Introduction to Linear Systems in the Frequency Domain.

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See Aster and Borchers, Time Series Analysis, chapter 2.

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Run matlab program sumcosine

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$$sinc(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin(x)}{x}, & x \neq 0 \end{cases}$$

$$\lim_{a \to 0} \frac{1}{a} sinc\left(\frac{x}{a}\right) = \delta(x)$$

Fourier theory tells us we can decompose a reasonably well behaved signal in time (or space) into its frequency components (complex sinusoids).

Recall, $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i2\pi ft} = \cos(2\pi ft) + isin(2\pi ft)$$

Where f is in Hz (cycles/s) and t is in seconds

Sometimes, we use radians (units of 1/seconds)

$$e^{i\omega t} = \cos(\omega t) + isin(\omega t)$$

Given the impulse response, $h(t)$, of a linear system h, we can find it's response, $g(t)$, to a complex unit sinusoid, $e^{i2\pi ft}$, by convolution.

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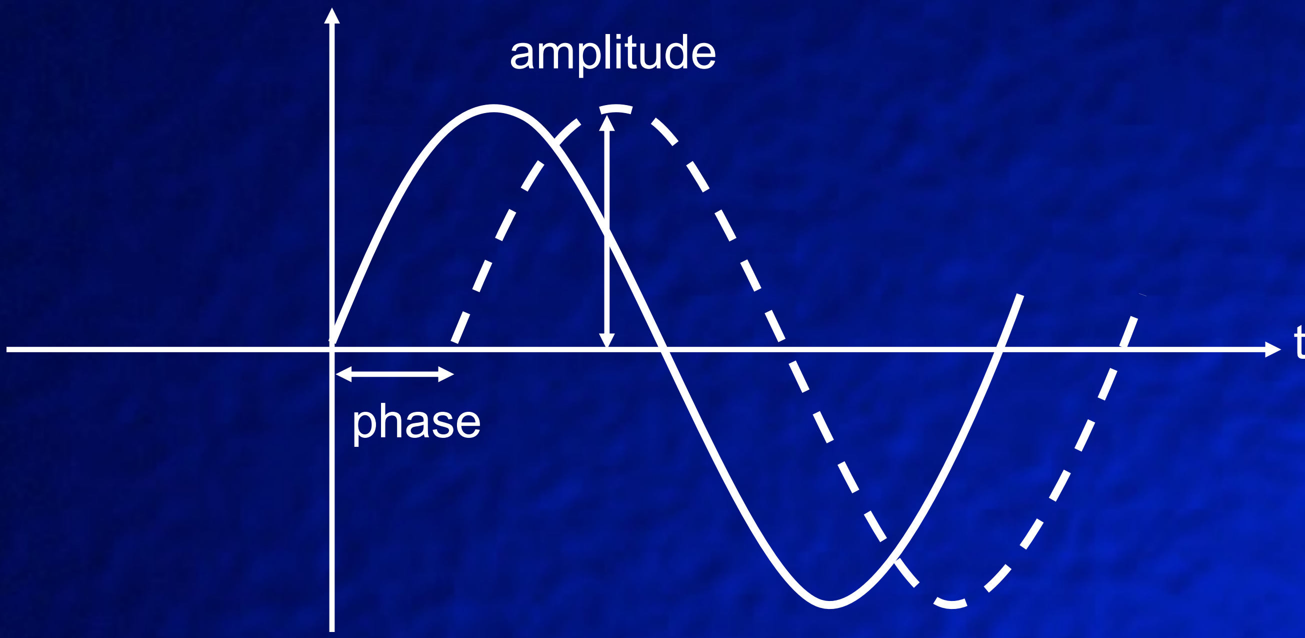
This tells us that the response of a linear system to a complex sinusoid remains unchanged in functional form and only the input's amplitude and phase are modified.


amplitude
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In fact, $g(t) = e^{i2\pi ft} \int_{-\infty}^{\infty} h(\tau) e^{-i2\pi f\tau} d\tau \equiv e^{i2\pi ft} H(f)$

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And in general, the Fourier Transform (not to be confused with the Fourier series) is thus defined for an arbitrary $\phi(t)$ as,

Forward Transform $$\Phi(f) = F[\phi(t)] \equiv \int_{-\infty}^{\infty} \phi(t)e^{-i2\pi ft} dt$$

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Another way to view the inverse transform is decomposing $\phi(t)$ into it's individual frequency components each with its own amplitude and phase.

It is common in the physical sciences to characterize linear systems as a differential equation. And in general,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1 \frac{d^1 y}{dt^1} + a_0 y$$

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Let $x(t) = e^{i2\pi f t}$ (a complex sinusoid input), then $y(t) = \Phi(f) e^{i2\pi f t}$

Where in this case, $x(t)$ is the input to the linear system characterized by the differential equation with frequency domain impulse response $\Phi(f)$ and output $y(t)$.

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi f t} dt = F[\phi(t)]$$

Now plug in our values for x and y.

$$a_n \frac{d^n}{dt^n}\left[\Phi(f)e^{i2\pi ft}\right] + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}\left[\Phi(f)e^{i2\pi ft}\right] + a_{n-2}\frac{d^{n-2}}{dt^{n-2}}\left[\Phi(f)e^{i2\pi ft}\right] + \cdots$$

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$$\Phi(f)\left\{a_n(i2\pi f)^n e^{i2\pi ft} + a_{n-1}(i2\pi f)^{n-1}e^{i2\pi ft} + \cdots + a_1(i2\pi f)^1 e^{i2\pi ft} + a_0 e^{i2\pi ft}\right\}$$

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Roots of polynomial are the zeros

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Recall the quadratic equation, $ax^2 + bx + c = 0$, has roots at $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

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So if we can model our physical system with a differential equation and determine the weights, a and b, then we can predict the output y, for any given input x by convolving the input with the impulse response of the system.

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So if we can model our physical system with a differential equation and determine the weights, a and b, then we can predict the output y, for any given input x by convolving the input with the impulse response of the system.

This is a remarkable result. It says we can model a system, make a prediction, and then test the expected result against measured data.

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Recap

For a linear system with impulse response, $\phi(t)$, and a complex sinusoid input, the output is:

$$g(t) = \phi(t) * e^{i2\pi ft} = \Phi(f)e^{i2\pi ft}$$

and

$$\Phi(f) = F[\phi(t)] = \int_{-\infty}^{\infty} \phi(t)e^{-i2\pi ft}dt$$

$$\phi(t) = F^{-1}[\Phi(f)] = \int_{-\infty}^{\infty} \Phi(f)e^{i2\pi ft}df$$

Recap (cont.)

And for a generic differential equation, we can use a complex sinusoid for the input to solve for the impulse response of the system compactly represented by poles and zeros.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{d^1 y}{dt^1} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{d^1 x}{dt^1} + b_0 x$$

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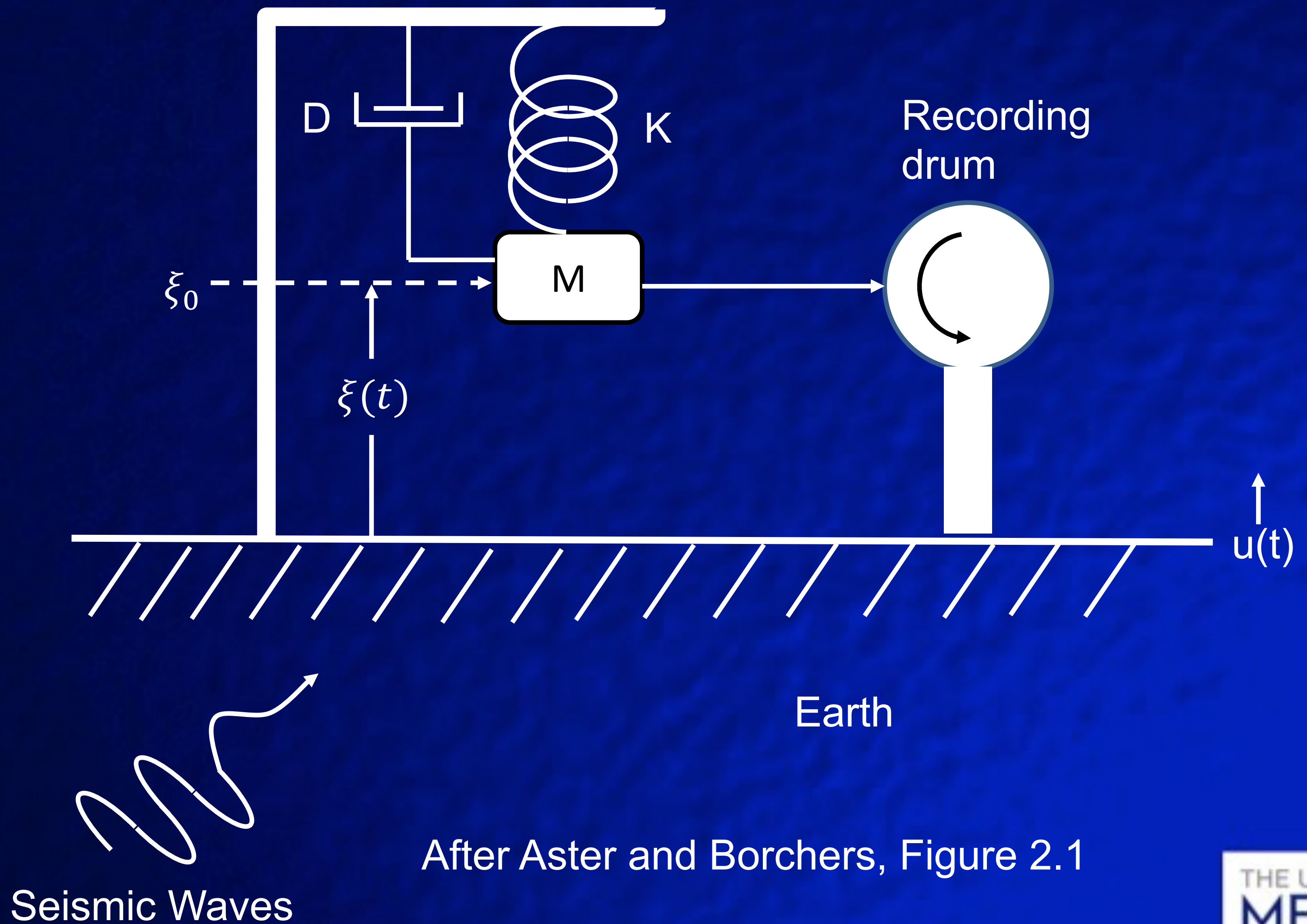
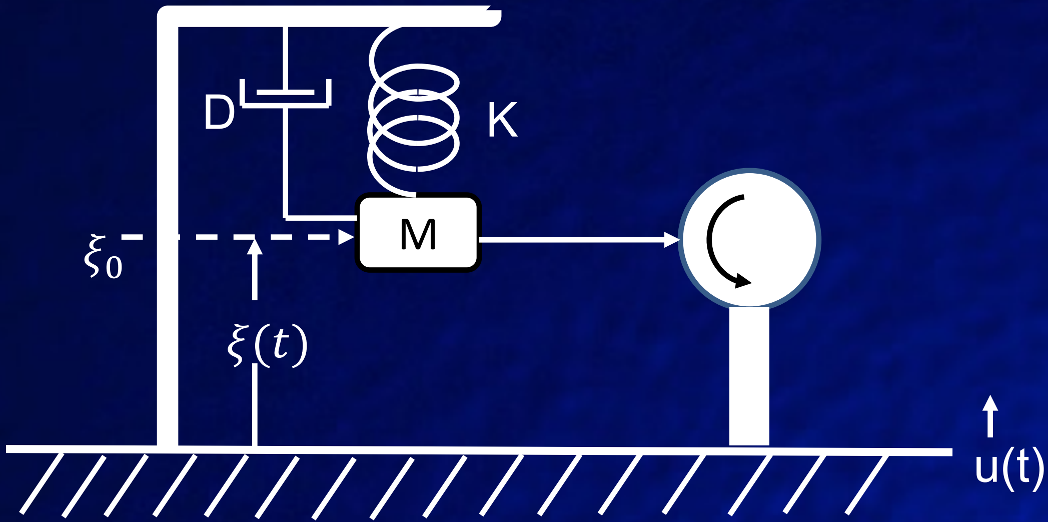
Roots of polynomial are the poles (where the denominator is 0)

A Mechanical Seismometer



After Aster and Borchers, Figure 2.1



$K = spring\ constant$

$D = dashpot\ damping$

$M = seismometer\ mass$

$\xi_0 = mass\ equilibrium\ position$

$\xi(t) = mass\ position\ wrt\ surface\ of\ the\ earth\ as\ a\ function\ of\ time.$

$u(t) = displacement\ of\ the\ surface\ as\ a\ function\ of\ time$

Newton's second law, $F_{up} = Ma_{up}$ upward force is positive

Recall Hook's law, $F(x) = -kx$

Damping depends on velocity $\frac{d}{dt}\xi(t)$

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Recall Hook's law, $F(x) = -kx$

Damping depends on velocity $\frac{d}{dt}\xi(t)$

The force of the spring depends on displacement from equilibrium, $\xi(t) - \xi_0$

$$\overbrace{\underbrace{-D\frac{d\xi(t)}{dt}} - \underbrace{K[\xi(t) - \xi_0]}}^{F} = \overbrace{\underbrace{M\frac{d^2}{dt^2}[\xi(t) + u(t)]}}^{Ma}$$

Damping force operates in the opposing direction to velocity.

Force of the spring.

Acceleration is second derivative of the mass position

$$-D\frac{d\xi(t)}{dt} - K[\xi(t) - \xi_0] = M\frac{d^2}{dt^2}[\xi(t) + u(t)]$$

Let $z(t) = \xi(t) - \xi_0$, and $\frac{dz}{dt} = \frac{d}{dt}\xi(t) = \dot{z}$

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$$-D\dot{z} - kz = M\ddot{z} + M\ddot{u}$$

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$$-D\dot{z} - kz = M\ddot{z} + M\ddot{u} \Rightarrow M\ddot{z} + D\dot{z} + kz = -M\ddot{u}$$

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$$\ddot{z} + \frac{D}{M}\dot{z} + \frac{K}{M}z = -\ddot{u}$$

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Let $2\zeta = \frac{D}{M}$ ⇒ The seismometer damping coefficient.

and $\omega_s^2 = \frac{K}{M}$ ⇒ The natural period of the seismometer.

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Let $2\zeta = \frac{D}{M}$ $\Rightarrow$ The seismometer damping coefficient.

and $\omega_s^2 = \frac{K}{M}$ $\Rightarrow$ The natural period of the seismometer.

then $\ddot{z} + 2\zeta\dot{z} + \omega_s^2 z = -\ddot{u}$ $\Rightarrow$ The differential equation describing our linear system, a mechanical seismometer.

$u(t)$ is the input

$z(t)$ is the output

$$\ddot{z} + 2\zeta\dot{z} + \omega_s^2 z = -\ddot{u}$$

$$u(t) \longrightarrow \boxed{\phi} \longrightarrow z(t)$$

$u(t)$ is the input

$z(t)$ is the output

$$\ddot{z} + 2\zeta\dot{z} + \omega_s^2 z = -\ddot{u}$$

Let the input $u(t) = e^{i2\pi ft}$, a complex sinusoid

Then the output $z(t) = \Phi(f)e^{i2\pi ft}$ the complex sinusoid input multiplied by the frequency domain impulse response of the linear system.

$$e^{i2\pi ft} \longrightarrow \boxed{\phi} \longrightarrow z(t)$$

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$$\ddot{u} = (i2\pi f)^2 e^{i2\pi f t} = -\omega^2 e^{i\omega t}$$

$\dot{z} = i\omega e^{i\omega t}\Phi(\omega)$ and $\ddot{z} = -\omega^2 e^{i\omega t}\Phi(\omega)$

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$$-\omega^2 e^{i\omega t}\Phi(\omega) + 2\zeta i\omega e^{i\omega t}\Phi(\omega) + \omega_s^2 e^{i\omega t}\Phi(\omega) = \omega^2 e^{i\omega t}$$

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$$\Phi(\omega) = \frac{\omega^2}{-\omega^2 + 2\zeta i\omega + \omega_s^2}$$

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$$\Phi(\omega) = \frac{-\omega^2}{\omega^2 - 2\zeta i\omega - \omega_s^2}$$

This is the frequency domain impulse response of our system (aka transfer function); the transfer function (or response) of the mechanical seismometer.

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$$-\omega^2 e^{i\omega t}\Phi(\omega) + 2\zeta i\omega e^{i\omega t}\Phi(\omega) + \omega_s^2 e^{i\omega t}\Phi(\omega) = \omega^2 e^{i\omega t}$$

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How many poles are there?

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$$\Phi(\omega) = \frac{\omega^2}{-\omega^2 + 2\zeta i\omega + \omega_s^2}$$

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How many poles are there?

How many zeros?

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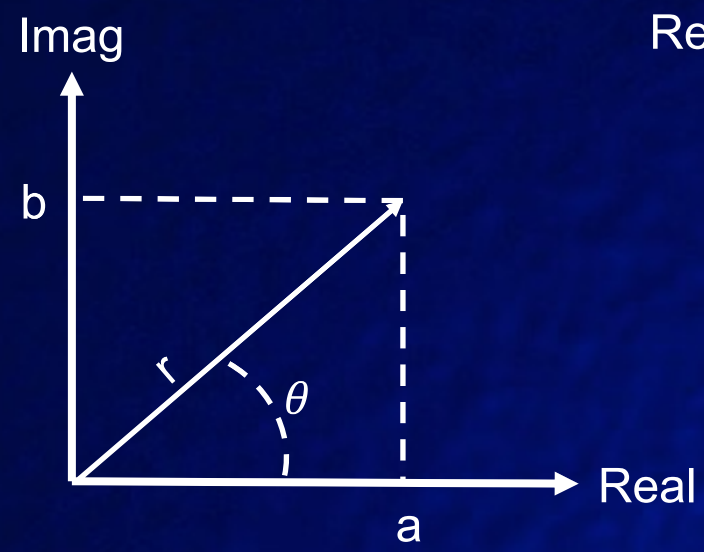
Recall



$$z = a + ib$$
$$r = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{(c^2+d^2)}$$

We can now find the amplitude response, $|\Phi(\omega)|$, and the phase response, $\theta(\omega)$.

After some algebra we find,

$$|\Phi(\omega)| = \frac{\omega^2}{\left[(\omega^2 - \omega_s^2)^2 + 4\zeta^2\omega^2\right]^{1/2}}$$

and

$$\theta(\omega) = \pi - \tan^{-1}\left(\frac{-2\zeta\omega}{\omega^2 - \omega_s^2}\right)$$

The amplitude response.

The phase response.

After some algebra we find,

$$|\Phi(\omega)| = \frac{\omega^2}{\left[(\omega^2 - \omega_s^2)^2 + 4\zeta^2\omega^2\right]^{1/2}} \quad \text{and} \quad \theta(\omega) = \pi - \tan^{-1}\left(\frac{-2\zeta\omega}{\omega^2 - \omega_s^2}\right)$$

The amplitude response. The phase response.

We can gain insight into the behavior of this linear system, the seismometer, by examining the behavior of the impulse response at extremes of high and low frequencies.

$$\lim_{\omega \gg \omega_s} |\Phi(\omega)| \approx \frac{\omega^2}{\left[\omega^4 + 4\zeta^2\omega^2\right]^{1/2}} = \frac{\omega}{\sqrt{\omega^2 + 4\zeta}} \approx 1$$

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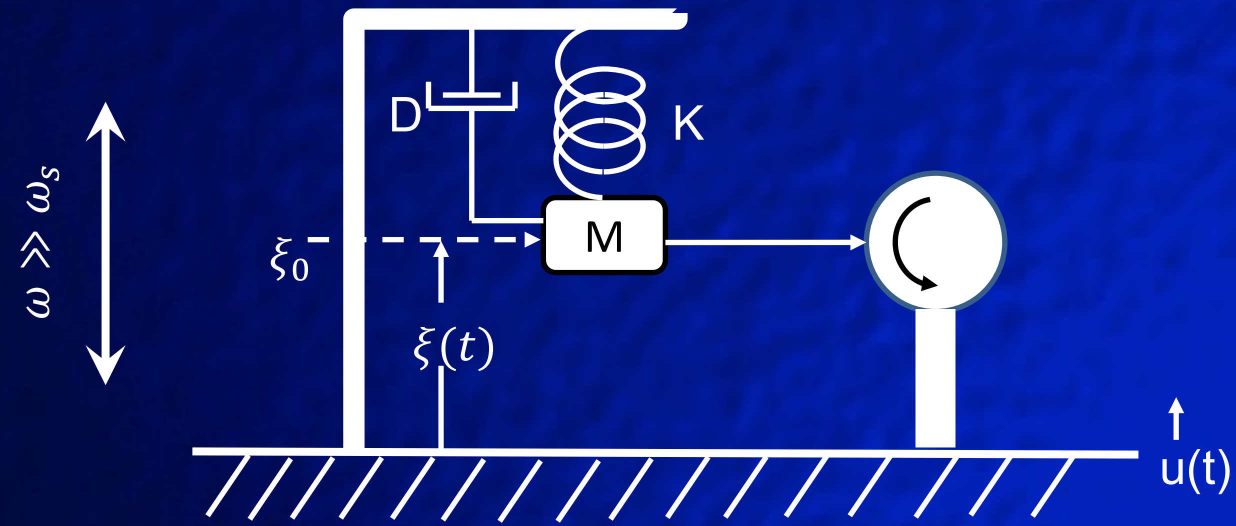
The seismometer mass is moving perfectly in sync with the input ground motion but 180° out of phase. The frequency is so high making the up and down motion so fast, that the mass is essentially still from an external reference frame while the rest of the pendulum moves up and down with the ground motion u(t). That is $\xi \approx -u$.



Now for very low frequencies,

$$\lim_{\omega \ll \omega_s} |\Phi(\omega)| = \frac{\omega^2}{\omega_s^2 \sqrt{\left(\frac{\omega^2}{\omega_s^2} - 1\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_s}\right)^2}} \approx \frac{\omega^2}{\omega_s^2} \approx 0$$

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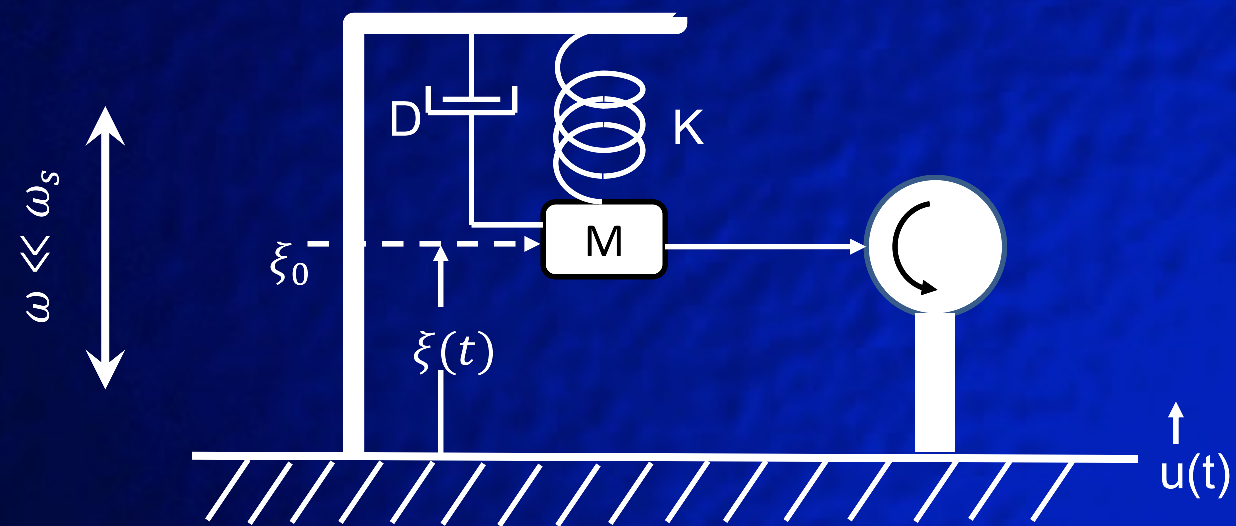
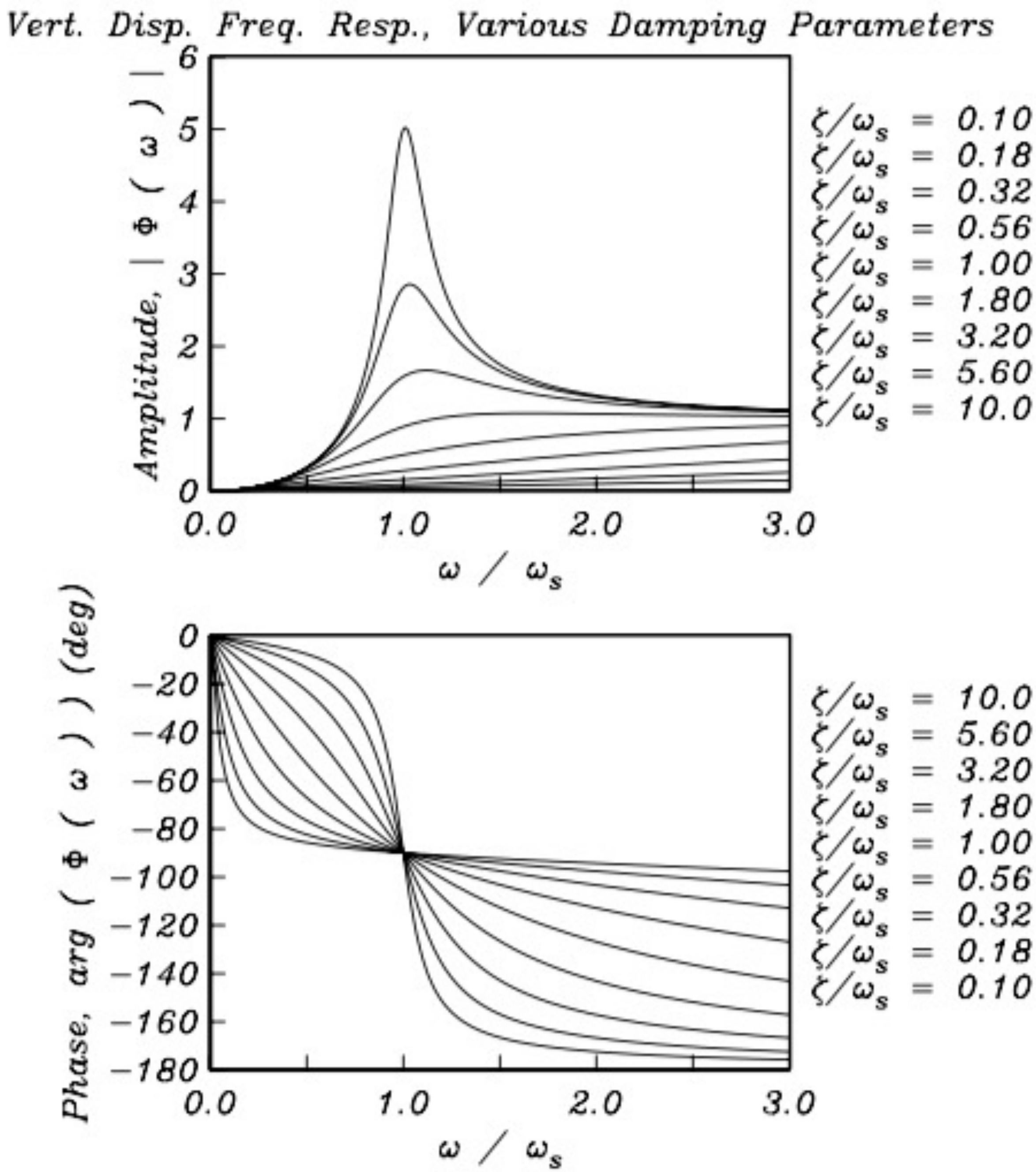
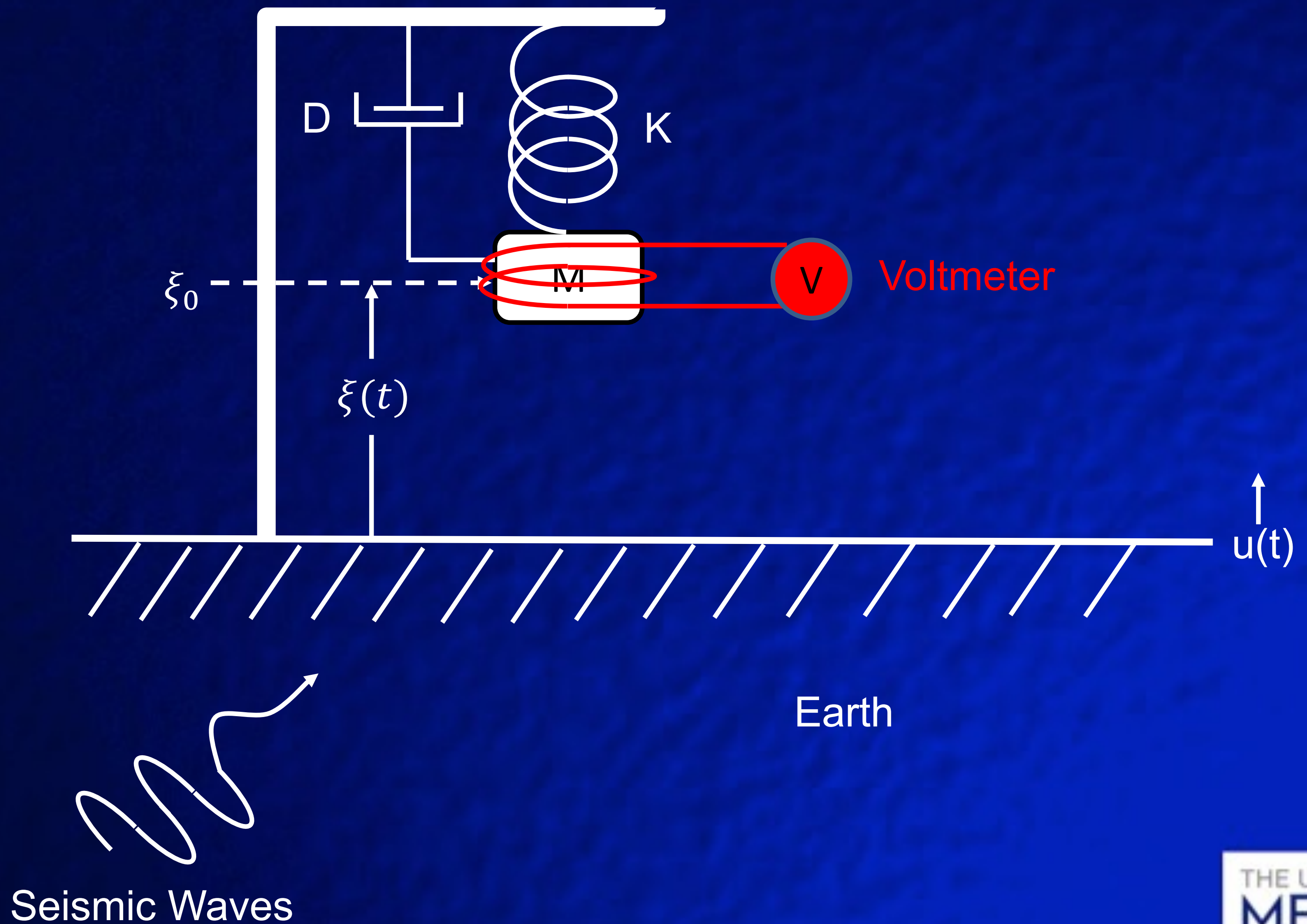
It stops responding at all! Picture moving the frame up and down very slowly. Move it so slow that the spring never stretches or compresses. So that $\zeta(t) = \zeta_0$, a constant independent of u(t). That is z(t)=0.





Vert. Disp. Freq. Resp., Various Damping Parameters
Amplitude, | Φ (ω) |
0
1
2
3
4
5
6
0.0
1.0
2.0
3.0
ω / ω_s
ζ/ω_s = 0.10
ζ/ω_s = 0.18
ζ/ω_s = 0.32
ζ/ω_s = 0.56
ζ/ω_s = 1.00
ζ/ω_s = 1.80
ζ/ω_s = 3.20
ζ/ω_s = 5.60
ζ/ω_s = 10.0
Phase, arg (Φ (ω)) (deg)
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
0.0
1.0
2.0
3.0
ω / ω_s
ζ/ω_s = 10.0
ζ/ω_s = 5.60
ζ/ω_s = 3.20
ζ/ω_s = 1.80
ζ/ω_s = 1.00
ζ/ω_s = 0.56
ζ/ω_s = 0.32
ζ/ω_s = 0.18
ζ/ω_s = 0.10

Convert our Mechanical Seismometer to an induction output



We're now measuring the mass velocity ($\dot{z}$) from a displacement input u(t).

Use the same complex sinusoid input, $u(t) = e^{i\omega t}$

The displacement output is still, $\mathrm{z} = \Phi_{disp}(f)e^{i\omega t}$

And $\dot{z} = i\omega e^{i\omega t}\Phi_{disp}(f)$

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We'll see later that if $\phi(t) * x(t) = y(t)$, then $\Phi(f) = \frac{Y(f)}{X(f)}$

Thus, $\Phi_{disp}(f) = \frac{z(\omega)}{u(\omega)}$ and $\Phi_{induction}(f) = \frac{\dot{z}(\omega)}{u(\omega)} = i\omega\Phi_{disp}(f)$

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Thus, $\Phi_{disp}(f) = \frac{z(\omega)}{u(\omega)}$ and $\Phi_{induction}(f) = \frac{\dot{z}(\omega)}{u(\omega)} = i\omega\Phi_{disp}(f)$

$$\Phi_{induct}(\omega) = \frac{-i\omega^3}{\omega^2 - 2\zeta i\omega - \omega_s^2}$$

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That means that to convert $\Phi(\omega)$ to velocity output, we need only multiply by $i\omega$; "add" a zero.

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That means that to convert $\Phi(\omega)$ to velocity output, we need only multiply by $i\omega$; “add” a zero.

To measure velocity output as a function of velocity input, divide by $i\omega$; remove a zero.

$$\frac{\dot{z}}{\dot{u}} = \frac{-\omega^2}{\omega^2 - 2\zeta i\omega - \omega_s^2}$$

Parseval's Theorem

The Fourier Transform conserves energy. $E \propto A^2$

$$\phi(t) = F^{-1}[\Phi(f)] = \int_{-\infty}^{\infty} \Phi(f) e^{i2\pi f t} df$$

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$$\phi^*(t) = \{F^{-1}[\Phi(f)]\}^* = \left\{\int_{-\infty}^{\infty} \Phi(f) e^{i2\pi f t} df\right\}^*$$

Where * is the complex conjugate.

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$$= \int_{-\infty}^{\infty} \Phi^*(f) e^{-i2\pi ft} df$$

Energy = $\int_{-\infty}^{\infty} \phi(t)\phi^*(t)dt$

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Parseval's Theorem, cont.

The Fourier Transform conserves energy. $E \propto A^2$

Energy = $\int_{-\infty}^{\infty} \phi(t)\phi^*(t)dt$

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Parseval's Theorem, cont.

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$$= \int_{-\infty}^{\infty} \Phi^*(f)\Phi(f)df$$

The sum of the squares in time is equal to the sum of the squares in frequency.

Parseval's Theorem, cont.

The Fourier Transform conserves energy. $E \propto A^2$

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The sum of the squares in time is equal to the sum of the squares in frequency.

The FT conserves energy.

Also a nice way to check your code.

Properties of the Fourier Transform

The scaling property

$$F[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\alpha t) e^{-i2\pi f t} dt$$

Let $\tau = \alpha t$, then $d\tau = \alpha dt$

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$$F[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f \tau/\alpha} \frac{d\tau}{\alpha}$$

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$$= \frac{1}{\alpha} \Phi\left(\frac{f}{\alpha}\right)$$

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$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi \frac{f}{\alpha} \tau} d\tau$$

$$= -\frac{1}{\alpha} \Phi\left(\frac{f}{\alpha}\right)$$

If α is negative (flips the integration limits)

Properties of the Fourier Transform

The scaling property

$$F[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\alpha t) e^{-i2\pi f t} dt$$

Let $\tau = \alpha t$, then $d\tau = \alpha dt$

$$F[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f \tau/\alpha} \frac{d\tau}{\alpha}$$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi \frac{f}{\alpha} \tau} d\tau$$

$$= \frac{1}{|\alpha|} \Phi\left(\frac{f}{\alpha}\right) \longrightarrow$$

Narrow in t is wide in f (e.g. a delta in time is a cosine in frequency).

Shifting Property

$$F[\phi(t - t_0)] = \int_{-\infty}^{\infty} \phi(t - t_0) e^{-i2\pi f t} dt$$

Let $\tau = t - t_0$, then $d\tau = dt$

Shifting Property

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$$F[\phi(t-t_0)] = \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f(\tau - t_0)} d\tau$$

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$$= e^{i2\pi f t_0} \Phi(f) \Longrightarrow$$ A phase shift

Table of FT properties.

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Recall even and odd functions.

Even functions, e.g. a cosine, has symmetry such that $f(t) = f(-t)$.

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$$odd \cdot odd = even$$
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$$even \cdot even = even$$
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$$\int even \neq 0$$

$$\int odd = 0$$

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Let $u = 2\pi f t,\;\; du = 2\pi f dt$

$$= \int_{?}^{?} \cos(u) \frac{du}{2\pi f}$$

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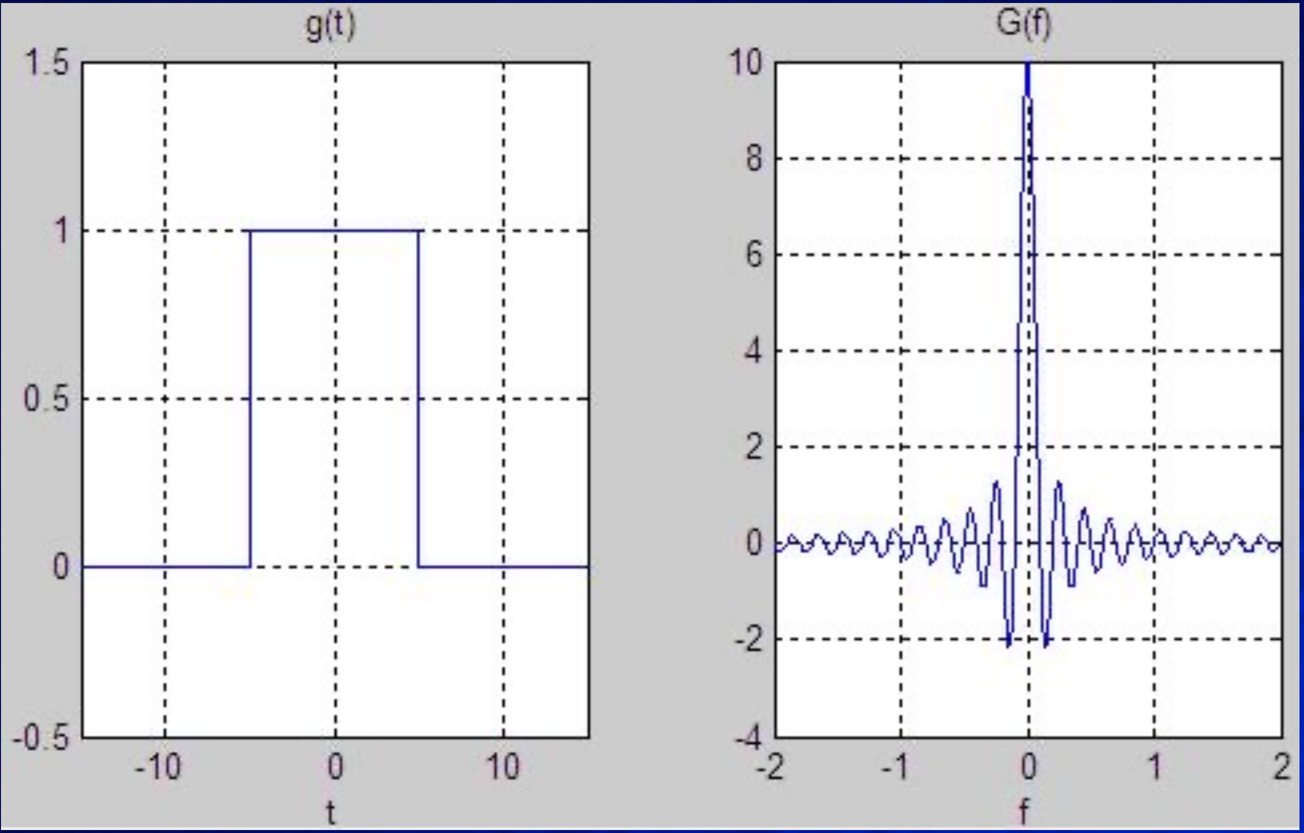
$$sinc(0) = 1$$



Sharp edges produce oscillations in the fourier transform. (more on that later)

Pictorial Dictionary of Fourier Transforms (Bracewell pp 411).

http://www.ceri.memphis.edu/people/mwithers/CERI7106/other/BracewellFTPictorialDictionary.pdf

Please do not distribute the pictorial dictionary beyond this class to avoid copyright violations.

$F[\delta(t)] = ?$ (recall the sifting property)

$$\int_a^b f(t)\delta(t - t_0)dt = f(t_0), \qquad where\ a \le t_0 \le b$$

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What is t_0 in this case?

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What is the amplitude as a function of f?
Remember z=a+ib.
What is the phase?

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$F^{-1}[1] = ?$

$F^{-1}[1] = \delta(t)$

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Hermetian Property

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If $\phi_1(t)$ is Hermetian, then $\phi_1(t) \star \phi_2(t) = \phi_1(t) * \phi_2(t)$

And if both ϕ_1 and ϕ_2 are Hermetian, then $\phi_1(t) \star \phi_2(t) = \phi_2(t) \star \phi_1(t)$.

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Convolution and the Fourier Transform

$$F[\phi_1(t) * \phi_2(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi_1(\tau)\phi_2(t-\tau)d\tau \right) e^{-i2\pi f t} dt$$

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Recall the shifting property.

$$F[\phi\,(t-\tau)] = \Phi(f)e^{-i2\pi f\tau}$$

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$$= \Phi_1(f)\Phi_2(f)$$

Convolution in the time domain is multiplication in the frequency domain (and vice versa).

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Differentiation

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}F^{-1}[\Phi(f)] = \frac{d}{dt}\int_{-\infty}^{\infty}\Phi(f)e^{i2\pi ft}df$$

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Differentiation in the time domain is multiplication by f in the frequency domain. (a high pass filter)

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It can be shown, with considerably more work, that integration in the time domain is, more or less, division in the frequency domain. (a low pass filter)

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$$F\left[\int_{-\infty}^{t}\phi(\tau)d\tau\right]=\frac{\Phi(f)}{i2\pi f}+\frac{\delta(f)}{2}\int_{-\infty}^{\infty}\phi(t)dt$$

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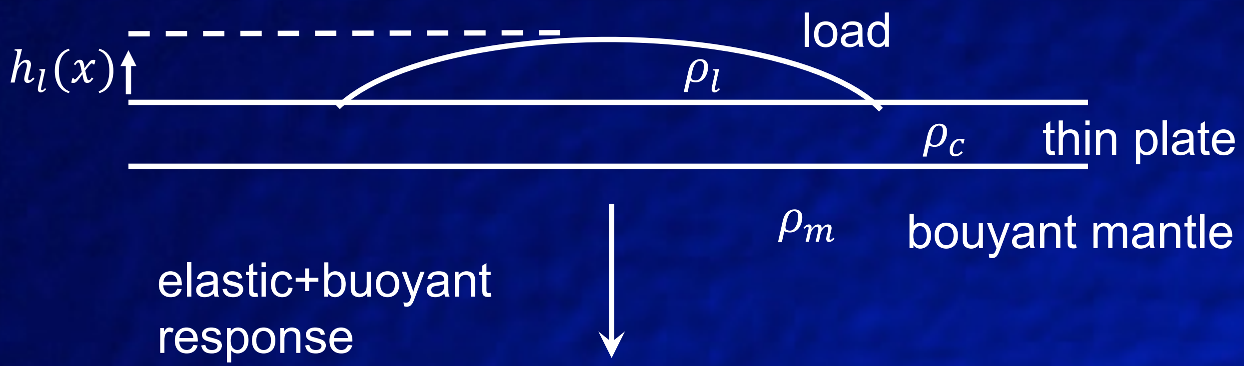
Note the integral is unstable because of division by f.

We wish to model, as a LTI system, the equilibrium elastic response of a loaded buoyant crust.



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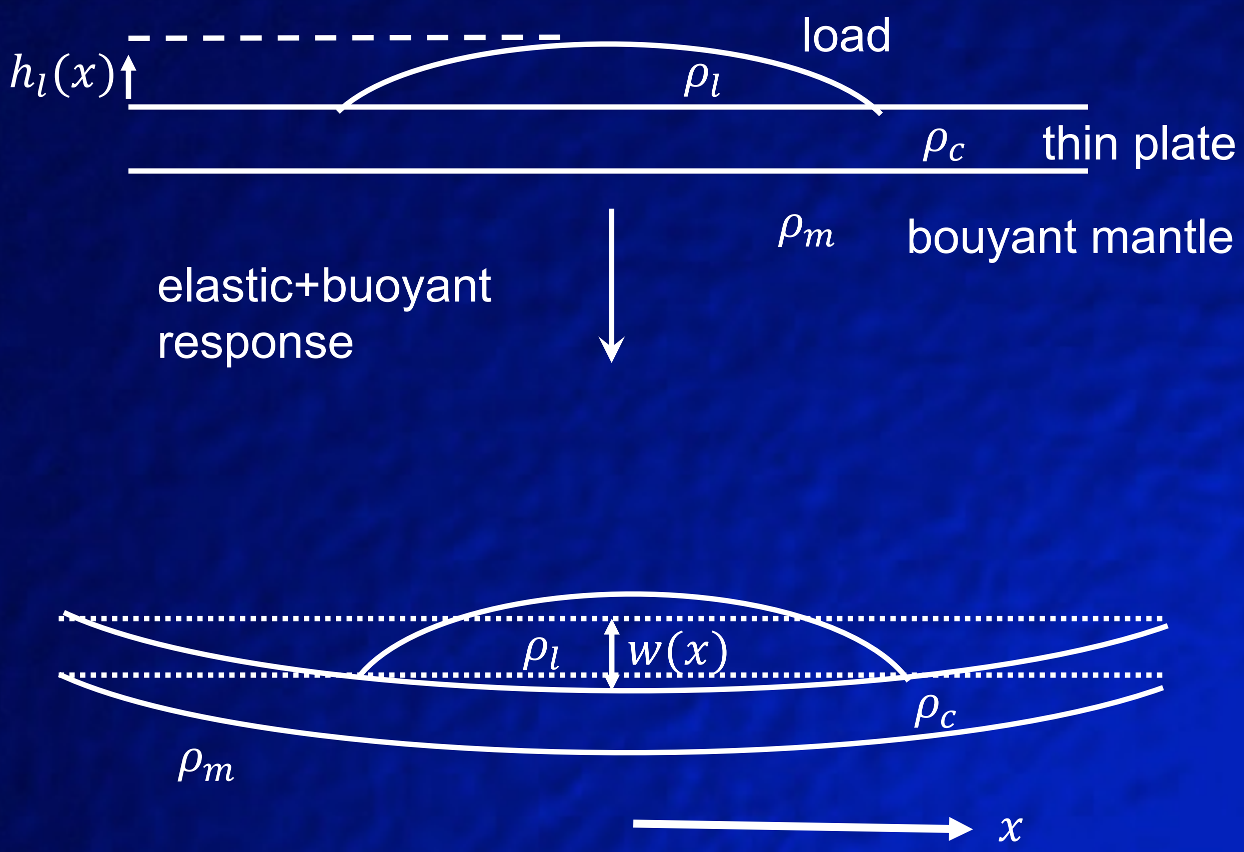

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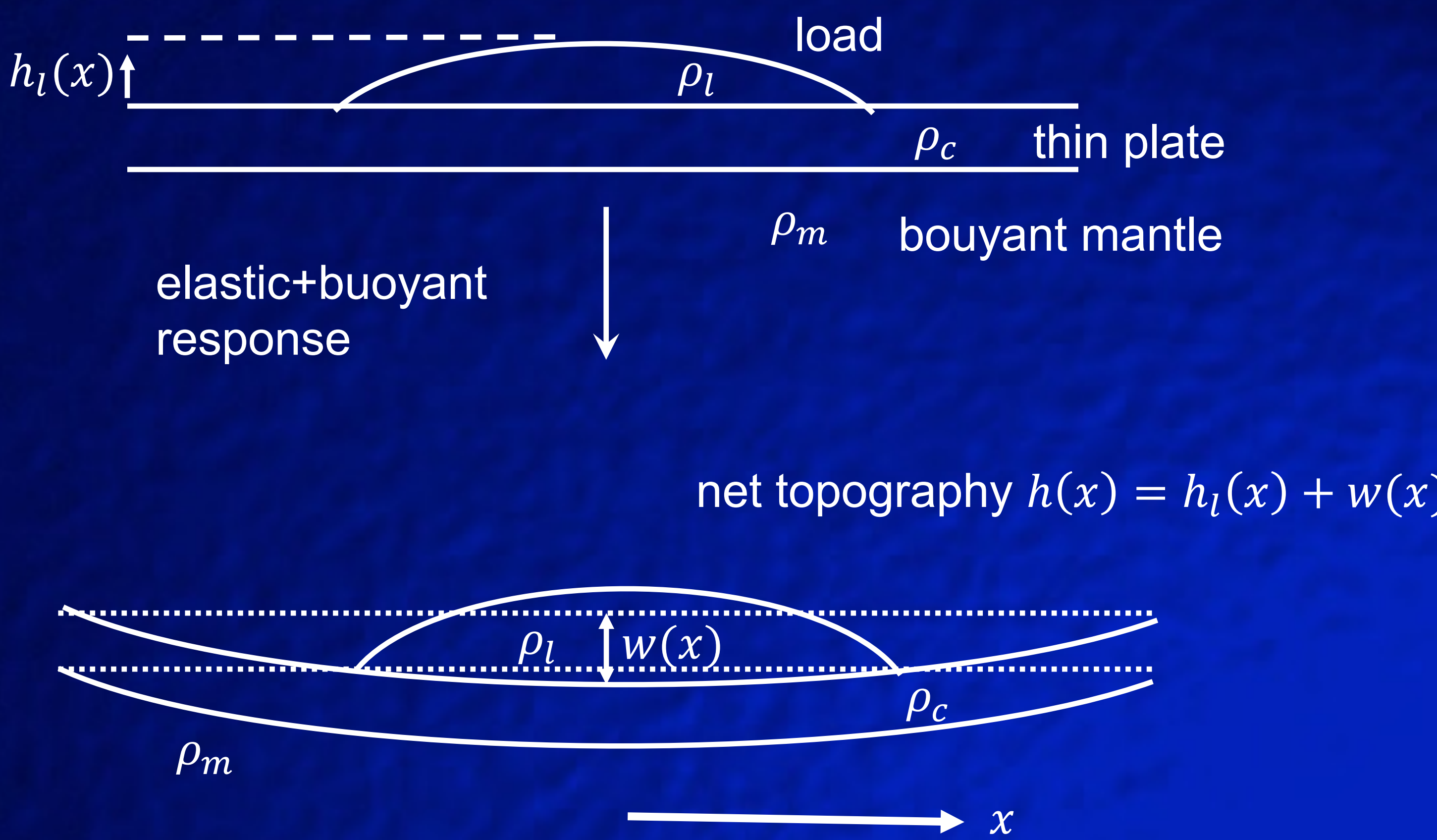

The downward force of the load is balanced by the upward buoyant force of the mantle plus the crustal rigidity resisting flexure.

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ν = poisson's ratio

$w(r)$ = deflection of the plate (or crust)

$p(r)$ = upward force (load + buoyancy)

Assume 1-d, $r \to x$ $D\nabla^4 w(x) = p(x)$

Assume 1-d, $r \rightarrow x$

$$D\nabla^4 w(x) = p(x)$$

load height

$$p(x) = \underbrace{-\rho_l g h_l(x)}_{\text{load force}} + \underbrace{B(x)}_{\text{bouyant force}}$$

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$$D\nabla^4 w(x) = p(x)$$

load height

$$p(x) = \underbrace{-\rho_l g h_l(x)}_{\text{load force}} + \underbrace{B(x)}_{\text{bouyant force}}$$

$$B(x) = -\rho_m g \overbrace{w(x)}^{\text{Plate height (downward flex is negative height).}}$$

Note that $B(x)$ is opposite of the direction of $w(x)$ hence negative. When the flexure, $w(x)$, is down (-), the buoyant force $B(x)$ is up (+).

$$p(x) = -\rho_l g h_l(x) - \rho_m g w(x)$$

$$\frac{\partial^4 w(x)}{\partial x^4} \cdot \frac{E\tau^3}{12(1-\nu^2)} = -g(\rho_l h_l(x) + \rho_m w(x))$$

$$\frac{\partial^4 w(x)}{\partial x^4} D + \rho_m g w(x) = -\rho_l g h_l(x)$$

This is the differential equation that models our linear system.

$$\text{hl} \longrightarrow \boxed{\phi} \longrightarrow \text{w}$$

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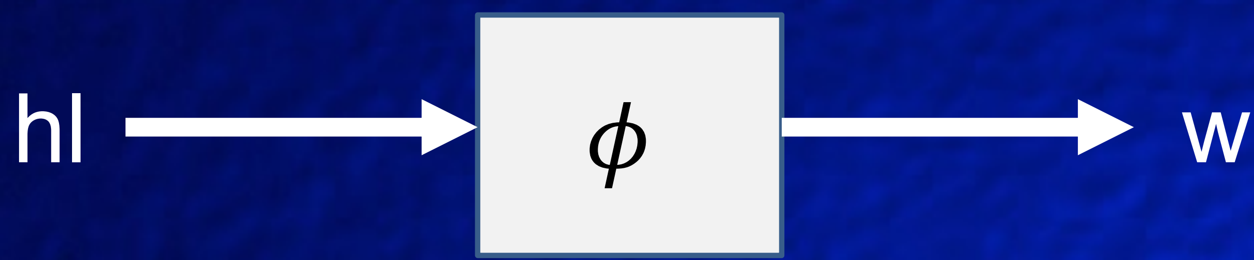

$$F\left[\frac{\partial^4 w(x)}{\partial x^4} D + \rho_m g w(x)\right] = F[-\rho_l g h_l(x)]$$

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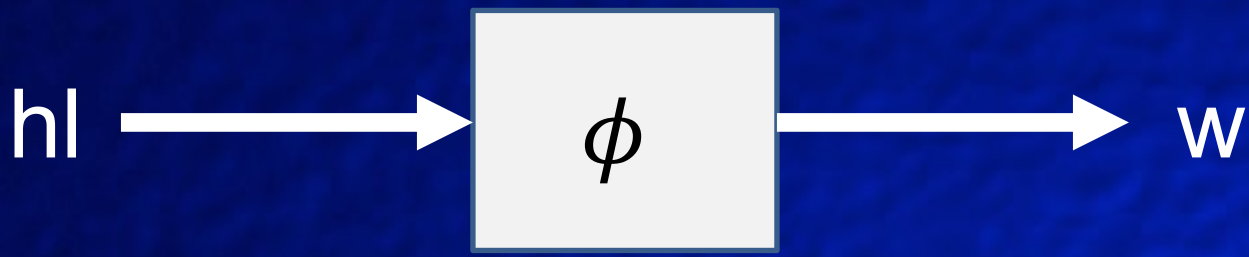

$$F\left[\frac{\partial^4 w(x)}{\partial x^4} D + \rho_m g w(x)\right] = F[-\rho_l g h_l(x)]$$

$$D \cdot F\left[\frac{\partial^4 w(x)}{\partial x^4}\right] + \rho_m g \cdot F[w(x)] = -\rho_l g \cdot F[h_l(x)]$$

Note that,

$$\phi(x) = F^{-1}[\Phi(k)] = \int_{-\infty}^{\infty} \Phi(k) e^{i2\pi kx} dk$$

$$\Phi(k) = F[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) e^{-i2\pi kx} dx$$

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Wavenumber, $k^* = 2\pi k = \frac{2\pi}{\lambda}$ ($f \cdot k$ = freq. $\cdot$ wavenumber)

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Wavenumber, $k^* = 2\pi k = \frac{2\pi}{\lambda}$ ($f \cdot k$ = freq. $\cdot$ wavenumber)

Not unlike $\omega = 2\pi f$

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$$D \cdot F\left[\frac{\partial^4 w(x)}{\partial x^4}\right] + \rho_m g \cdot F[w(x)] = -\rho_l g \cdot F[h_l(x)]$$

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$W(k)$ = crustal deformation response in the spatial frequency domain.

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$$W(k) = \frac{-g\rho_l H_l(k)}{\rho_m g + D(2\pi i k)^4} = \frac{-g\rho_l H_l(k)}{\rho_m g\left(1 + \frac{D(2\pi i k)^4}{\rho_m g}\right)}$$

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$$D \cdot F\left[\frac{\partial^4 w(x)}{\partial x^4}\right] + \rho_m g \cdot F[w(x)] = -\rho_l g \cdot F[h_l(x)]$$

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$$= -H_l(k)\frac{\rho_l/\rho_m}{1 + \frac{16\pi^4 k^4 D}{g\rho_m}}$$

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$W(k)$ is the response of the crust to load $H_l(k)$

If $H_l(k) = 1$, then $h_l(x) = \delta(x)$

$$Q(k) = \frac{W(k)}{H_l(k)} = \frac{-\rho_l/\rho_m}{1 + \frac{16\pi^4 k^4 D}{g\rho_m}}$$

The impulse response of the system.

$$Q(k) = \frac{-\rho_l/\rho_m}{1 + \frac{16\pi^4 k^4 D}{g\rho_m}}$$

$k \gg 1 \Rightarrow$ rigidity dominates

$\lambda \ll 1 \Rightarrow$ narrow

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$k \gg 1$ → rigidity dominates

$\lambda \ll 1$ → narrow

$k \ll 1$ → bouyancy dominates

$\lambda \gg 1$ → wide

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$$Q(k) = \frac{-\rho_l/\rho_m}{1 + \dfrac{16\pi^4 k^4 D}{g\rho_m}}$$

$k \gg 1 \Rightarrow$ rigidity dominates $\qquad \lambda \ll 1 \Rightarrow$ narrow

$k \ll 1 \Rightarrow$ bouyancy dominates $\qquad \lambda \gg 1 \Rightarrow$ wide

Let $\alpha^4 = \frac{g\rho_m}{D}$

$$Q(k) = \frac{-\rho_l/\rho_m}{\dfrac{D}{g\rho_m}\left(\dfrac{g\rho_m}{D} + 16\pi^4 k^4\right)} = \frac{-\rho_l/\rho_m}{\dfrac{D}{g\rho_m}(\alpha^4 + 16\pi^4 k^4)}$$

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$$= \frac{-\rho_l}{\rho_m} \cdot \frac{g\rho_m}{D} \cdot \frac{1}{\alpha^4 + (2\pi k)^4}$$

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$$Q(k) = \frac{-\rho_l/\rho_m}{\frac{D}{g\rho_m}\left(\frac{g\rho_m}{D} + 16\pi^4 k^4\right)} = \frac{-\rho_l/\rho_m}{\frac{D}{g\rho_m}(\alpha^4 + 16\pi^4 k^4)}$$

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$\frac{1}{k^4}$ is even, $isin(2\pi kx)$ is odd $\quad even \cdot odd = odd \quad \int_{-a}^{a} odd = 0$

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$$q(x) = \frac{-g\rho_l}{D}\int_{-\infty}^{\infty}\frac{\cos(2\pi kx)}{\alpha^4 + (2\pi k)^4}dk$$

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$$q(x) = F^{-1}[Q(k)] = \frac{-g\rho_l}{D}\int_{-\infty}^{\infty}\frac{1}{\alpha^4 + (2\pi k)^4}e^{i2\pi kx}dk$$

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from tables

$$q(x) = \frac{-\sqrt{2}g\rho_l}{4\alpha^3 D}e^{-\alpha|x|/\sqrt{2}}\left[\sin\left(\frac{\alpha|x|}{\sqrt{2}}\right) + \cos\left(\frac{\alpha|x|}{\sqrt{2}}\right)\right]$$

$$q(x) = \frac{-\sqrt{2} g \rho_l}{4\alpha^3 D} e^{-\alpha|x|/\sqrt{2}} \left[\sin\left(\frac{\alpha|x|}{\sqrt{2}}\right) + \cos\left(\frac{\alpha|x|}{\sqrt{2}}\right) \right]$$

Given an arbitrary crustal load, $h_l(x)$, $w(x) = q(x) * h_l(x)$, and

$$h(x) = h_l(x) + w(x) = h_l(x) + q(x) * h_l(x)$$

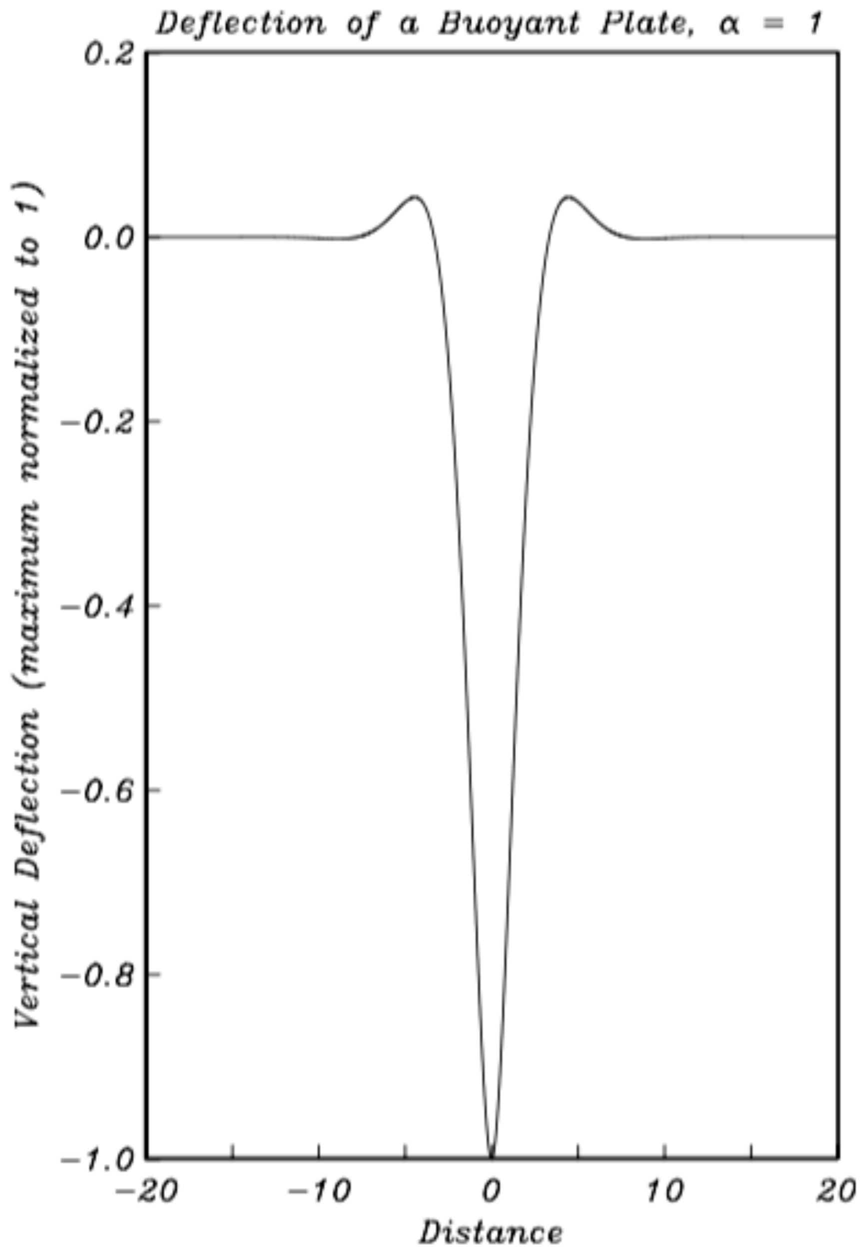

Figure 6: Response of a Buoyant, Rigid Plate to an Spatial Impulse Load

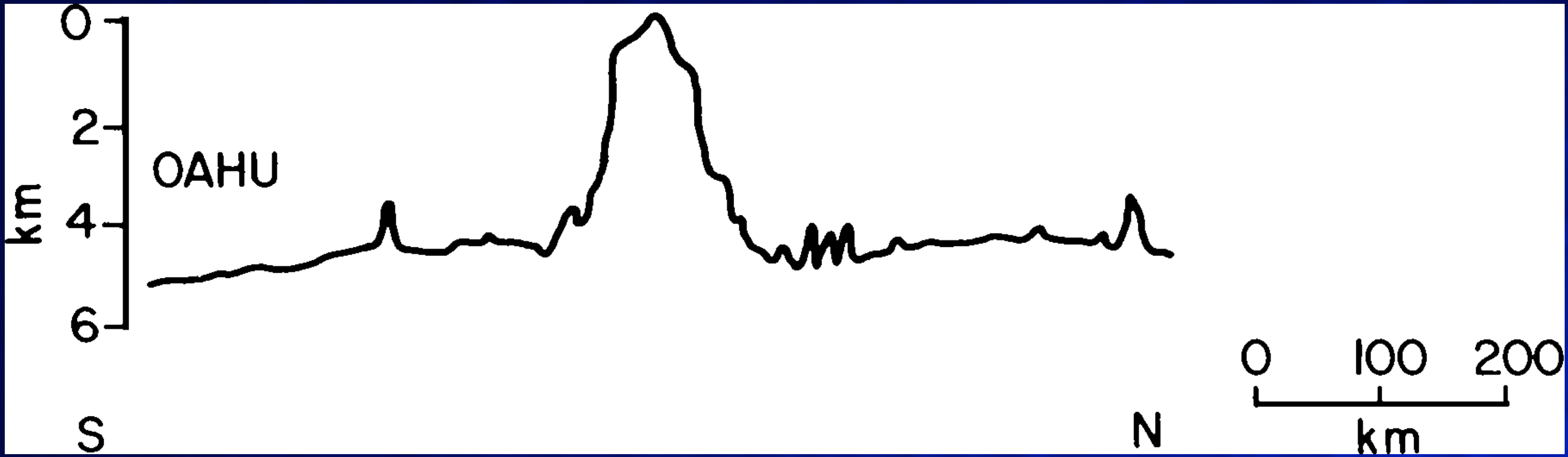

Figure 3.28 A bathymetric profile across the Hawaiian archipelago.

Turcotte and Schubert, Geodynamics 2nd Edition, page 222.

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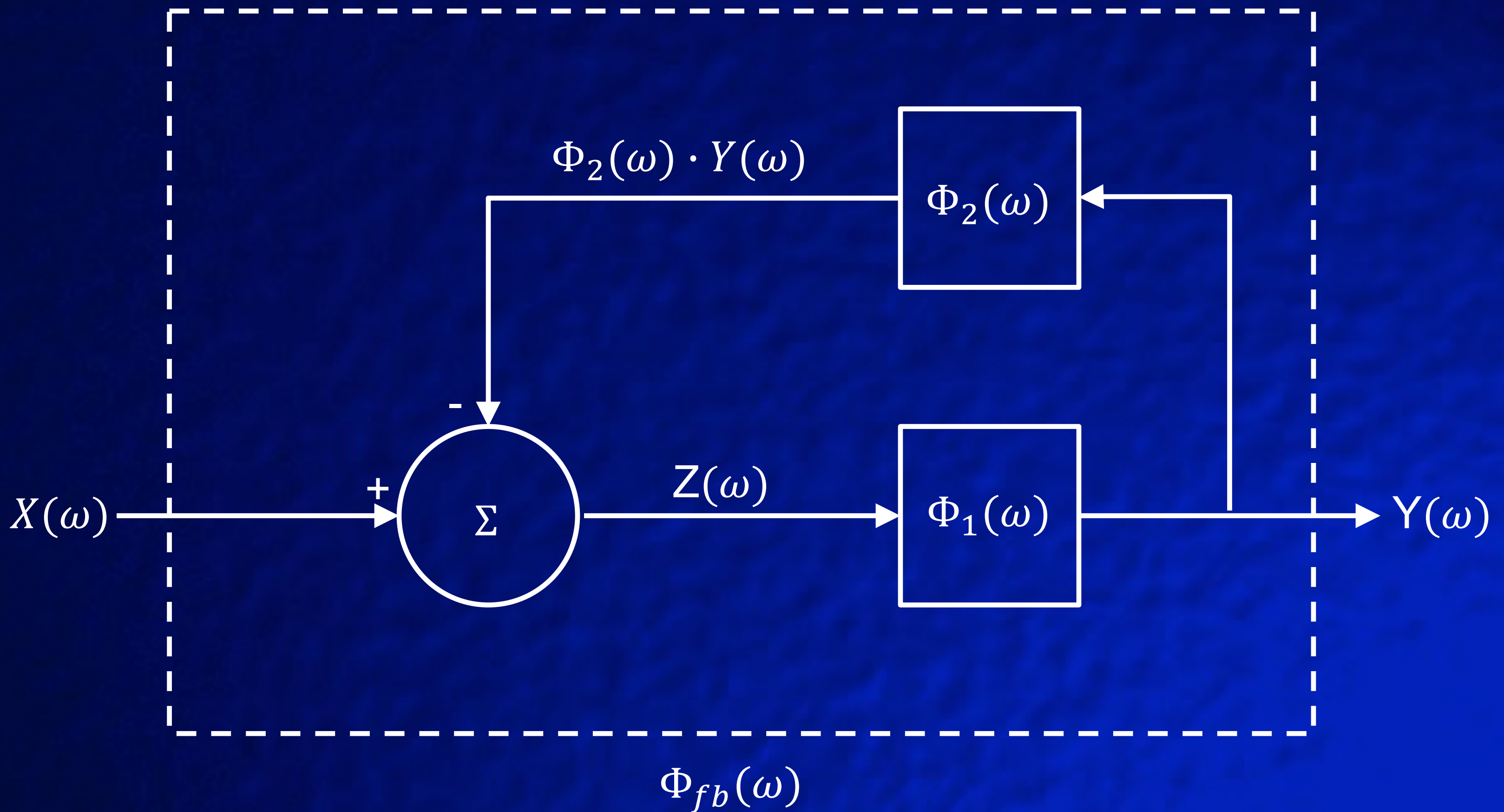
Feedback


$\Phi_2(\omega) \cdot Y(\omega)$
$\Phi_2(\omega)$
-
+
$X(\omega)$
Σ
$Z(\omega)$
$\Phi_1(\omega)$
$Y(\omega)$
$\Phi_{fb}(\omega)$

$$Y(\omega) = Z(\omega)\Phi_1(\omega) \Longrightarrow Z(\omega) = \frac{Y(\omega)}{\Phi_1(\omega)}$$

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$$Z(\omega) = X(\omega) - \Phi_2(\omega)Y(\omega)$$

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$$Y(\omega) = Z(\omega)\Phi_1(\omega) \Longrightarrow Z(\omega) = \frac{Y(\omega)}{\Phi_1(\omega)}$$

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$$Y(\omega) = \big(X(\omega) - \Phi_2(\omega)Y(\omega)\big)\Phi_1(\omega)$$

$$Y(\omega) = Z(\omega)\Phi_1(\omega) \quad \Rightarrow \quad Z(\omega) = \frac{Y(\omega)}{\Phi_1(\omega)}$$

$$Z(\omega) = X(\omega) - \Phi_2(\omega)Y(\omega) \qquad \frac{Y(\omega)}{\Phi_1(\omega)} = X(\omega) - \Phi_2(\omega)Y(\omega)$$

$$Y(\omega) = \big(X(\omega) - \Phi_2(\omega)Y(\omega)\big)\Phi_1(\omega)$$

$$Y(\omega)\big(1 + \Phi_1(\omega)\Phi_2(\omega)\big) = X(\omega)\Phi_1(\omega)$$

$$Y(\omega) = Z(\omega)\Phi_1(\omega) \Rightarrow Z(\omega) = \frac{Y(\omega)}{\Phi_1(\omega)}$$

$$Z(\omega) = X(\omega) - \Phi_2(\omega)Y(\omega) \qquad \frac{Y(\omega)}{\Phi_1(\omega)} = X(\omega) - \Phi_2(\omega)Y(\omega)$$

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$$Y(\omega)\big(1 + \Phi_1(\omega)\Phi_2(\omega)\big) = X(\omega)\Phi_1(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)} = \Phi_{fb}(\omega)$$

$$Y(\omega) = Z(\omega)\Phi_1(\omega) \quad \Longrightarrow \quad Z(\omega) = \frac{Y(\omega)}{\Phi_1(\omega)}$$

$$Z(\omega) = X(\omega) - \Phi_2(\omega)Y(\omega) \qquad \frac{Y(\omega)}{\Phi_1(\omega)} = X(\omega) - \Phi_2(\omega)Y(\omega)$$

$$Y(\omega) = \big(X(\omega) - \Phi_2(\omega)Y(\omega)\big)\Phi_1(\omega)$$

$$Y(\omega)\big(1 + \Phi_1(\omega)\Phi_2(\omega)\big) = X(\omega)\Phi_1(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)} = \Phi_{fb}(\omega)$$

The current input affects future output.

In practice, there will be slight delays, or phase shifts.

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$$\Phi_{fb}(\omega) = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)}$$

Let Φ_1 be a mechanical seismometer, $\Phi_1(\omega) = \frac{-\omega^2}{\omega^2 - 2i\zeta\omega - \omega_s^2}$

Let Φ_2 be a constant, $\Phi_2(\omega) = \alpha$

$$\Phi_{fb}(\omega) = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)}$$

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The poles are now $\omega_{fb} = \frac{i\zeta \pm \sqrt{(1-\alpha)\omega_s^2 - \zeta^2}}{1-\alpha}$ instead of $\omega_p = i\zeta \pm \sqrt{\omega_s^2 - \zeta^2}$

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$$\Phi_{fb}(\omega) = \frac{-\omega^2/(\omega^2 - 2i\zeta\omega - \omega_s^2)}{1 - \alpha\omega^2/(\omega^2 - 2i\zeta\omega - \omega_s^2)} = \frac{-\omega^2}{(1-\alpha)\omega^2 - 2i\zeta\omega - \omega_s^2}$$

The poles are now $\omega_{fb} = \frac{i\zeta \pm \sqrt{(1-\alpha)\omega_s^2 - \zeta^2}}{1-\alpha}$ instead of $\omega_p = i\zeta \pm \sqrt{\omega_s^2 - \zeta^2}$

So we essentially lower the ”corner” of the seismometer by about $\sqrt{(1-\alpha)}$.

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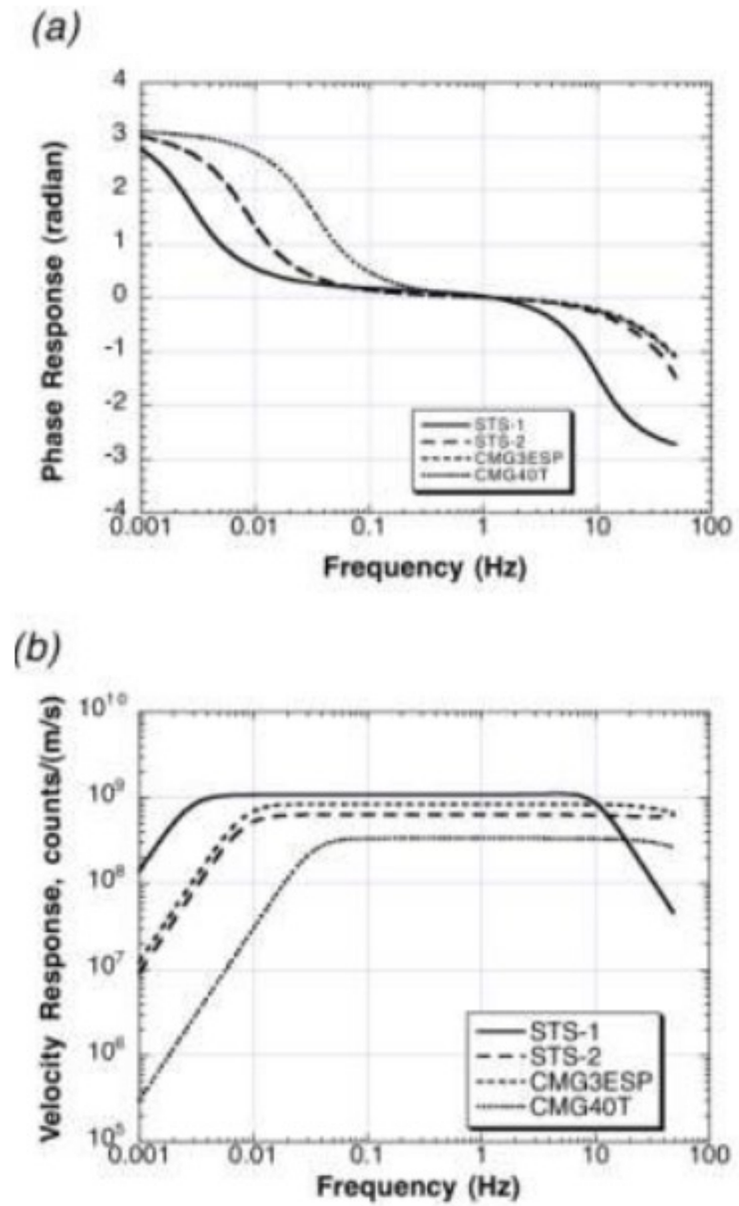

Figure 12: Phase and Velocity responses of some modern seismograph systems used in PASSCAL, GSN, and other networks, c/o the Incorporated Research Institutions for Seismology (IRIS).

https://www.passcal.nmt.edu/content/instrumentation/sensors/sensor-comparison-chart