frac{f}{r}\right)$

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \Pi\left(\frac{f}{r}\right)$

$$F^{-1}[\Psi(f)] = \phi(t) * F^{-1}\left[\frac{1}{r}\Pi\left(\frac{f}{r}\right)\right]$$

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \prod\left(\frac{f}{r}\right)$

$$F^{-1}[\Psi(f)] = \phi(t) * F^{-1}\left[\frac{1}{r}\prod\left(\frac{f}{r}\right)\right]$$

$$= \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\prod\left(\frac{f}{r}\right) e^{i2\pi f t} df$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \Pi\left(\frac{f}{r}\right)$

$$F^{-1}[\Psi(f)] = \phi(t) * F^{-1}\left[\frac{1}{r}\Pi\left(\frac{f}{r}\right)\right]$$

$$= \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi\left(\frac{f}{r}\right) e^{i2\pi ft} df \qquad v = \frac{f}{r} \qquad dv = \frac{1}{r} df$$

$$\psi(t) = \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi(v) e^{i2\pi vrt} r dv$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \Pi\left(\frac{f}{r}\right)$

$$F^{-1}[\Psi(f)] = \phi(t) * F^{-1}\left[\frac{1}{r}\Pi\left(\frac{f}{r}\right)\right]$$

$$= \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi\left(\frac{f}{r}\right) e^{i2\pi f t} df \qquad v = \frac{f}{r} \qquad dv = \frac{1}{r} df$$

$$\psi(t) = \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi(v) e^{i2\pi v r t} r dv$$

$$= \phi(t) * \Pi(rt)$$

Likewise, if $\Psi(f) = \Phi(f) \cdot \frac{1}{r} \cdot \Pi\left(\frac{f}{r}\right)$

$$F^{-1}[\Psi(f)] = \phi(t) * F^{-1}\left[\frac{1}{r}\Pi\left(\frac{f}{r}\right)\right]$$

$$= \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi\left(\frac{f}{r}\right) e^{i2\pi f t} df \qquad v = \frac{f}{r} \qquad dv = \frac{1}{r} df$$

$$\psi(t) = \phi(t) * \int_{-\infty}^{\infty} \frac{1}{r}\Pi(v) e^{i2\pi v r t} r dv$$

$$= \phi(t) * \Pi(rt)$$

Digitizing in t makes $\Phi(f)$ periodic (replicates).

Digitizing in f makes $\phi(t)$ periodic.

From Aster’s notes:

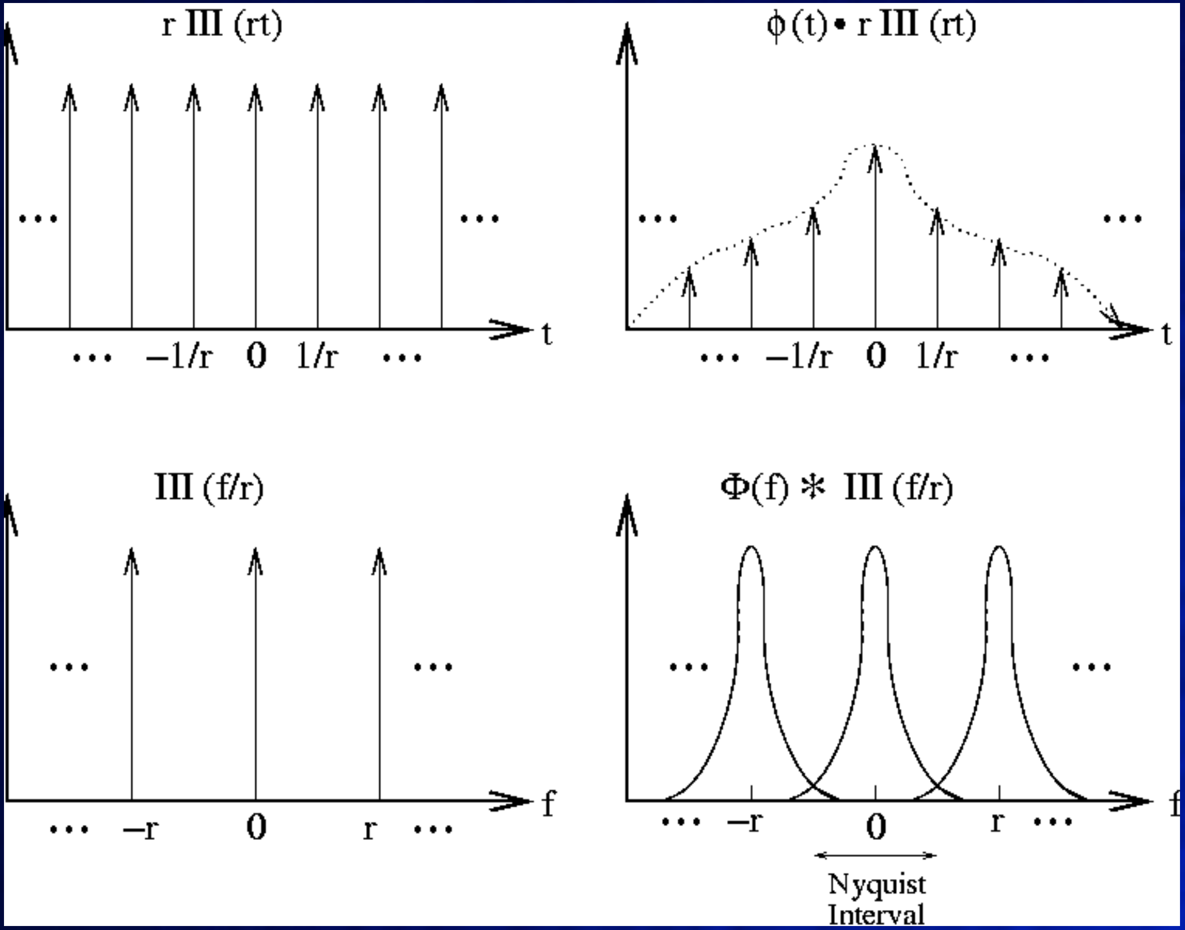
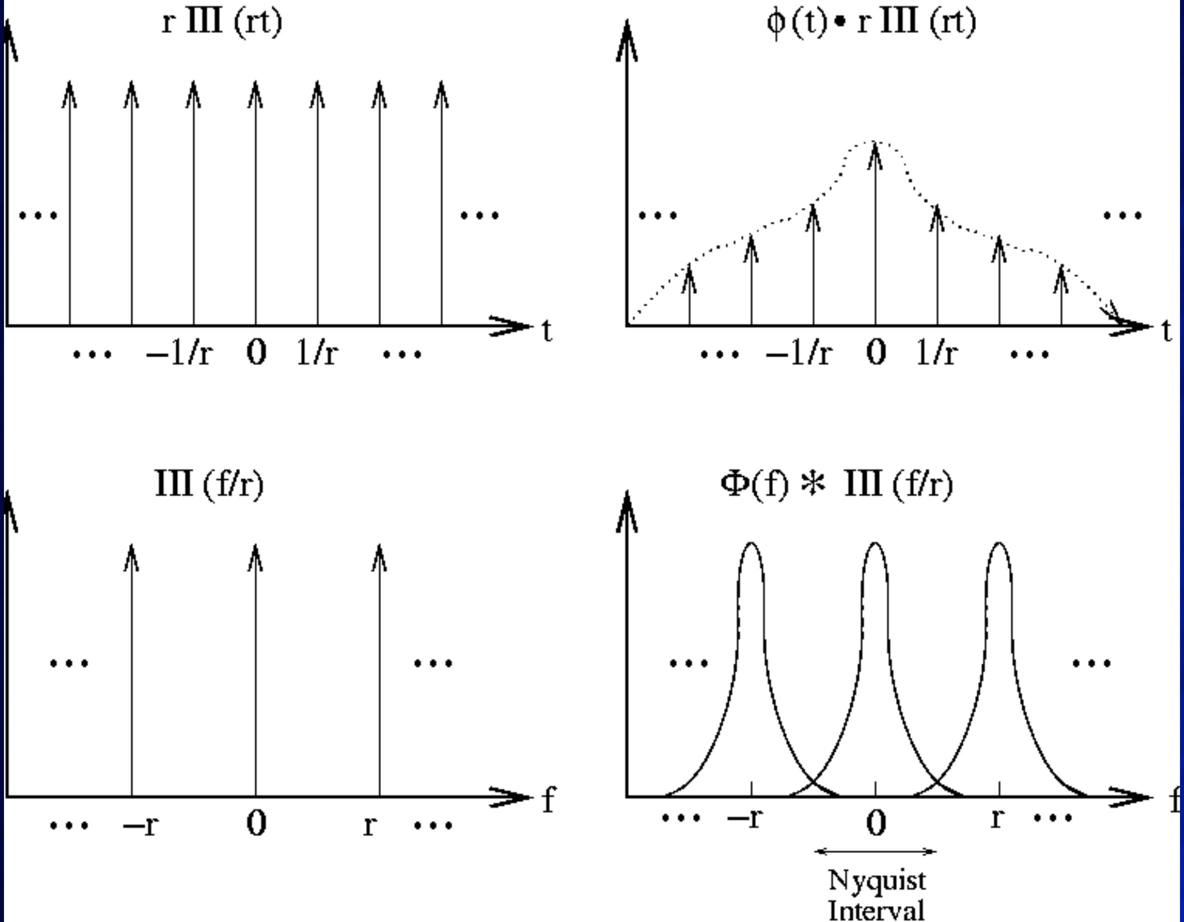


Figure 1: The Shah function and its Fourier Transform; Fourier Transform of a Sampled Function (slightly aliased)

From Aster’s notes:

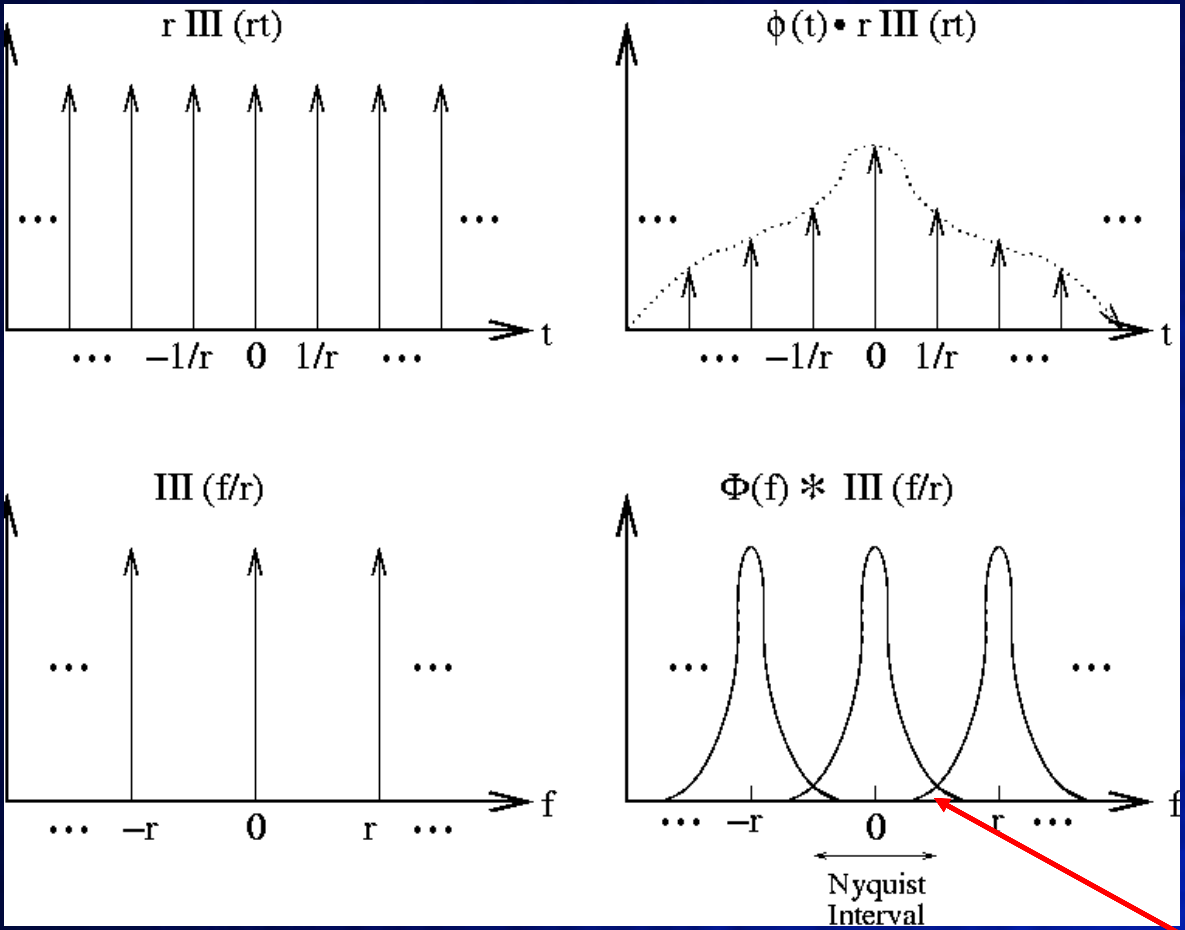


$\phi(t)$ is completely recoverable from $\Phi(f)$ if $\Phi(f)$ is band limited between $\pm\frac{r}{2}$.

Figure 1: The Shah function and its Fourier Transform;
Fourier Transform of a Sampled Function (slightly aliased)

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

From Aster’s notes:



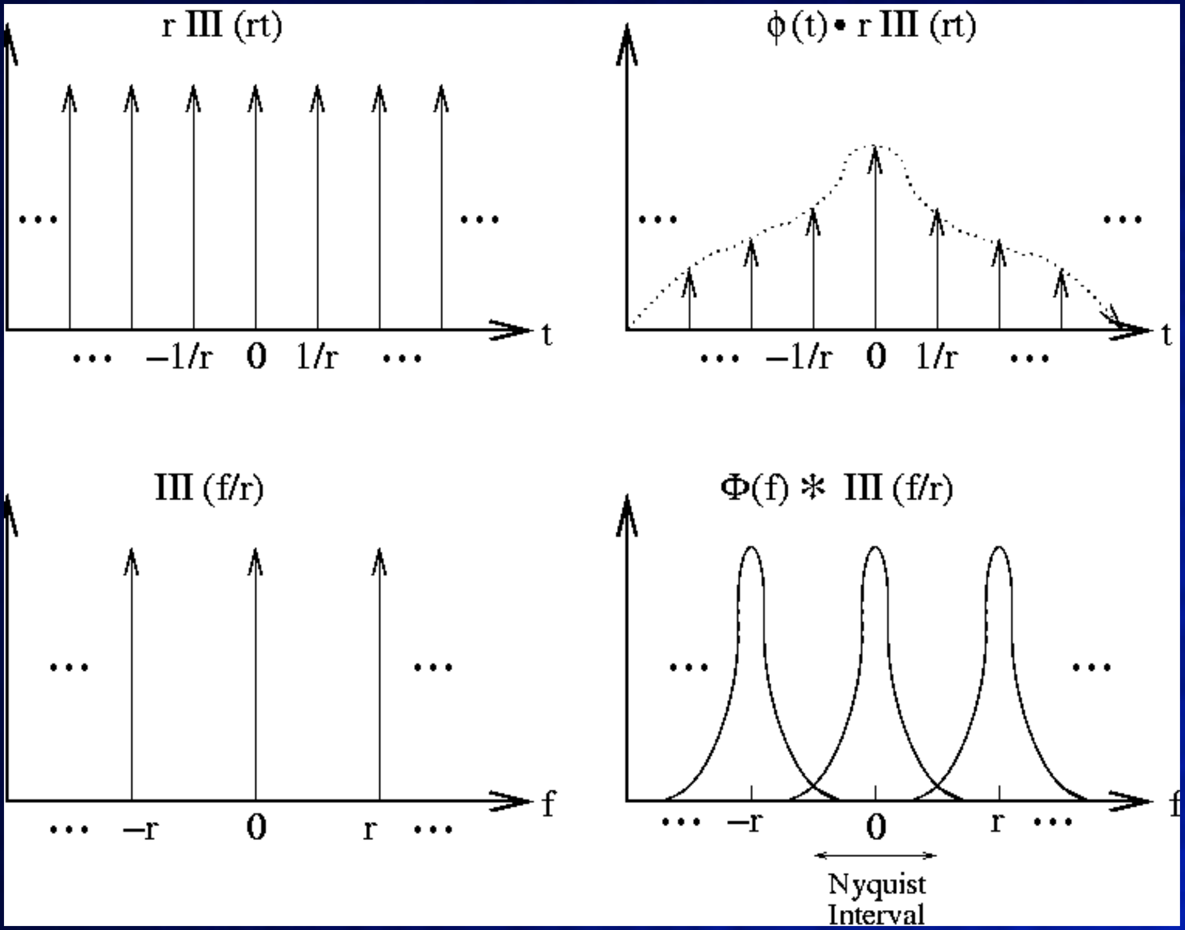
$\phi(t)$ is completely recoverable from $\Phi(f)$ if $\Phi(f)$ is band limited between $\pm\frac{r}{2}$.

If $\Phi(f)$ is not band limited between $\pm\frac{r}{2}$ then it is said to be aliased.

Aster and Borchers Figure 3.1: The Shah function and its Fourier Transform; Fourier Transform of a Sampled Function (slightly aliased)

Aliased overlap.

From Aster’s notes:



Aster and Borchers Figure 3.1: The Shah function and its Fourier Transform; Fourier Transform of a Sampled Function (slightly aliased)

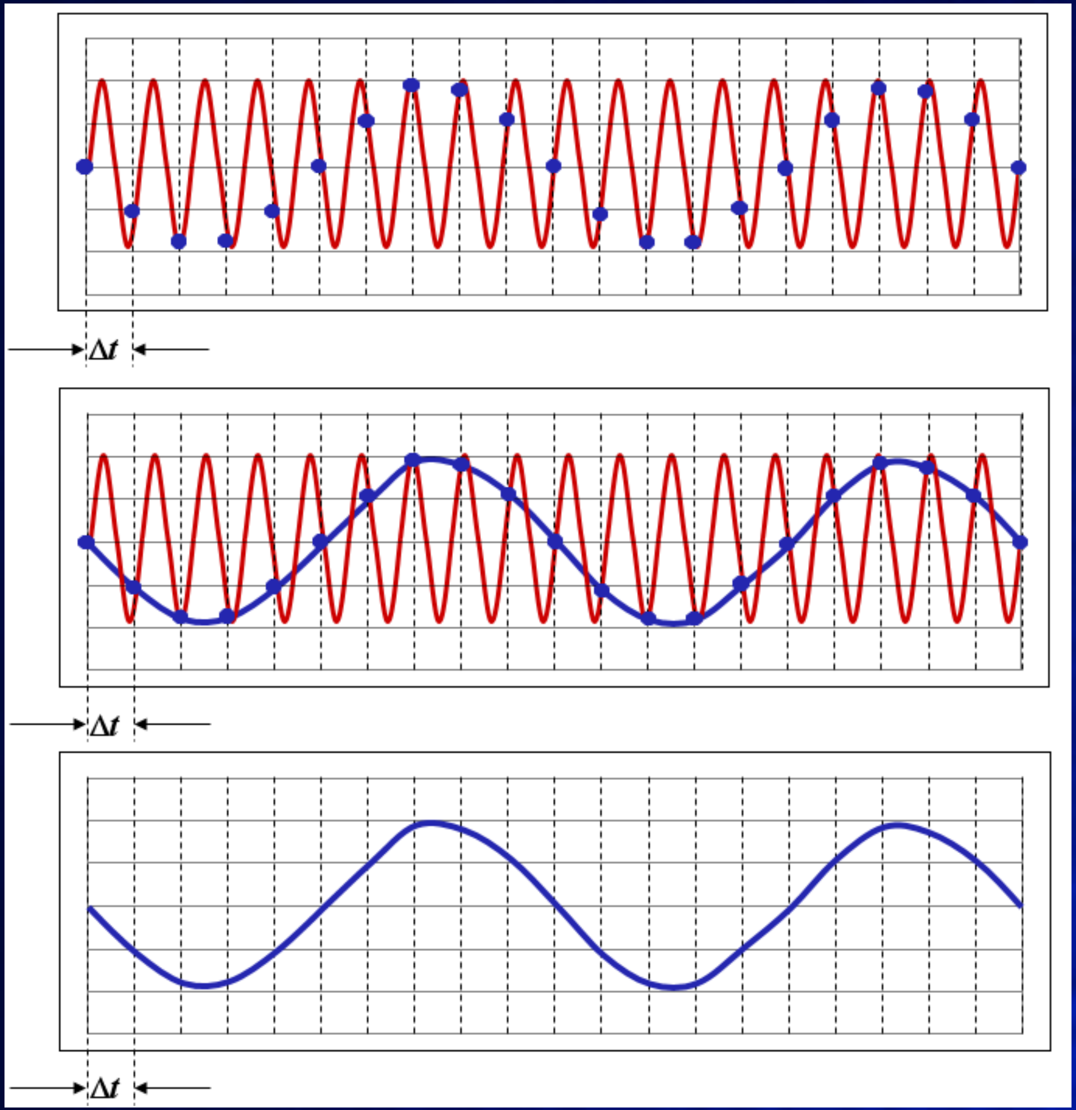
$\phi(t)$ is completely recoverable from $\Phi(f)$ if $\Phi(f)$ is band limited between $\pm\frac{r}{2}$.

If $\Phi(f)$ is not band limited between $\pm\frac{r}{2}$ then it is said to be aliased.

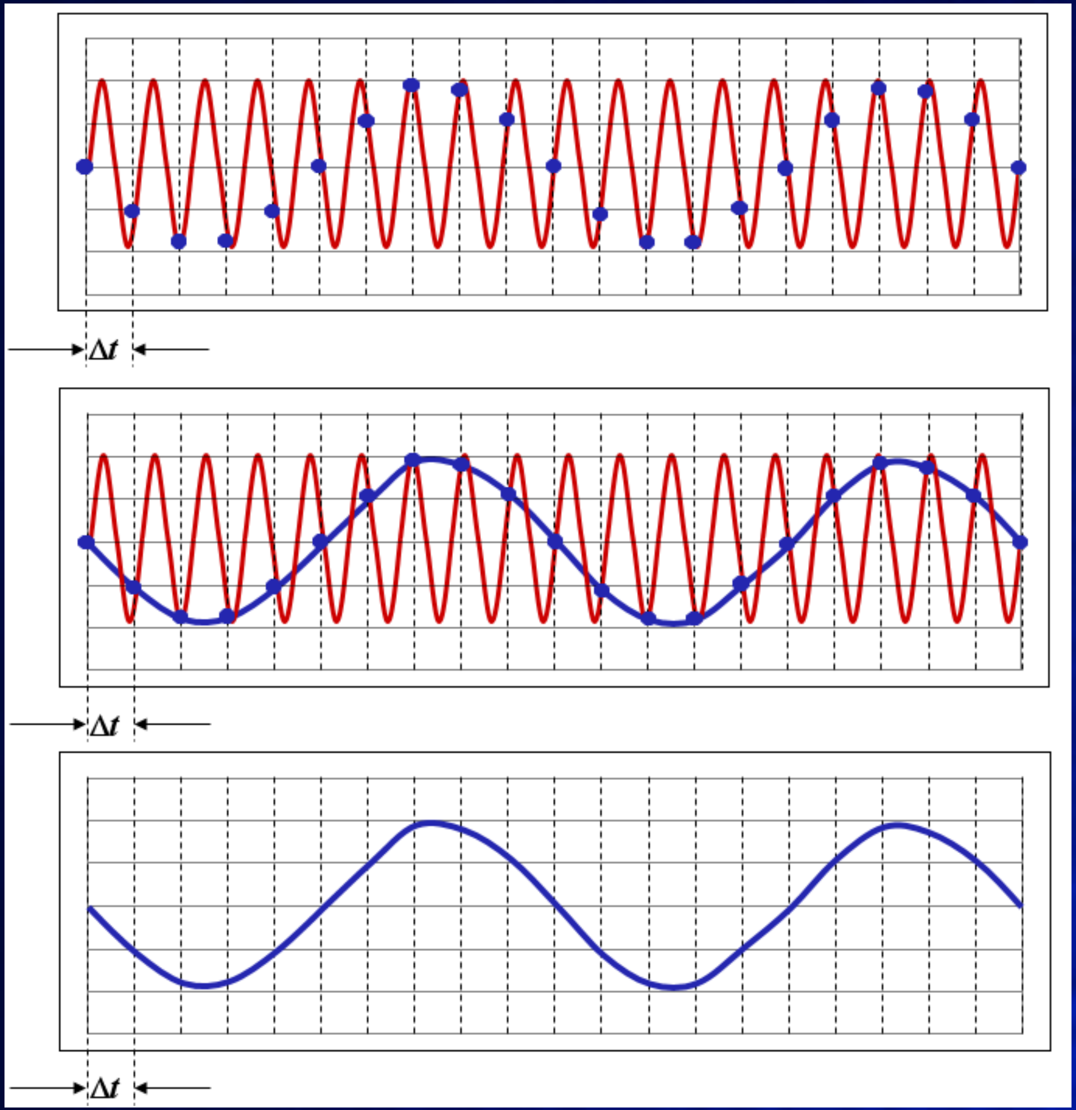
$$r = 2f_{max}$$

$$f_N = 2f_{max}$$

$$N \Longrightarrow Nyquist$$

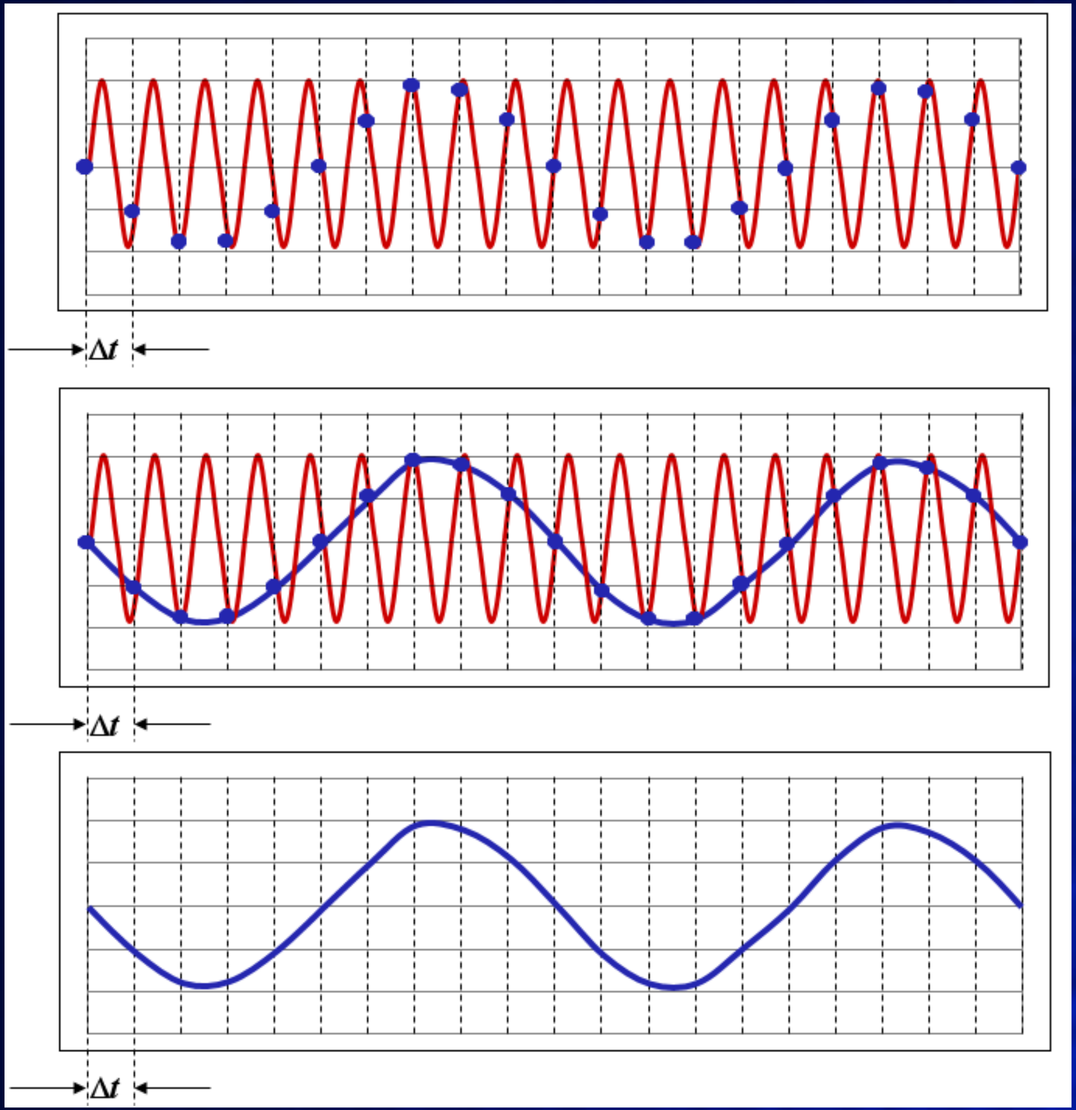


Need at least two samples per cycle to avoid aliasing (though at two samples per cycle, the amplitude will be incorrect without perfect phase alignment).



Need at least two samples per cycle to avoid aliasing (though the amplitude will be incorrect without perfect phase alignment).

Aliased signals map to $(-f_{max}, f_{max})$ by $f_a = \pm(r - f)$ where f_a is the aliased frequency on $(-f_{max}, f_{max})$, r is the sample rate (or Nyquist frequency) and f is the true unaliased frequency.



Need at least two samples per cycle to avoid aliasing (though the amplitude will be incorrect without perfect phase alignment).

Aliased signals map to $(-f_{max}, f_{max})$ by $f_a = \pm(r - f)$ where f_a is the aliased frequency on $(-f_{max}, f_{max})$, r is the sample rate (or Nyquist frequency) and f is the true unaliased frequency.

e.g. $f = 60Hz, r = 100sps,$
$f_a = \pm 40Hz$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

$$x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

Here $n \Longrightarrow t$ and $k \Longrightarrow \tau$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

$$x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

Here $n \Longrightarrow t$ and $k \Longrightarrow \tau$

Let $s_n = e^{i2\pi f n}$

$$g_n = x_n * s_n = \sum_{k=-\infty}^{\infty} x_k e^{i2\pi f(n-k)}$$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

$$x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

Here $n \Longrightarrow t$ and $k \Longrightarrow \tau$

Let $s_n = e^{i2\pi fn}$

$$g_n = x_n * s_n = \sum_{k=-\infty}^{\infty} x_k e^{i2\pi f(n-k)} = e^{i2\pi fn} \sum_{k=-\infty}^{\infty} x_k e^{-i2\pi fk}$$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

$$x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

Here $n \Longrightarrow t$ and $k \Longrightarrow \tau$

Let $s_n = e^{i2\pi fn}$

$$g_n = x_n * s_n = \sum_{k=-\infty}^{\infty} x_k e^{i2\pi(n-k)} = e^{i2\pi fn} \sum_{k=-\infty}^{\infty} x_k e^{-i2\pi fk}$$

$$= e^{i2\pi fn} X(f)$$

Discrete time equivalents.

$$\delta_{n-m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$$H_{n-m} = \begin{cases} 1, & n \geq m \\ 0, & n < m \end{cases}$$

$$x_n * y_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

Here $n \Longrightarrow t$ and $k \Longrightarrow \tau$

Let $s_n = e^{i2\pi fn}$

$$g_n = x_n * s_n = \sum_{k=-\infty}^{\infty} x_k e^{i2\pi(n-k)} = e^{i2\pi fn} \sum_{k=-\infty}^{\infty} x_k e^{-i2\pi fk}$$

$$= e^{i2\pi fn} X(f)$$

Much like

$$g(t) = \phi(t) * e^{i2\pi ft} = \int_{-\infty}^{\infty} \phi(\tau) e^{i2\pi f(t-\tau)} d\tau$$

$$= e^{i2\pi ft} \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f\tau} d\tau = e^{i2\pi ft} \Phi(f)$$

We previously found, $X(f) = F[x_n] = F[r\Pi(rt)x(t)]$

We previously found, $X(f) = F[x_n] = F[r\Pi(rt)x(t)]$

$$X(f) = rF\left[\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\right]$$

We previously found, $X(f) = F[x_n] = F[r\Pi(rt)x(t)]$

$$X(f) = rF\left[\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\right] = r\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\, e^{-i2\pi ft}dt$$

We previously found, $X(f) = F[x_n] = F[r\Pi(rt)x(t)]$

$$X(f) = rF\left[\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\right] = r\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\,e^{-i2\pi ft}dt$$

The integral is 0 for $rt - n \neq 0$ and 1 for $t = {}^{n}/_{r}$.

$$X(f) = r\sum_{n=-\infty}^{\infty} x\left(\frac{n}{r}\right)e^{-i2\pi f^{n}/_{r}}$$

We previously found, $X(f) = F[x_n] = F[r\Pi(rt)x(t)]$

$$X(f) = rF\left[\sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\right] = r\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(rt-n)x(t)\, e^{-i2\pi f t}dt$$

The integral is 0 for $rt - n \neq 0$ and 1 for $t = {}^{n}/_{r}$.

$$X(f) = r\sum_{n=-\infty}^{\infty} x\left(\frac{n}{r}\right) e^{-i2\pi f {}^{n}/_{r}}$$

Let $r = 1$ for simplicity which implies $f_{max} = {}^{1}/_{2}$.

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi f n}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi f n}$$

We set $f_{max} = \frac{1}{2}$ which means $X(f)$ is band limited between $\left(-\frac{1}{2}, \frac{1}{2}\right)$ so we can recover the original sequence with the inverse Fourier Transform.

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn}$$

We set $f_{max} = \frac{1}{2}$ which means $X(f)$ is band limited between $\left(-\frac{1}{2}, \frac{1}{2}\right)$ so we can recover the original sequence with the inverse Fourier Transform.

$$x_n = F^{-1}[X(f)] = \int_{-1/2}^{1/2} X(f)e^{i2\pi fn}df$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn}$$

We set $f_{max} = \frac{1}{2}$ which means $X(f)$ is band limited between $\left(-\frac{1}{2}, \frac{1}{2}\right)$ so we can recover the original sequence with the inverse Fourier Tansform.

$$x_n = F^{-1}[X(f)] = \int_{-{}^{1}/_{2}}^{{}^{1}/_{2}} X(f)e^{i2\pi fn}df$$

While this is a transform pair (x_n and $X(f)$), it is not symmetric and n is discrete but infinite and f is continuous but finite.

We have this less than ideal transform pair,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn} \qquad x_n = \int_{-1/2}^{1/2} X(f)e^{i2\pi fn}df$$

We have this less than ideal transform pair,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn} \qquad x_n = \int_{-1/2}^{1/2} X(f)e^{i2\pi fn}df$$

We require $X(f)$ be bandlimited and discrete which makes x_n periodic in N (recall the convolution with the shah function).

We have this less than ideal transform pair,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn} \qquad x_n = \int_{-{}^{1}/_{2}}^{{}^{1}/_{2}} X(f)e^{i2\pi fn}df$$

We require $X(f)$ be bandlimited and discrete which makes x_n periodic in N (recall the convolution with the shah function).

That makes $f = {}^{k}/_{N}$ which lets $X(f)$ become X_k periodic. Then x_n on the Nth interval may be fully recovered from X_k also on the Nth interval.

We have this less than ideal transform pair,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi fn} \qquad x_n = \int_{-1/2}^{1/2} X(f)e^{i2\pi fn}df$$

We require $X(f)$ be bandlimited and discrete which makes x_n periodic in N (recall the convolution with the shah function).

That makes $f = {}^{k}/_{N}$ which lets $X(f)$ become X_k periodic. Then x_n on the Nth interval may be fully recovered from X_k also on the Nth interval.

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N} = IDFT[X_k] \qquad X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} = DFT[x_n]$$

This is the discrete Fourier Transform.

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}$$

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$DFT[x_n] = DFT\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]$$

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$DFT[x_n] = DFT\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]$$

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right] e^{-i2\pi nm/N}$$

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$DFT[x_n] = DFT\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]$$

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right] e^{-i2\pi nm/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1} e^{i2\pi n(k-m)/N}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$DFT[x_n] = DFT\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]$$

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right] e^{-i2\pi nm/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1} e^{i2\pi n(k-m)/N}$$

De Moivre’s theorem says, $(e^x)^n = e_1^x \cdot e_2^x \cdot e_3^x \cdots e_n^x = e^{nx}$ (where n is a positive integer)

THE UNIVERSITY OF
MEMPHIS.
Center for Earthquake Research
and Information

To verify the transform pair we apply the forward DFT both sides of the IDFT (though we must take care with our choice of summation indices since k is already used for the sequence X_k).

$$DFT[x_n] = DFT\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]$$

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right] e^{-i2\pi nm/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1} e^{i2\pi n(k-m)/N}$$

De Moivre’s theorem says, $(e^x)^n = e_1^x \cdot e_2^x \cdot e_3^x \cdots e_n^x = e^{nx}$ (where n is a positive integer)

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1} \left(e^{i2\pi(k-m)/N}\right)^n$$

When $(k - m)$ is a multiple of N, ${}^{(k-m)}/_{N}$ is an integer and,

$$e^{i2\pi(k-m)/N} = e^{i2\pi j} = 1$$

$$\sin(j2\pi) = 0$$
$$\cos(j2\pi) = 1$$

For integer j

When $(k-m)$ is a multiple of N, ${}^{(k-m)}\!/_{N}$ is an integer and,

$$e^{i2\pi(k-m)/N} = e^{i2\pi j} = 1$$

$$\sin(j2\pi) = 0$$
$$\cos(j2\pi) = 1$$

For integer j

so that $$\sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n = \sum_{n=0}^{N-1} 1^n = N$$

For values of $(k-m)$ that are multiples of N.

When $(k - m)$ is a multiple of N, ${}^{(k-m)}/_{N}$ is an integer and,

$$e^{i2\pi(k-m)/N} = e^{i2\pi j} = 1$$

$$\sin(j2\pi) = 0$$
$$\cos(j2\pi) = 1$$

For integer j

so that $$\sum_{n=0}^{N-1} \left(e^{i2\pi(k-m)/N}\right)^n = \sum_{n=0}^{N-1} 1^n = N$$

For values of $(k - m)$ that are multiples of N.

When $(k - m)$ is not a multiple of N it's a bit more complicated.

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Subtract rs from s

$$s - rs = a - ar^n$$

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$ Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$ Subtract rs from s

$$s - rs = a - ar^n \qquad \Longrightarrow \qquad s(1-r) = a(1-r^n)$$

$$s = a\frac{1-r^n}{1-r}$$

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Subtract rs from s

$$s - rs = a - ar^n \quad \Longrightarrow \quad s(1-r) = a(1-r^n)$$

$$s = a\frac{1-r^n}{1-r} \quad \Longrightarrow \quad \sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

THE UNIVERSITY OF
MEMPHIS.
Center for Earthquake Research
and Information

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Subtract rs from s

$$s - rs = a - ar^n \quad \Longrightarrow \quad s(1-r) = a(1-r^n)$$

$$s = a\frac{1-r^n}{1-r} \quad \Longrightarrow \quad \sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

$$\sum_{n=0}^{N-1} \left(e^{i2\pi(k-m)/N}\right)^n = \frac{1-\left(e^{i2\pi(k-m)/N}\right)^N}{1-e^{i2\pi(k-m)/N}}$$

Consider the geometric series,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r

$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Subtract rs from s

$$s - rs = a - ar^n \quad \Longrightarrow \quad s(1-r) = a(1-r^n)$$

$$s = a\frac{1-r^n}{1-r} \quad \Longrightarrow \quad \sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

$$\sum_{n=0}^{N-1} \left(e^{i2\pi(k-m)/N}\right)^n = \frac{1-\left(e^{i2\pi(k-m)/N}\right)^N}{1-e^{i2\pi(k-m)/N}} = \frac{1-e^{i2\pi(k-m)}}{1-e^{i2\pi(k-m)/N}}$$

Remember we're working with values of $(k-m)$ not integer multiples of N.

Both k and m are always integers so that $e^{i2\pi(k-m)}$ has an exponent that is always an integer multiple of 2π, so that,

Both k and m are always integers so that $e^{i2\pi(k-m)}$ has an exponent that is always an integer multiple of 2π, so that,

$$\frac{1-e^{i2\pi(k-m)}}{1-e^{i2\pi(k-m)/N}} = \frac{1-1}{1-e^{i2\pi(k-m)/N}} = 0$$

When $(k-m)$ are not integer multiples of N.

Both k and m are always integers so that $e^{i2\pi(k-m)}$ has an exponent that is always an integer multiple of 2π, so that,

$$\frac{1-e^{i2\pi(k-m)}}{1-e^{i2\pi(k-m)/N}} = \frac{1-1}{1-e^{i2\pi(k-m)/N}} = 0$$

When $(k-m)$ are not integer multiples of N.

So that

$$\sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n = \begin{cases} N \\ 0 \end{cases}$$

$(k-m)$ integer multiples of N

otherwise

Both k and m are always integers so that $e^{i2\pi(k-m)}$ has an exponent that is always an integer multiple of 2π, so that,

$$\frac{1-e^{i2\pi(k-m)}}{1-e^{i2\pi(k-m)/N}} = \frac{1-1}{1-e^{i2\pi(k-m)/N}} = 0$$

When $(k-m)$ are not integer multiples of N.

So that $$\sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n = \begin{cases} N & (k-m) \text{ integer multiples of N} \\ 0 & \text{otherwise} \end{cases}$$

We're only working with values of k and m on the interval $(0, N-1)$,

$$\sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n = N\delta_{k-m}$$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

Now back to the DFT.

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n$$

Now back to the DFT.

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k N\delta_{k-m} = X_m$$

Now back to the DFT.

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1}\left(e^{i2\pi(k-m)/N}\right)^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k N \delta_{k-m} = X_m$$

So we successfully verified our DFT/IDFT pair.

$$\Phi_k = \sum_{n=0}^{N-1} \phi_n e^{-i2\pi kn/N} \qquad \phi_n = \frac{1}{N}\sum_{k=0}^{N-1} \Phi_k e^{i2\pi kn/N}$$

Now back to the DFT.

$$\sum_{n=0}^{N-1} x_n e^{-i2\pi nm/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \sum_{n=0}^{N-1} \left(e^{i2\pi(k-m)/N}\right)^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k N \delta_{k-m} = X_m$$

So we successfully verified our DFT/IDFT pair.

$$\Phi_k = \sum_{n=0}^{N-1} \phi_n e^{-i2\pi kn/N} \qquad \phi_n = \frac{1}{N}\sum_{k=0}^{N-1} \Phi_k e^{i2\pi kn/N}$$

Similarly to the continuous FT, we can decompose ϕ_n into its constituent frequency components Φ_k, and we can reconstruct ϕ_n from those frequency components. Except here, both frequency and time are finite length, periodic, and discrete. (very handy for working with computers)

The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N}$$

The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{i2\pi nk/N}$$

THE UNIVERSITY OF MEMPHIS

Center for Earthquake Research and Information

The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{i2\pi nk/N}$$

$$e^{i2\pi n} = \overset{\overset{1}{\nearrow}}{\cos(2\pi n)} + i \overset{\overset{0}{\nearrow}}{\sin(2\pi n)} = 1$$

The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{i2\pi nk/N}$$

$$e^{i2\pi n} = \overset{1}{\overset{\nearrow}{\cos(2\pi n)}} + i\,\overset{0}{\overset{\nearrow}{\sin(2\pi n)}} = 1$$

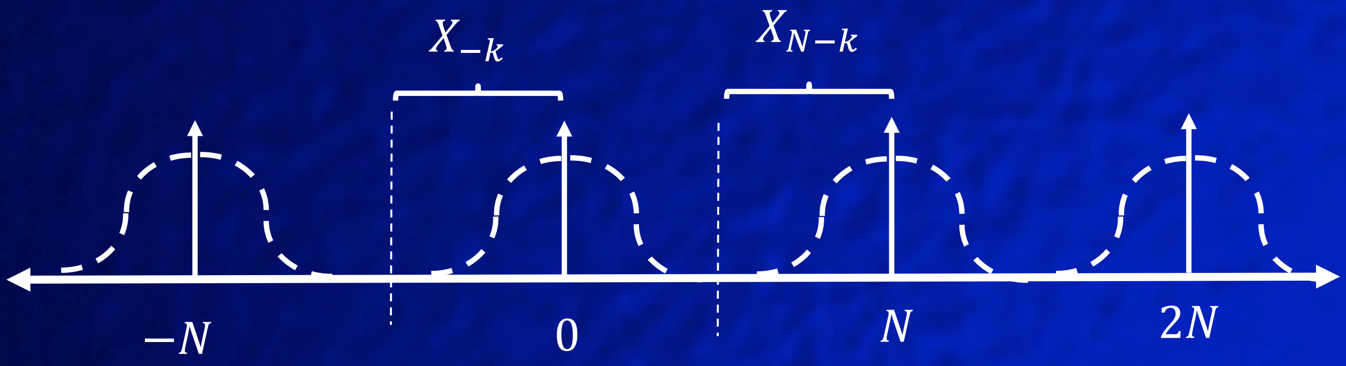
$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{i2\pi nk/N} = X_{-k}$$

The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{i2\pi nk/N}$$

$$e^{i2\pi n} = \overset{1}{\cos(2\pi n)} + i\,\overset{0}{\sin(2\pi n)} = 1$$

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{i2\pi nk/N} = X_{-k}$$

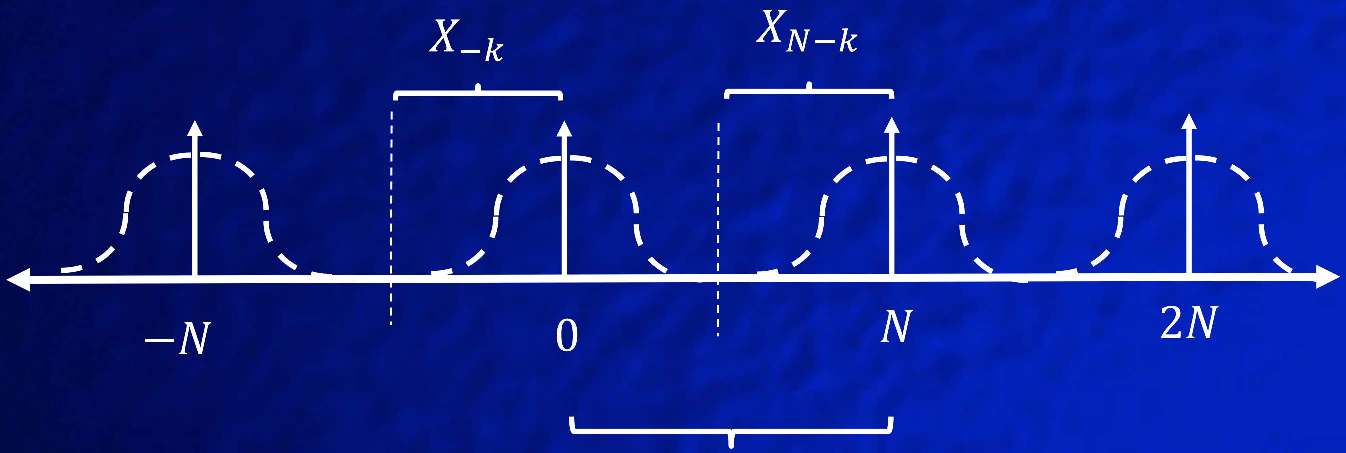


The DFT is periodic in N.

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n(N-k)/N} = \sum_{n=0}^{N-1} x_n e^{-i2\pi n} e^{i2\pi nk/N}$$

$$e^{i2\pi n} = \cancelto{1}{\cos(2\pi n)} + i\,\cancelto{0}{\sin(2\pi n)} = 1$$

$$X_{N-k} = \sum_{n=0}^{N-1} x_n e^{i2\pi nk/N} = X_{-k}$$



Matlab FFT vector.

Indexing

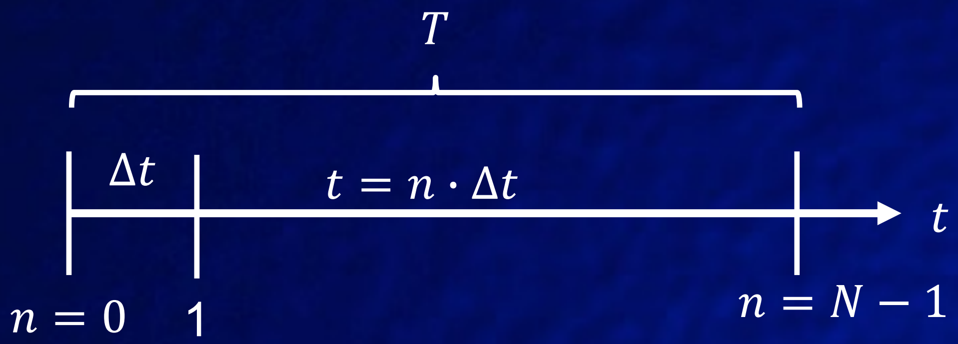
If N is even, plot $-\frac{N}{2} \leq k \leq \left(\frac{N}{2} - 1\right)$

If N is odd, plot $-\frac{(N-1)}{2} \leq k \leq \frac{(N-1)}{2}$

Indexing

If N is even, plot $-\frac{N}{2} \le k \le \left(\frac{N}{2} - 1\right)$

If N is odd, plot $-\frac{(N-1)}{2} \le k \le \frac{(N-1)}{2}$



Time resolution

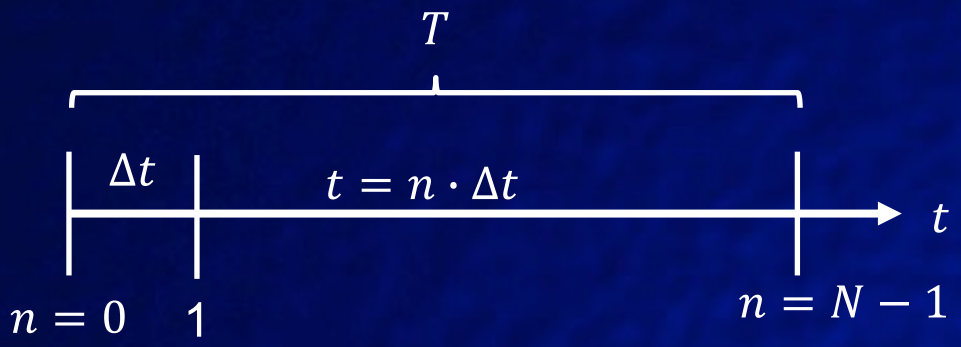
$$\Delta t = \frac{T}{N}, \qquad N = T \cdot r$$

$r = sample\ rate$, samps/sec

Indexing

If N is even, plot $-\frac{N}{2} \leq k \leq \left(\frac{N}{2} - 1\right)$

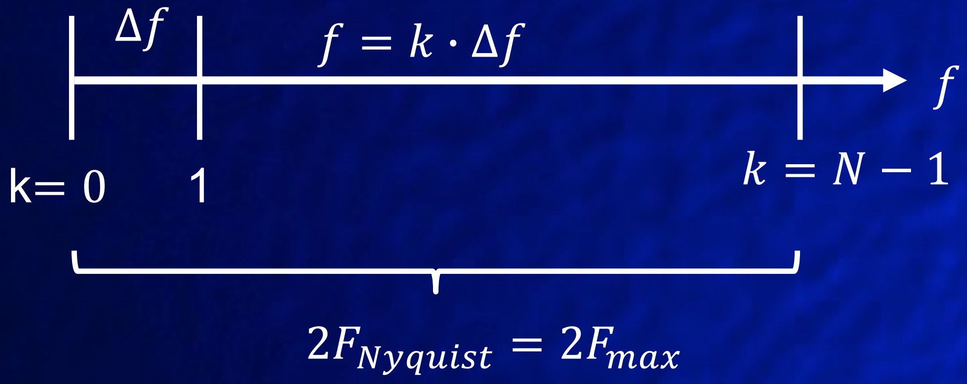
If N is odd, plot $-\frac{(N-1)}{2} \leq k \leq \frac{(N-1)}{2}$



Time resolution

$$\Delta t = \frac{T}{N}, \qquad N = T \cdot r$$

$r = sample\ rate$, samps/sec



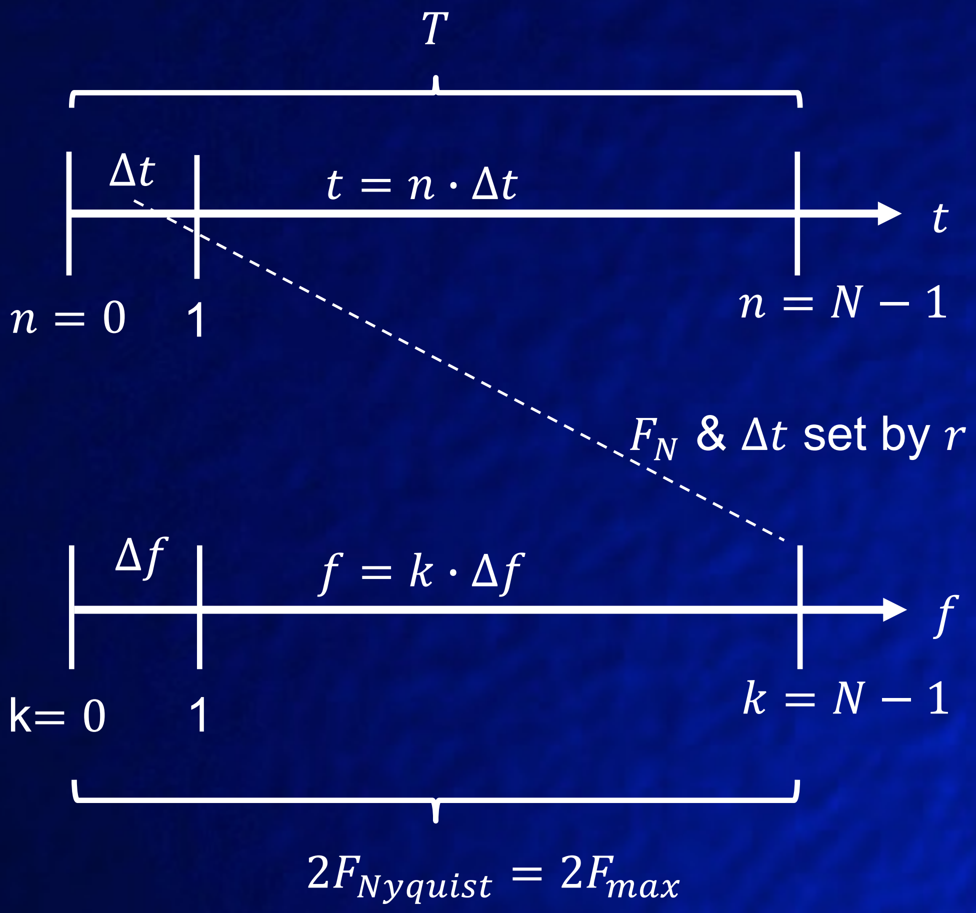
Frequency resolution

$$\Delta f = \frac{2F_{max}}{N} = \frac{r}{N}$$

Indexing

If N is even, plot $-\frac{N}{2} \leq k \leq \left(\frac{N}{2} - 1\right)$

If N is odd, plot $-\frac{(N-1)}{2} \leq k \leq \frac{(N-1)}{2}$



T

Δt

$t = n \cdot \Delta t$

t

$n = 0$ 1

$n = N - 1$

F_N & Δt set by r

Δf

$f = k \cdot \Delta f$

f

k$= 0$ 1

$k = N - 1$

$2F_{Nyquist} = 2F_{max}$

Time resolution

$$\Delta t = \frac{T}{N}, \qquad N = T \cdot r$$

$r = sample\ rate$, samps/sec

Frequency resolution

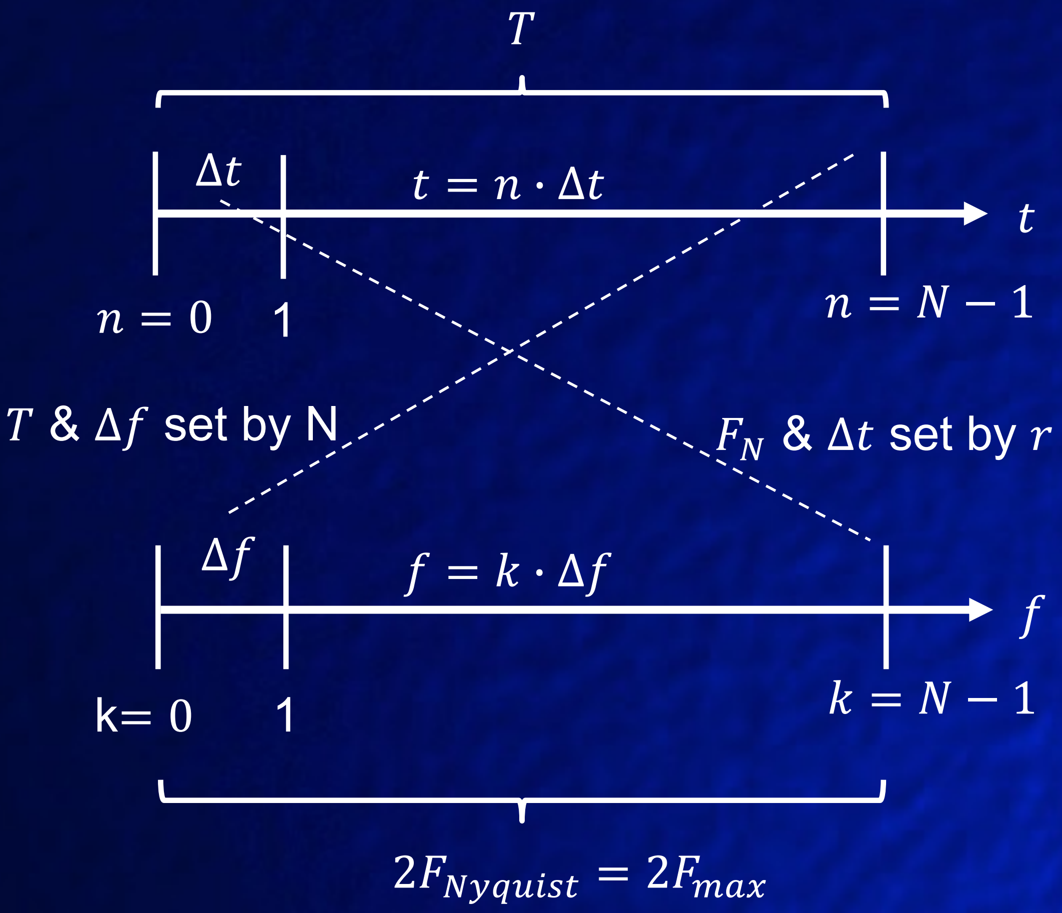
$$\Delta f = \frac{2F_{max}}{N} = \frac{r}{N}$$

F_N, or the frequency band, is limited by the sample rate.

Indexing

If N is even, plot $-\frac{N}{2} \leq k \leq \left(\frac{N}{2} - 1\right)$

If N is odd, plot $-\frac{(N-1)}{2} \leq k \leq \frac{(N-1)}{2}$



T

Δt

$t = n \cdot \Delta t$

t

$n = 0$ 1

$n = N - 1$

T & Δf set by N

F_N & Δt set by r

Δf

$f = k \cdot \Delta f$

f

k= 0 1

$k = N - 1$

$2F_{Nyquist} = 2F_{max}$

Time resolution

$$\Delta t = \frac{T}{N}, \qquad N = T \cdot r$$

$r = sample\ rate$, samps/sec

Frequency resolution

$$\Delta f = \frac{2F_{max}}{N} = \frac{r}{N}$$

F_N, or the frequency band, is limited by the sample rate.

Δf, or frequency resolution, is limited by N or T.

Discrete Convolution

Let $z_n = x_n * y_n$ $\qquad DFT[z_n] = X_k Y_k$ $\qquad z_n = IDFT[X_k Y_k] = x_n * y_n$

Discrete Convolution

Let $z_n = x_n * y_n$ $DFT[z_n] = X_k Y_k$ $z_n = IDFT[X_k Y_k] = x_n * y_n$

$$z_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y_k e^{i2\pi kn/N}$$

Discrete Convolution

Let $z_n = x_n * y_n$ $DFT[z_n] = X_k Y_k$ $z_n = IDFT[X_k Y_k] = x_n * y_n$

$$z_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y_k e^{i2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{l=0}^{N-1} x_l e^{-i2\pi kl/N}\right]\left[\sum_{m=0}^{N-1} y_m e^{-i2\pi km/N}\right] e^{i2\pi kn/N}$$

Discrete Convolution

Let $z_n = x_n * y_n$ $\quad DFT[z_n] = X_k Y_k$ $\quad z_n = IDFT[X_k Y_k] = x_n * y_n$

$$z_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y_k e^{i2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{\left[\sum_{l=0}^{N-1} x_l e^{-i2\pi kl/N}\right]}_{X_k = DFT[x_n]} \underbrace{\left[\sum_{m=0}^{N-1} y_m e^{-i2\pi km/N}\right]}_{Y_k = DFT[y_n]} e^{i2\pi kn/N}$$

Discrete Convolution

Let $z_n = x_n * y_n$ $\qquad DFT[z_n] = X_kY_k$ $\qquad z_n = IDFT[X_kY_k] = x_n * y_n$

$$z_n = \frac{1}{N}\sum_{k=0}^{N-1} X_kY_ke^{i2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1}\underbrace{\left[\sum_{l=0}^{N-1} x_le^{-i2\pi kl/N}\right]}_{X_k = DFT[x_n]}\underbrace{\left[\sum_{m=0}^{N-1} y_me^{-i2\pi km/N}\right]}_{Y_k = DFT[y_n]}e^{i2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} x_ly_me^{-i2\pi kl/N}e^{-i2\pi km/N}e^{i2\pi kn/N}$$

Discrete Convolution

Let $z_n = x_n * y_n$ $DFT[z_n] = X_k Y_k$ $z_n = IDFT[X_k Y_k] = x_n * y_n$

$$z_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y_k e^{i2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{\left[\sum_{l=0}^{N-1} x_l e^{-i2\pi kl/N}\right]}_{X_k = DFT[x_n]} \underbrace{\left[\sum_{m=0}^{N-1} y_m e^{-i2\pi km/N}\right]}_{Y_k = DFT[y_n]} e^{i2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} x_l y_m e^{-i2\pi kl/N} e^{-i2\pi km/N} e^{i2\pi kn/N}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \sum_{k=0}^{N-1} e^{i2\pi k(n-m-l)/N}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

From before, $$\sum_{k=0}^{N-1} e^{i2\pi(n-m-l)/N} = \begin{cases} N, & n-m-l=0 \\ 0, & \text{otherwise} \end{cases}$$

From before, $\sum_{k=0}^{N-1} e^{i2\pi(n-m-l)/N} = \begin{cases} N, & n-m-l=0 \\ 0, & \text{otherwise} \end{cases}$

$$z_n = \frac{1}{N} \sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \sum_{k=0}^{N-1} e^{i2\pi k(n-m-l)/N}$$

From before, $\displaystyle\sum_{k=0}^{N-1} e^{i2\pi(n-m-l)/N} = \begin{cases} N, & n-m-l=0 \\ 0, & \text{otherwise} \end{cases}$

$$z_n = \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \underbrace{\sum_{k=0}^{N-1} e^{i2\pi k(n-m-l)/N}}_{N\delta_{n-m-l}}$$

$$m = n - l$$

From before, $$\sum_{k=0}^{N-1} e^{i2\pi(n-m-l)/N} = \begin{cases} N, & n-m-l=0 \\ 0, & \text{otherwise} \end{cases}$$

$$z_n = \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \underbrace{\sum_{k=0}^{N-1} e^{i2\pi k(n-m-l)/N}}_{N\delta_{n-m-l}}$$

$m = n - l$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \left[N\delta_{n-m-l}\right]$$

Sifts values of $m = n - l$

From before, $\sum_{k=0}^{N-1} e^{i2\pi(n-m-l)/N} = \begin{cases} N, & n-m-l=0 \\ 0, & \text{otherwise} \end{cases}$

$$z_n = \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \underbrace{\sum_{k=0}^{N-1} e^{i2\pi k(n-m-l)/N}}_{N\delta_{n-m-l}}$$

$m = n - l$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m \left[N\delta_{n-m-l}\right]$$

Sifts values of $m = n - l$

$$= \sum_{l=0}^{N-1} x_l y_{n-l}$$

The discrete convolution.

Linear Convolution: $z_n = \sum_{l=-\infty}^{\infty} x_l y_{n-l}$

THE UNIVERSITY OF MEMPHIS

Center for Earthquake Research and Information

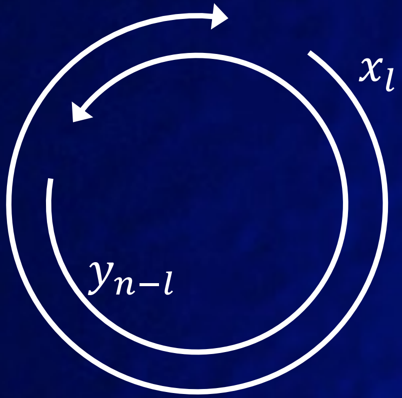
Linear Convolution: $$z_n = \sum_{l=-\infty}^{\infty} x_l y_{n-l}$$

Circular Convolution: $$z_n = \sum_{l=0}^{N-1} x_l y_{n-l}$$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

Linear Convolution: $$z_n = \sum_{l=-\infty}^{\infty} x_l y_{n-l}$$

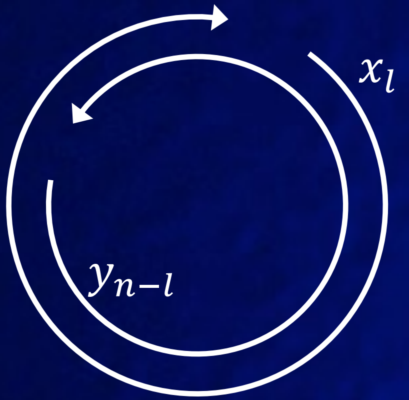
Circular Convolution: $$z_n = \sum_{l=0}^{N-1} x_l y_{n-l}$$



Periodicity can be graphically modeled by putting the data vectors on a circle.

Linear Convolution: $z_n = \sum_{l=-\infty}^{\infty} x_l y_{n-l}$

Circular Convolution: $z_n = \sum_{l=0}^{N-1} x_l y_{n-l}$

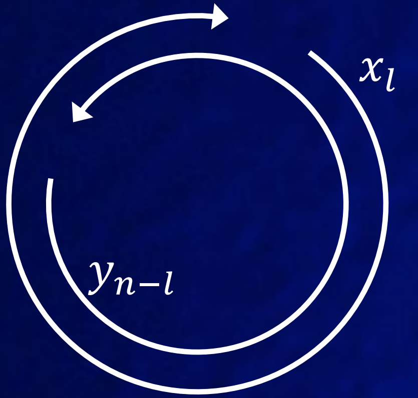


Periodicity can be graphically modeled by putting the data vectors on a circle.

Multiply and sum, then rotate one point. Repeat for $N-1$ points.

Linear Convolution: $z_n = \sum_{l=-\infty}^{\infty} x_l y_{n-l}$

Circular Convolution: $z_n = \sum_{l=0}^{N-1} x_l y_{n-l}$



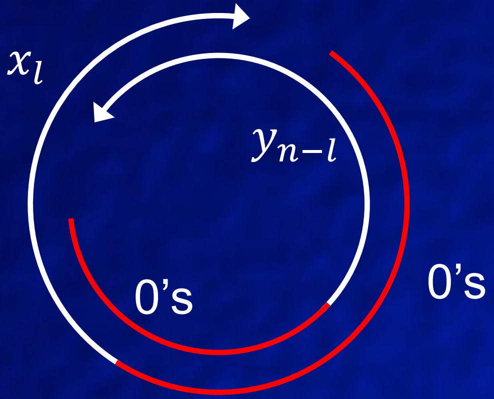
Periodicity can be graphically modeled by putting the data vectors on a circle.

Multiply and sum, then rotate one point. Repeat for $N-1$ points.

We almost always want linear convolution but only have $N-1$ and summing over $(-\infty, \infty)$ is not feasible for discrete finite length series. So we approximate linear convolution using circular convolution by padding with zeros.

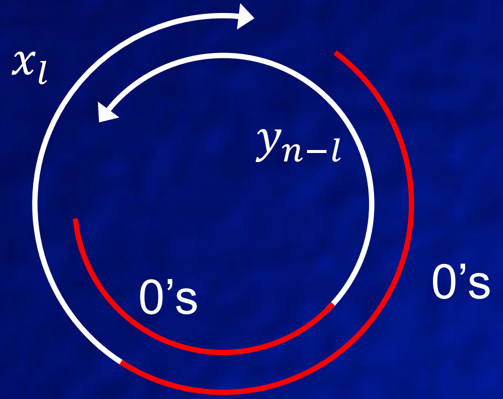
If x_n has N points and y_n has M points, we pad x with M points and y with N points then keep the first $N + M - 1$ values in the convolution.

If x_n has N points and y_n has M points, we pad x with M points and y with N points then keep the first $N + M - 1$ values in the convolution.



$$z_n = \sum_{l=0}^{N+M-1} x_l y_{n-l}$$

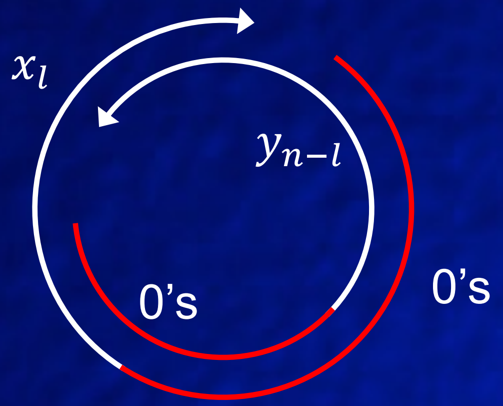
If x_n has N points and y_n has M points, we pad x with M points and y with N points then keep the first $N + M - 1$ values in the convolution.



$$z_n = \sum_{l=0}^{N+M-1} x_l y_{n-l}$$

Another way of looking at it is windowing x and y by a boxcar prior to convolution so that all points outside the window are 0.

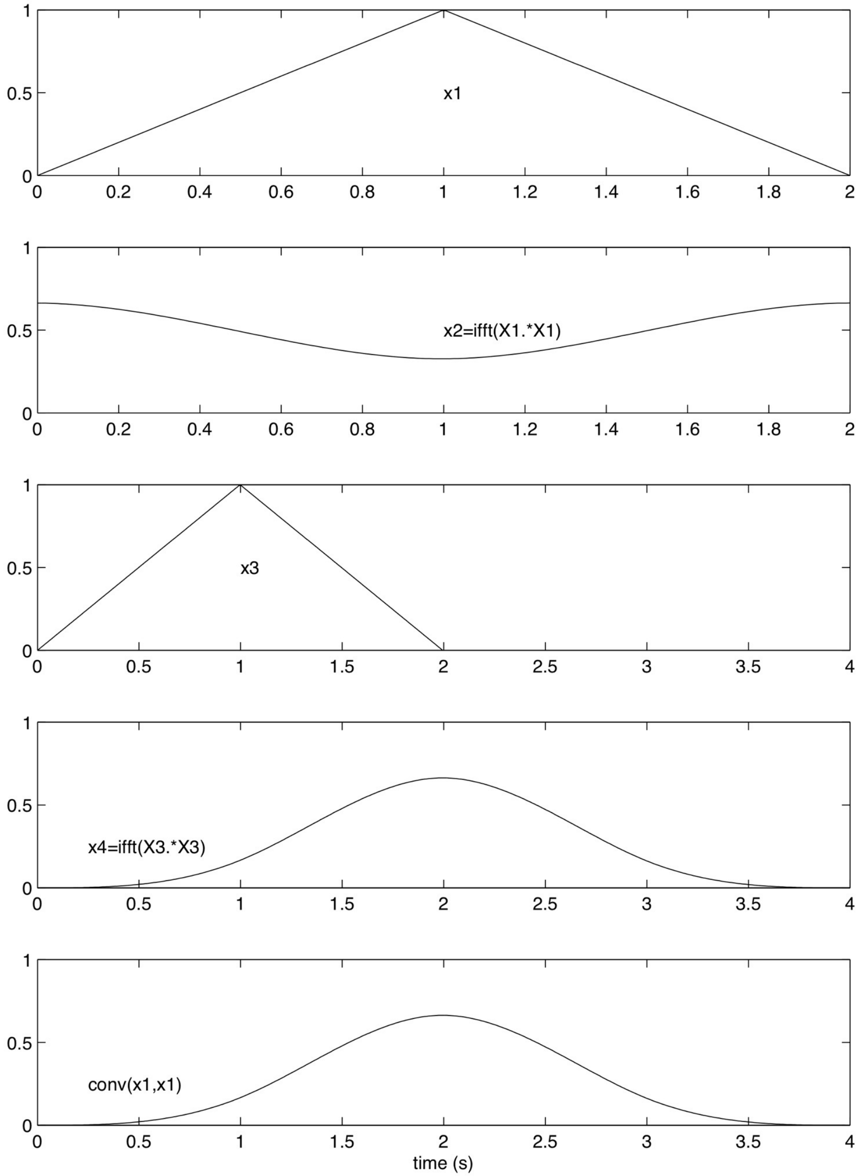
If x_n has N points and y_n has M points, we pad x with M points and y with N points then keep the first $N + M - 1$ values in the convolution.



$$z_n = \sum_{l=0}^{N+M-1} x_l y_{n-l}$$

Another way of looking at it is windowing x and y by a boxcar prior to convolution so that all points outside the window are 0.

Our data are finite length, so we’re still performing circular convolution, but we approximate the linear convolution by windowing the data.



The matlab *conv* command takes care of the padding for you and returns $N + M - 1$, points.

Figure to the left is from wrap.m that we reviewed earlier.

More discrete analogs to the continuous FT.

$$DFT\left[x_{n-n_0}\right] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

More discrete analogs to the continuous FT.

Let $n - n_0 = l, \; n = l + n_o$

$$DFT\left[x_{n-n_0}\right] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

More discrete analogs to the continuous FT.

$$DFT\left[x_{n-n_0}\right] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

Let $n - n_0 = l,\ n = l + n_o$

For $n = 0,\ l = -n_0$

For $n = N - 1,\ l = N - 1 - n_0$

THE UNIVERSITY OF MEMPHIS.
Center for Earthquake Research and Information

More discrete analogs to the continuous FT.

$$DFT[x_{n-n_0}] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

Let $n - n_0 = l,\ n = l + n_o$

For $n = 0,\ l = -n_0$

For $n = N - 1,\ l = N - 1 - n_0$

$$= \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi k(l+n_0)/N}$$

More discrete analogs to the continuous FT.

$$DFT[x_{n-n_0}] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

Let $n - n_0 = l,\ n = l + n_o$

For $n = 0,\ l = -n_0$

For $n = N - 1,\ l = N - 1 - n_0$

$$= \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi k(l+n_0)/N} = e^{-i2\pi kn_0/N} \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi kl/N}$$

More discrete analogs to the continuous FT.

$$DFT\left[x_{n-n_0}\right] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

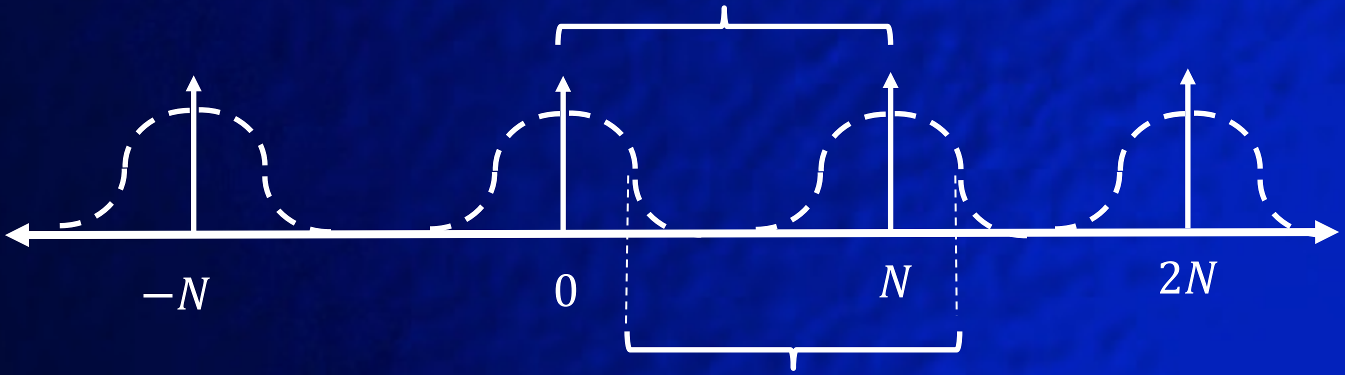
Let $n - n_0 = l, \;\; n = l + n_o$

For $n = 0, \;\; l = -n_0$

For $n = N - 1, \;\; l = N - 1 - n_0$

$$= \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi k(l+n_0)/N} = e^{-i2\pi kn_0/N} \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi kl/N}$$

Periodicity says it doesn't matter what N-length interval we sum over so,



More discrete analogs to the continuous FT.

$$DFT\left[x_{n-n_0}\right] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

Let $n - n_0 = l, \;\; n = l + n_o$

For $n = 0, \;\; l = -n_0$

For $n = N - 1, \;\; l = N - 1 - n_0$

$$= \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi k(l+n_0)/N} = e^{-i2\pi kn_0/N} \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi kl/N}$$

Periodicity says it doesn’t matter what N-length interval we sum over so,

$$DFT\left[x_{n-n_0}\right] = e^{-i2\pi kn_0/N} \sum_{l=0}^{N-1} x_l e^{-i2\pi kl/N}$$

More discrete analogs to the continuous FT.

$$DFT[x_{n-n_0}] = \sum_{n=0}^{N-1} x_{n-n_0} e^{-i2\pi kn/N}$$

Let $n - n_0 = l, \;\; n = l + n_o$

For $n = 0, \;\; l = -n_0$

For $n = N-1, \;\; l = N - 1 - n_0$

$$= \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi k(l+n_0)/N} = e^{-i2\pi kn_0/N} \sum_{l=-n_0}^{N-n_0-1} x_l e^{-i2\pi kl/N}$$

Periodicity says it doesn’t matter what N-length interval we sum over so,

$$DFT[x_{n-n_0}] = e^{-i2\pi kn_0/N} \sum_{l=0}^{N-1} x_l e^{-i2\pi kl/N} = e^{-i2\pi kn_0/N} X_k$$

Time shift ⟷ Phase shift

THE UNIVERSITY OF MEMPHIS

Center for Earthquake Research and Information

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^*$$

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l e^{i2\pi ln/N}\right]^*$$

$$\sum_{n=0}^{N-1}|x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l e^{i2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l^* e^{-i2\pi ln/N}\right]$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l e^{i2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l^* e^{-i2\pi ln/N}\right]$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X_k X_l^* e^{i2\pi n(k-l)/N}$$

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l e^{i2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l^* e^{-i2\pi ln/N}\right]$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X_k X_l^* e^{i2\pi n(k-l)/N} = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X_k X_l^* \left(e^{i2\pi(k-l)/N}\right)^n$$

$$\sum_{n=0}^{N-1}|x_n|^2=\sum_{n=0}^{N-1}x_n x_n^* \ =\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1}X_l e^{i2\pi ln/N}\right]^*$$

$$=\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1}X_l^* e^{-i2\pi ln/N}\right]$$

$$=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}X_k X_l^* e^{i2\pi n(k-l)/N}=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}X_k X_l^*\left(e^{i2\pi(k-l)/N}\right)^n$$

$$=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}X_k\sum_{l=0}^{N-1}X_l^*\delta_{k-l}$$

$$\sum_{n=0}^{N-1}|x_n|^2 = \sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l e^{i2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1} X_l^* e^{-i2\pi ln/N}\right]$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X_k X_l^* e^{i2\pi n(k-l)/N} = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X_k X_l^* \left(e^{i2\pi(k-l)/N}\right)^n$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X_k \sum_{l=0}^{N-1} X_l^* \delta_{k-l} = \frac{1}{N^2}\sum_{k=0}^{N-1} X_k X_k^* \sum_{n=0}^{N-1} 1$$

$$\sum_{n=0}^{N-1}|x_n|^2=\sum_{n=0}^{N-1}x_n x_n^* =\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1}X_l e^{i2\pi ln/N}\right]^*$$

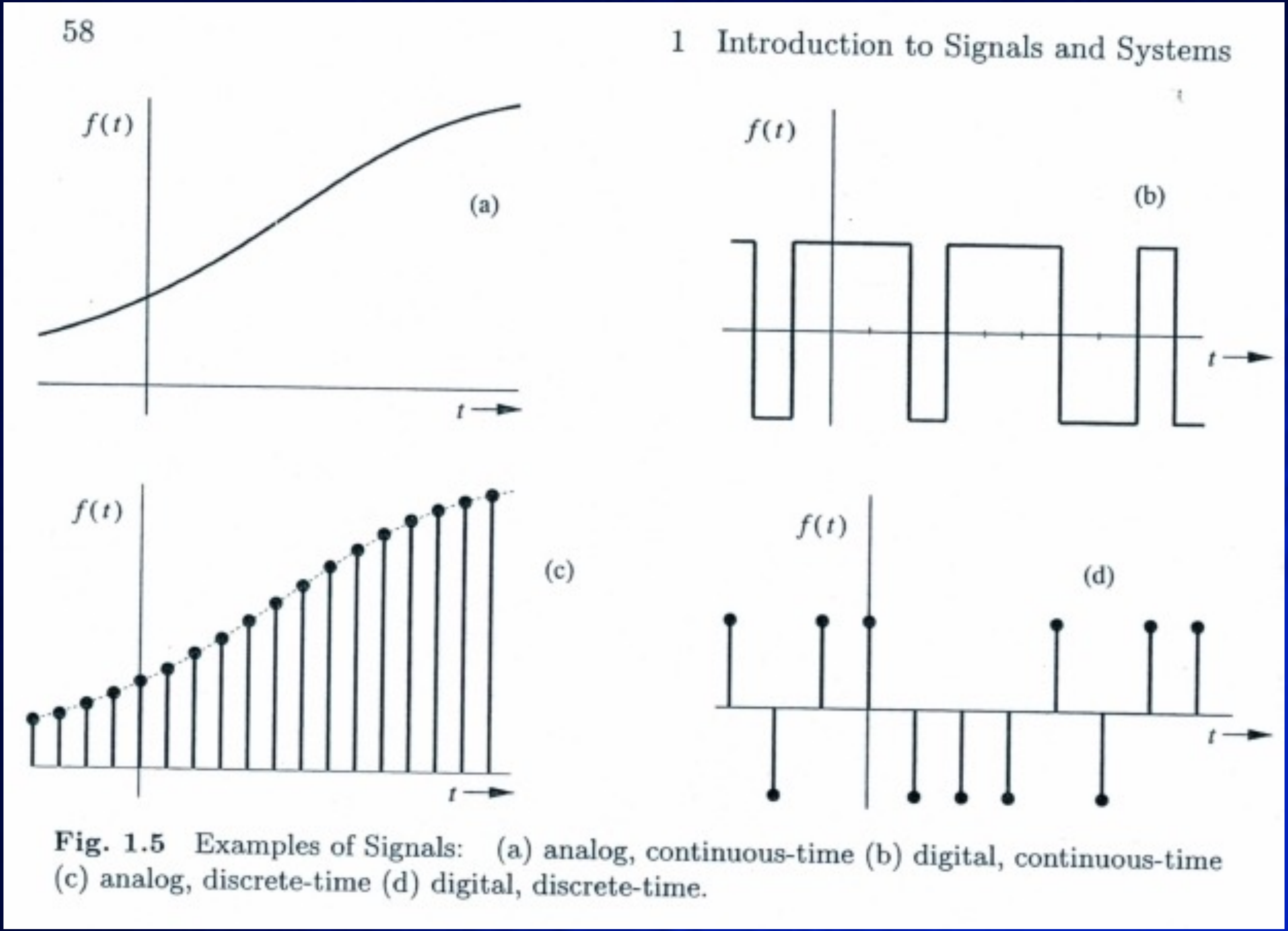
$$=\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X_k e^{i2\pi kn/N}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1}X_l^* e^{-i2\pi ln/N}\right]$$

$$=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}X_k X_l^* e^{i2\pi n(k-l)/N}=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}X_k X_l^*\left(e^{i2\pi(k-l)/N}\right)^n$$

$$=\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}X_k\sum_{l=0}^{N-1}X_l^*\delta_{k-l}=\frac{1}{N^2}\sum_{k=0}^{N-1}X_k X_k^*\sum_{n=0}^{N-1}1$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}|X_k|^2$$

Discrete Analog to Parseval.



58 1 Introduction to Signals and Systems

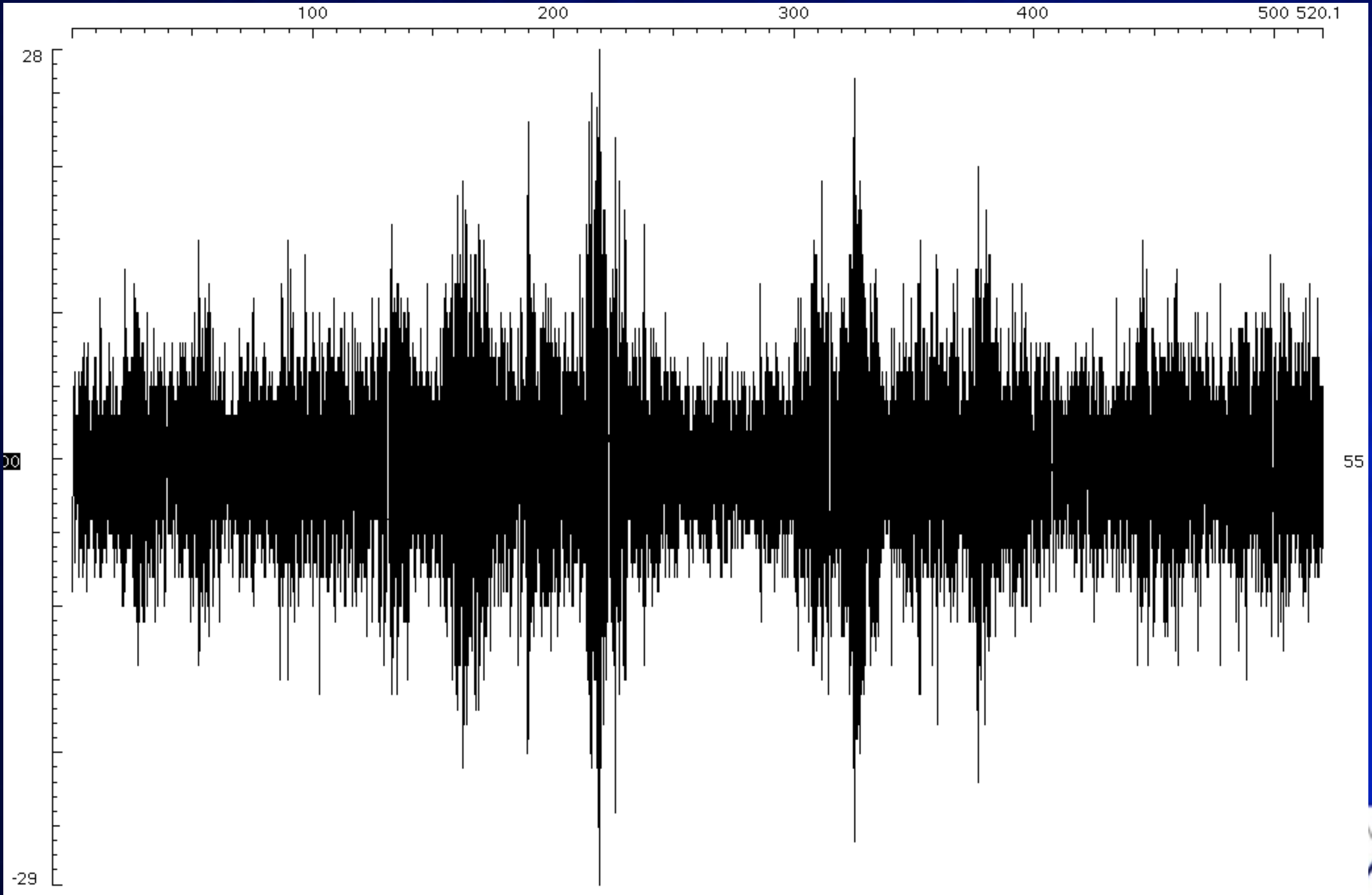
Fig. 1.5 Examples of Signals: (a) analog, continuous-time (b) digital, continuous-time (c) analog, discrete-time (d) digital, discrete-time.

From Signal Processing and Linear Systems, B.P. Lathi, 1998.

THE UNIVERSITY OF MEMPHIS
Center for Earthquake Research and Information

Example of amplitude resolution CATM.EHZ.NM.00

$2^{12} = \pm 2048$ $2^{24} = \pm 8{,}388{,}608$



Examples of Discrete Processes from Signal Processing and Linear Systems by B.P. Lathi, 1998, Section 8.5, pp 562-564.

Discrete systems don’t necessarily need to be digitized versions of continuous systems.

Examples of Discrete Processes from Signal Processing and Linear Systems by B.P. Lathi, 1998, Section 8.5, pp 562-564.

Discrete systems don't necessarily need to be digitized versions of continuous systems.

Consider a person who makes bank deposits at regular intervals, T (e.g. once each month). The bank pays interest on the balance during T. We wish to find the output (the account balance) of the "system" to the input (the deposit).

Examples of Discrete Processes from Signal Processing and Linear Systems by B.P. Lathi, 1998, Section 8.5, pp 562-564.

Discrete systems don’t necessarily need to be digitized versions of continuous systems.

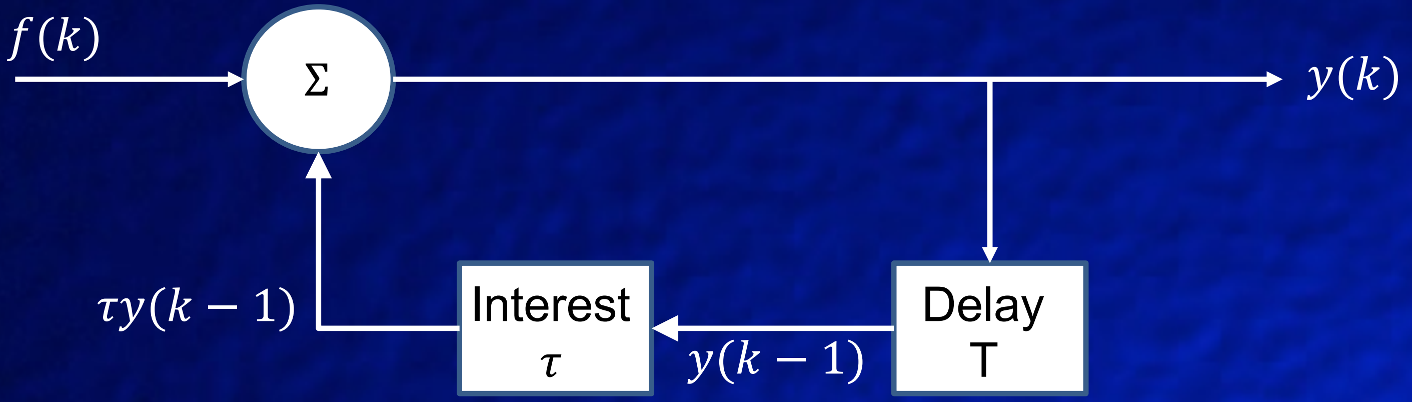
Consider a person who makes bank deposits at regular intervals, T (e.g. once each month). The bank pays interest on the balance during T. We wish to find the output (the account balance) of the “system” to the input (the deposit).

$f(k) =$ deposit made at k^{th} interval

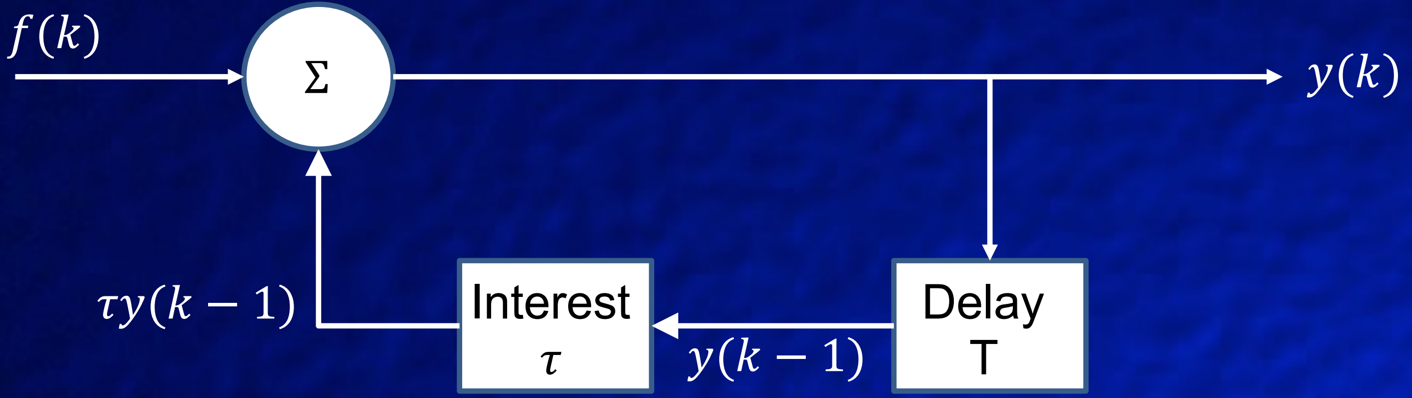
$y(k) =$ account balance at k immediately after the deposit

τ = interest rate per dollar per T

The balance, $y(k)$, is the sum of the previous balance, $y(k-1)$, the interest earned on $y(k-1)$ during T, and the deposit $f(k)$.



The balance, $y(k)$, is the sum of the previous balance, $y(k-1)$, the interest earned on $y(k-1)$ during T, and the deposit $f(k)$.



$$y(k) = y(k-1) + \tau y(k-1) + f(k)$$

$$= (1+\tau)y(k-1) + f(k)$$

If we assume monthly deposits are a constant, D, then

$$y_0 = D$$

If we assume monthly deposits are a constant, D, then

$$y_0 = D$$

$$y_1 = \tau y_0 + D = \tau D + D$$

If we assume monthly deposits are a constant, D, then

$$y_0 = D$$

$$y_1 = \tau y_0 + D = \tau D + D$$

$$y_2 = \tau y_1 + D = \tau^2 D + \tau D + D$$

If we assume monthly deposits are a constant, D, then

$$y_0 = D$$

$$y_1 = \tau y_0 + D = \tau D + D$$

$$y_2 = \tau y_1 + D = \tau^2 D + \tau D + D$$

$$y_N = \sum_{n=0}^{N-1} \tau^n D$$

$f(k)$ students register for a class in the k^{th} semester and must buy a book. Books last three semesters. $^{1}/_{4}$ of the books from the previous semester are sold as used. $y(k)$ is the number of new books sold.

$f(k)$ students register for a class in the k^{th} semester and must buy a book. Books last three semesters. $^1/_4$ of the books from the previous semester are sold as used. $y(k)$ is the number of new books sold.

$y(k) =$ number of new books sold in semester k.

$f(k)$ students register for a class in the k^{th} semester and must buy a book. Books last three semesters. $^1/_4$ of the books from the previous semester are sold as used. $y(k)$ is the number of new books sold.

$y(k) =$ number of new books sold in semester k.

$y(k-1) =$ number of new books sold in semester $k-1$.

$^1/_4\, y(k-1)$ of these new books are sold as used in semester k.

$f(k)$ students register for a class in the k^{th} semester and must buy a book. Books last three semesters. $^1/_4$ of the books from the previous semester are sold as used. $y(k)$ is the number of new books sold.

$y(k) =$ number of new books sold in semester k.

$y(k-1) =$ number of new books sold in semester $k-1$.

$^1/_4\, y(k-1)$ of these new books are sold as used in semester k.

$y(k-2) =$ number of new books sold in semester $k-2$.

$^1/_4\, y(k-2)$ of these new books are sold as used in semester k-1 and $^1/_4$ of those, or $^1/_{16}\, y(k-2)$ are sold as used in semester k.

$f(k)$ students register for a class in the k^{th} semester and must buy a book. Books last three semesters. ${}^1/_4$ of the books from the previous semester are sold as used. $y(k)$ is the number of new books sold.

$y(k)$ = number of new books sold in semester k.

$y(k-1)$ = number of new books sold in semester $k-1$.

${}^1/_4\, y(k-1)$ of these new books are sold as used in semester k.

$y(k-2)$ = number of new books sold in semester $k-2$.

${}^1/_4\, y(k-2)$ of these new books are sold as used in semester k-1 and ${}^1/_4$ of those, or ${}^1/_{16}\, y(k-2)$ are sold as used in semester k.

$$f(k) = y(k) + \frac{1}{4}y(k-1) + \frac{1}{16}y(k-2)$$

$$f(k) = y(k) + \frac{1}{4}y(k-1) + \frac{1}{16}y(k-2)$$