Power Spectra

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 4.

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This is also the FT of the autocorrelation,

$$\Phi(f)\Phi^*(f) = F[\phi(t) * \phi^*(-t)] = F[\phi(t) \star \phi^*(t)]$$

Note that, $\Phi^*(f) = [\Phi(f)]^* = (F[\phi(t)])^*$

$$= \left(\int_{-\infty}^{\infty} \phi(t) e^{-i2\pi f t} dt\right)^* = \int_{-\infty}^{\infty} \phi^*(t) e^{-i2\pi f(-t)} dt$$

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Now τ is just a dummy variable of integration so it doesn't matter if we call it τ or t anymore.

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We often refer to noise that has equal power at all frequencies as "white noise". This stems from the fact that white light contains equal contributions from all colors (or frequencies of electromagnetic radiation). More on noise later.

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Some signals such as tides, seismic noise, wind, etc last, essentially forever from a human perspective (or at least for a very long time) and thus have infinite energy,

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Power is energy per unit time ${}^{E}/_{T}$ and for stationary signals we can estimate the power of the signal using,

$$P_T = \frac{1}{T} \int_{-T/2}^{T/2} |\phi(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} \left| \phi(t) \Pi\left({}^{t}/_{T} \right) \right|^2 dt$$

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The PSD will have units of ${}^{u^2}/_{Hz}$ so it really is a spectral density that shows how power is distributed over each frequency.

$$PSD[\phi(t)] = \lim_{T\to\infty} \frac{1}{T}\Phi_T(f)\Phi_T^*(f), \text{ where } \Phi_T(f) = F[\phi(t)\Pi(^t/_T)]$$

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That convolution causes our estimate of the power at each frequency to include contributions of the power from neighboring frequencies weighted by the sinc function.

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Choose a smoother window.

Run Aster matlab program, leaktest.

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Different sensors and data may have different units (e.g. velocity or acceleration) so always be sure to properly label your units as "db wrt $u^2/_{Hz}$" where u are your units. For example "db wrt $(^{micron}/_{s})^2/_{Hz}$" or "db wrt $\left(^{m}/_{s^2}\right)^2/_{Hz}$" or "db wrt $^{m^2}/_{s^4}/_{Hz}$"

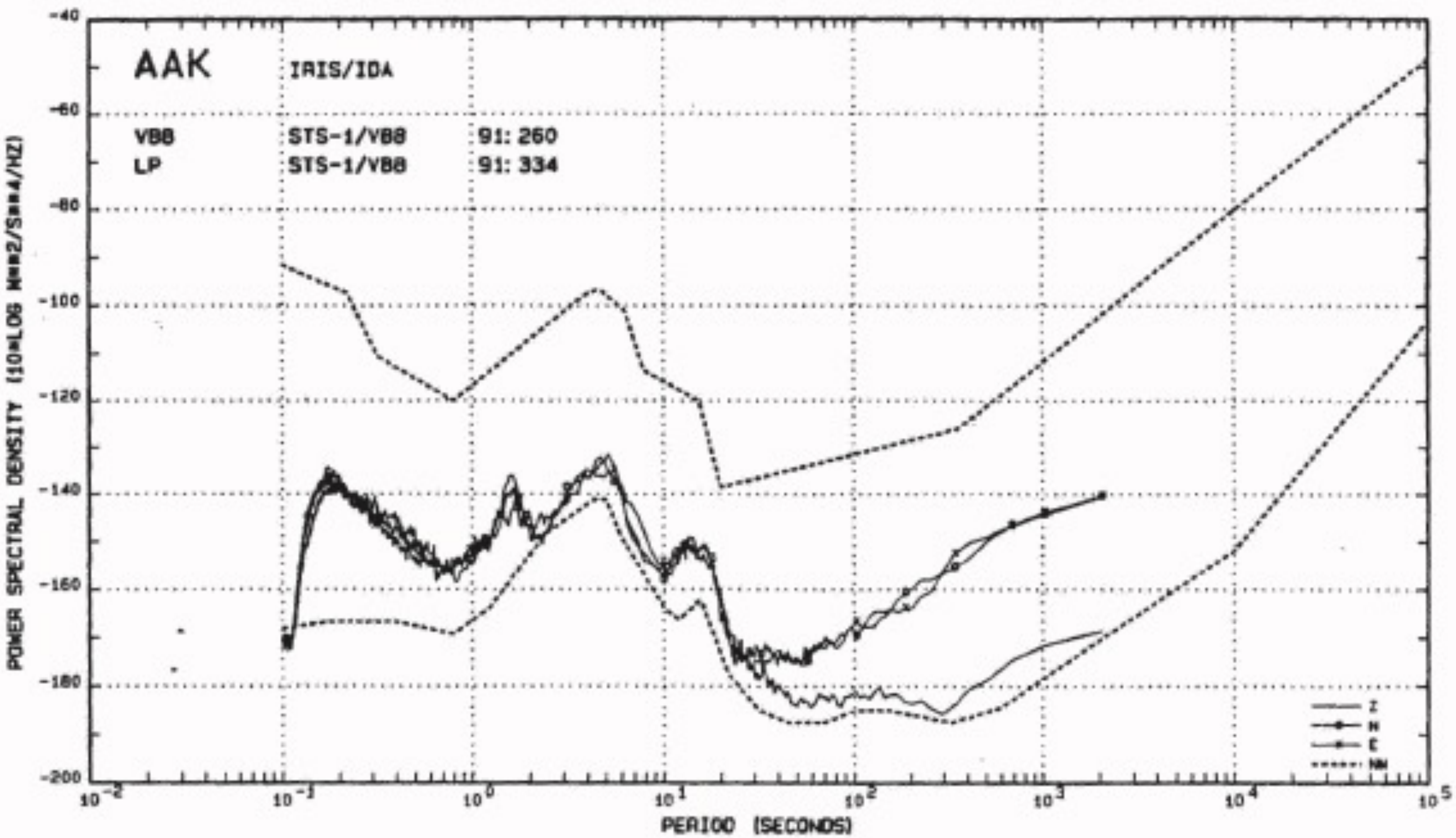

Figure 14: Earth Acceleration Power Spectral Density for background noise at the Ala Archa IRIS/IDA station as a function of period. Z, N, E refer to vertical, north, and east seismometer components. Curves labeled NM are the empirical noise model bounds of Peterson (1994) denoting to extremal PSD values from stations installed around the world. The reference (0 db) level is (1 m/s2)2/Hz. PSD estimates were obtained using Welch’s method.

It is sometimes useful to track changes in the PSD over time. We just calculate the PSD estimate multiple times advancing the window by some T for each new PSD. We then plot the results in 3-d using color. This is called a spectragram.

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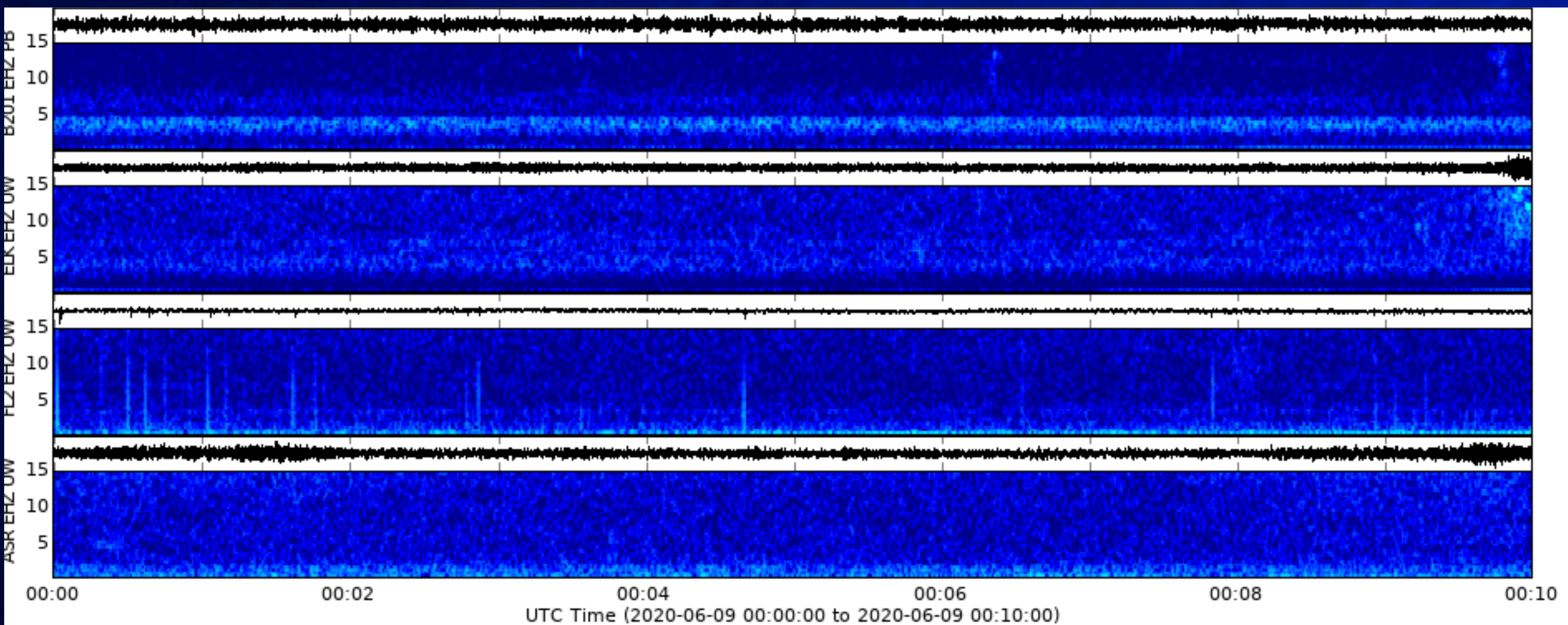
The PNSN has nice continuously updating spectragrams for their volcano monitoring. This is the one for Mt St. Helens.



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Welch's Method.

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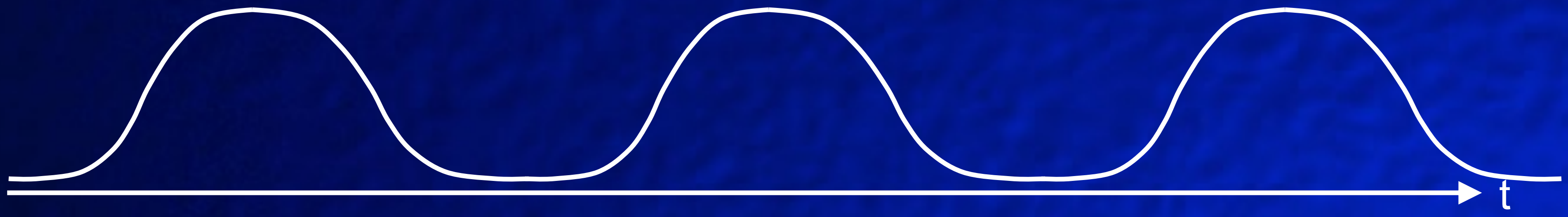
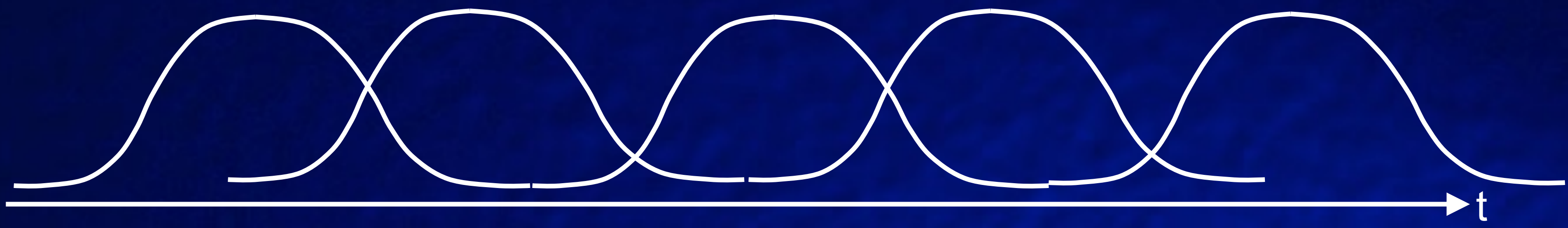


But we lose a lot of data by down weighting at the edge of the windows.

So we allow the windows to overlap.



This is Welch's method. It is common to overlap windows by 50%.

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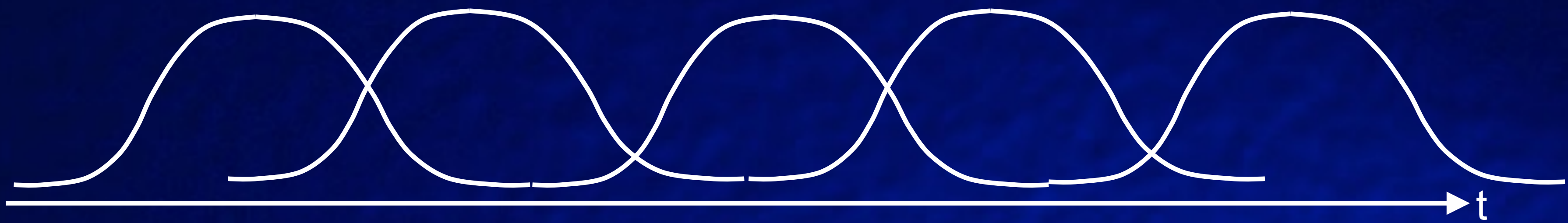

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And there's a handy matlab program called pwelch to do this.

[Pxx,F] = pwelch(X,window,Noverlap,nfft,Fs)
Pxx = psd (one-sided is default)
F = frequency vector (Hz)
X = data vector, real
Window = type of window (Hamming is default)
Noverlap = number of points of overlap
nfft = number of points in each fft
Fs = sample rate

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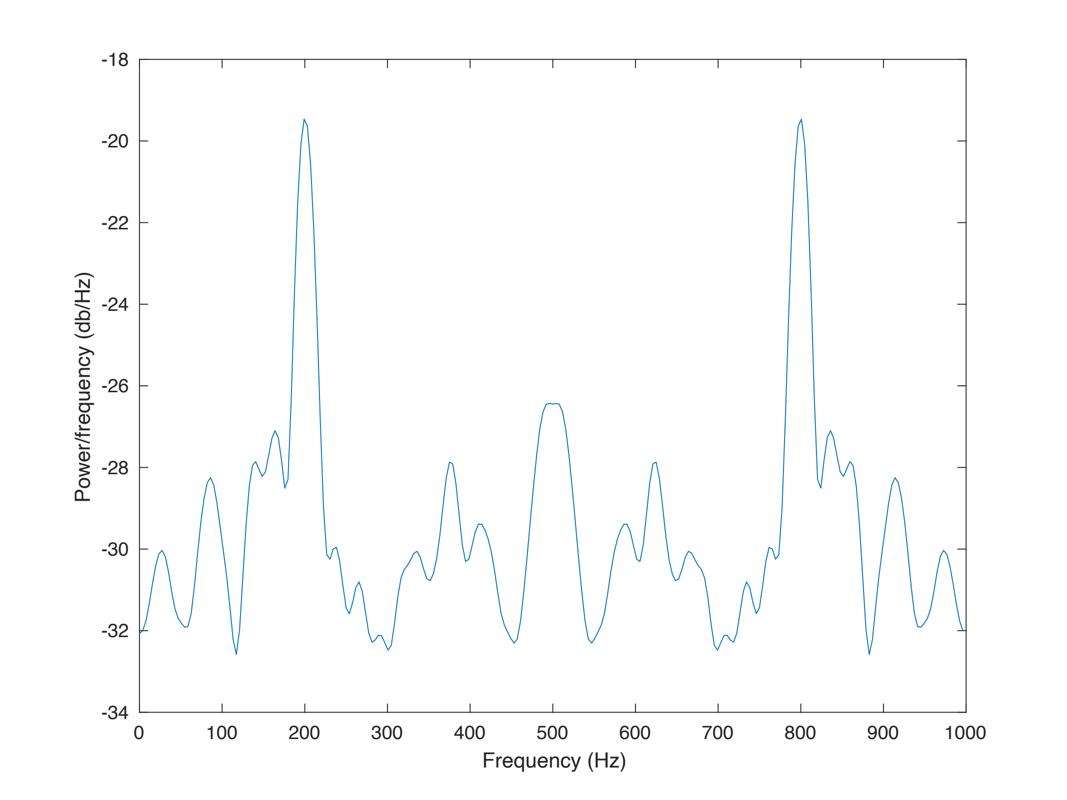
Example:

```
Fs = 1000; t=0:1/Fs:.296;
x = cos(2*pi*t*200)+randn(size(t)); %A cosine of 200Hz plus noise.
[Pxx,F]=pwelch(x,[],[],[],Fs,'twosided'); %Uses default window, overlap, and NFFT
```



Multitaper Spectral Estimation similarly to Welch's Method computes an average PSD over several windows. This method, while computationally much more expensive, reduces data and bandwidth loss from windowing by operating on the same data segment T.

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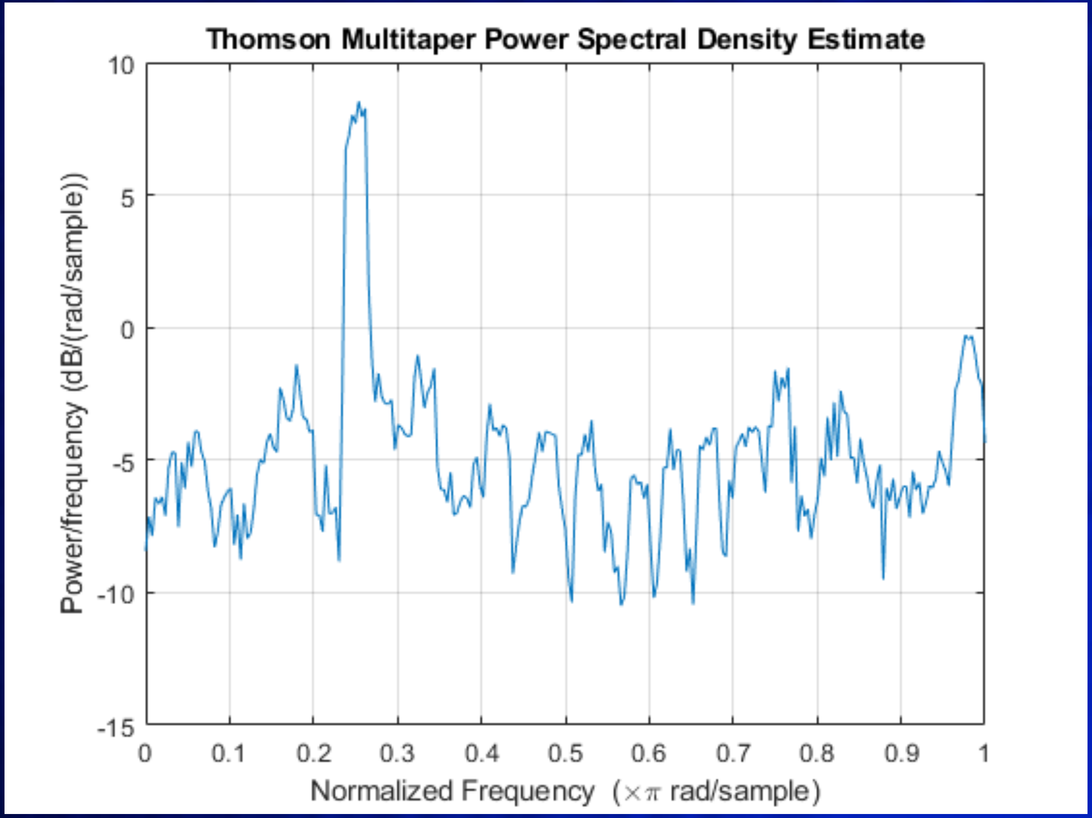
A suite of prolate spheroidal tapers is used to window the data.

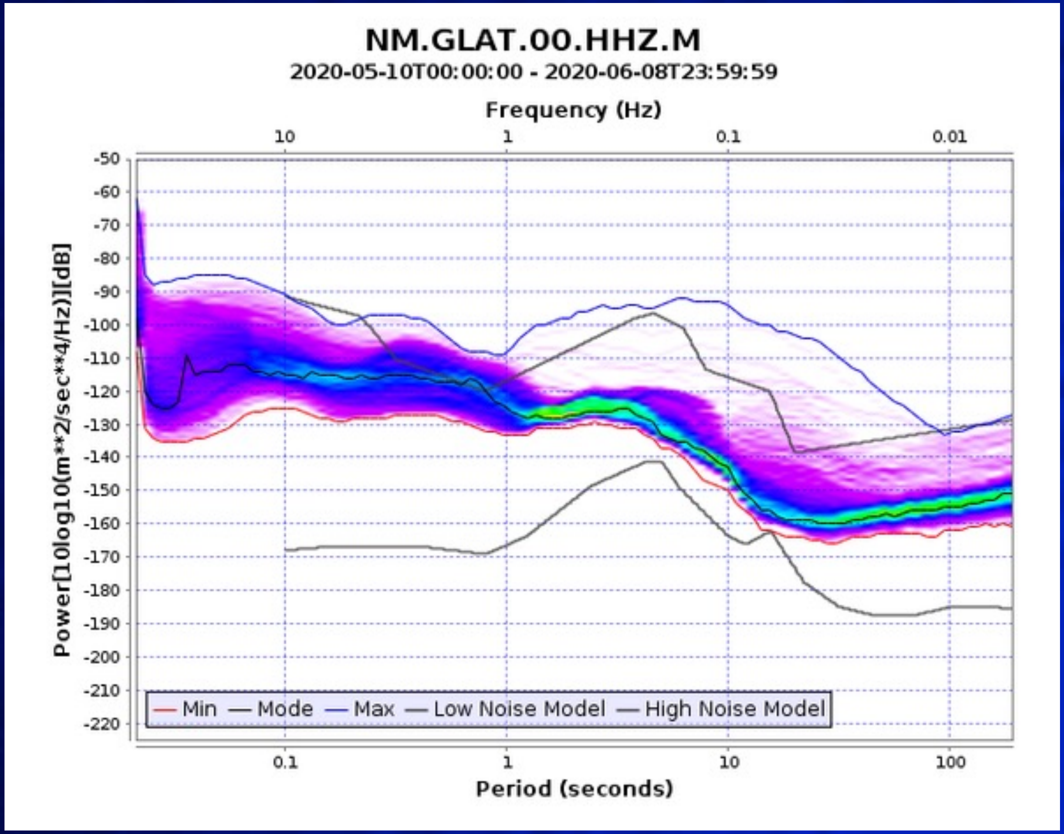
The matlab pmtm command computes the PSD using the multitaper prolate spheroid method. https://www.mathworks.com/help/signal/ref/pmtm.html

```
n = 0:319; x = cos(pi/4*n)+randn(size(n)); pxx = pmtm(x);
```



PSD PDF's are composed of combining many PSD's into an empirical Probability Density Function. They are useful for observing long term trends and signal characteristics of a seismic station.

IRIS has a tool to produce PDF plots for any channel in there archive. Start here and choose PDF Plot. https://ds.iris.edu/mustang/databrowser/



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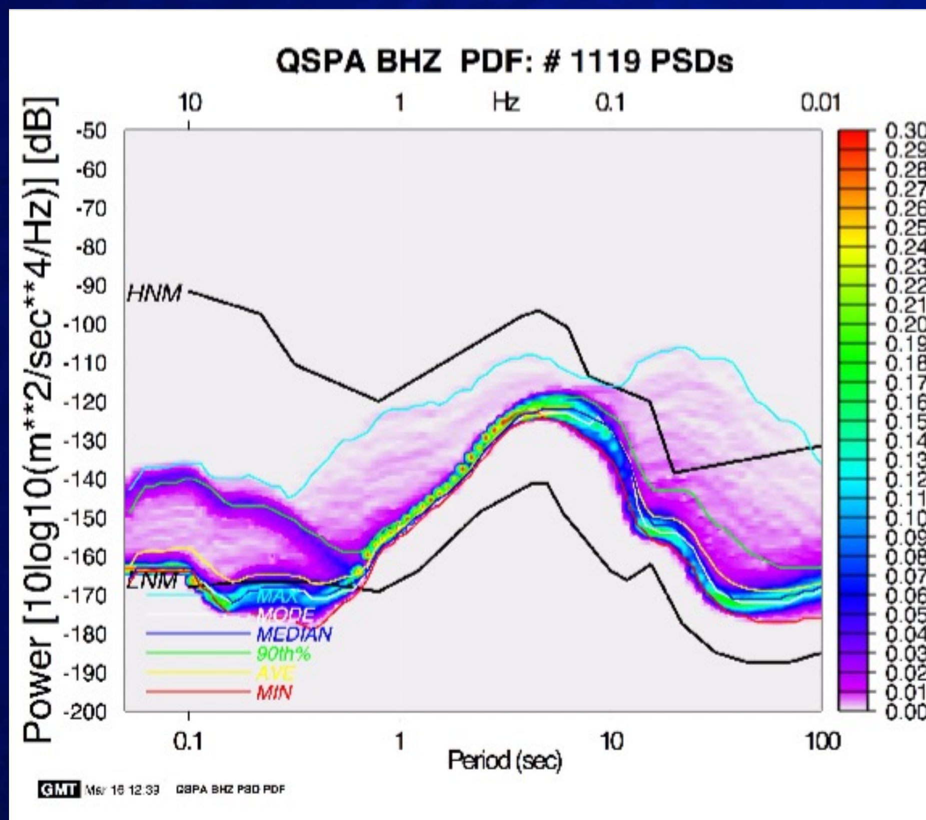
It is useful for finding local noise sources, instrument problems (especially ephemeral ones), and other long term trends in the data.

From Aster, “As a final indication of the great utility of the PSD, the Figure (15) shows processed PSDs from a broadband seismometer (Guralp CMG-3Tb) located in a 255-m deep borehole in the polar icecap near the South Pole. A great many of 1-hour data length, 50% overlap, PSDs using a hamming taper, were calculated from the month of May, 2003, and the resulting individual PSDs were used to assemble an empirical probability density function for the signal characteristics at he station The bifurcation of the high frequency noise is caused by intermittent periods where tractors are moving snow near the station. Pink misty areas concentrated around 1 and 20 s are PSDs that include teleseismic earthquake signals. At short periods this is among the quietest stations on Earth. “



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The High and Low Noise Models (HNM and LNM) are constructed using many PSD's from the Global Seismic Network (or some superset that includes the GSN) and extracting the highest and lowest amplitudes at each frequency across all PSD's.

The HNM and LNM may contain data points from many different stations. Each point represents the highest or lowest recorded for that frequency in the entire global dataset. (No station will be as good across all frequencies as the LNM though it is possible to construct a station that is noisier than the HNM at all frequencies).