Introduction to Linear Systems in the Time Domain.

Mitch Withers, Res. Assoc. Prof., Univ. of Memphis

See Aster and Borchers, Time Series Analysis, chapter 1.

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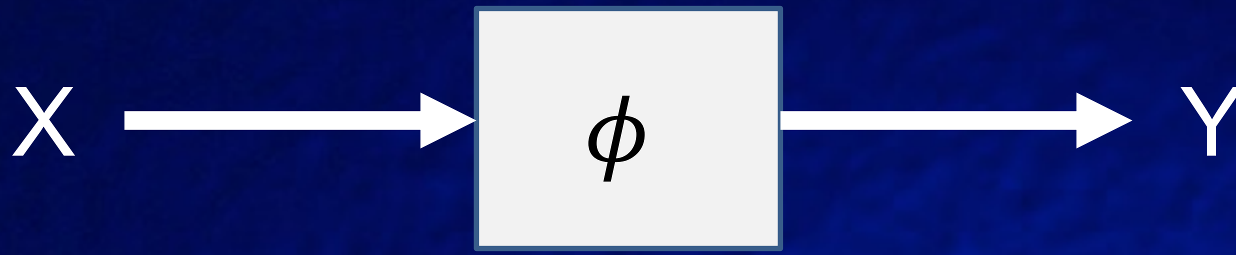

- We will learn signal processing from linear systems perspective.
- For a given x(t), the linear system, ϕ, transforms it to some output y(t).
- $y(t) = \phi[x(t)]$, where ϕ is said to be operating on x to produce y.
- An example is a mass suspended by a spring

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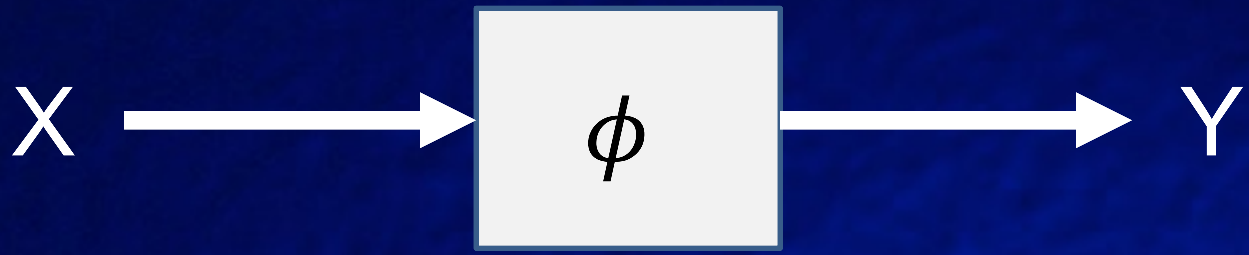

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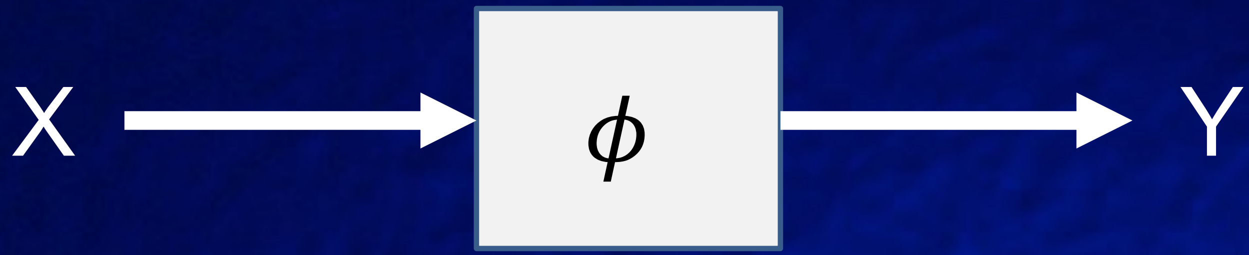

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- → non-linear

- A child on a swing is another example of a linear system.
- A small push will result in linear motion but a big push may not.
- With a small push the ropes holding the swing remain taught and the motion is that of a simple pendulum. Push a little harder and it swings a proportionally longer arc.
- Swinging too high may cause the ropes to no longer be taught. At that point the response of the swing depends on the input which makes it non-linear.
- In other words, the response ø, of a linear system is independent of the input.

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$$\phi[x(t) + y(t)] = \phi[x(t)] + \phi[y(t)]$$

Phi operating on the sum of the inputs is equal to the sum of outputs of phi operating individually on the inputs.

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Properties of a linear system

- Superposition
- Scaling

$$\phi[\alpha x(t)] = \alpha\phi[x(t)]$$

Which is really just a special case of superposition

$$\phi[\alpha x(t)] = \phi[x(t) + x(t) + x(t) + \cdots] = \alpha\phi[x(t)]$$

where α is an integer in this case. The property is also true with non-integer multiples.

Note that $\phi[\alpha x(t)] = \alpha\phi[x(t)]$ implies $\phi[0] = 0$
That is, 0 input produces 0 output.

Let x(t)=0
then, $\phi[\alpha x(t)] = \phi[\alpha \cdot 0] = \alpha\phi[0]$
and, $\phi[\alpha \cdot 0] = \phi[0]$
therefor $\phi[0] = \alpha\phi[0]$
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This means that 0 input gives 0 output regardless of ϕ.
That is ϕ doesn't add energy.

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LTI

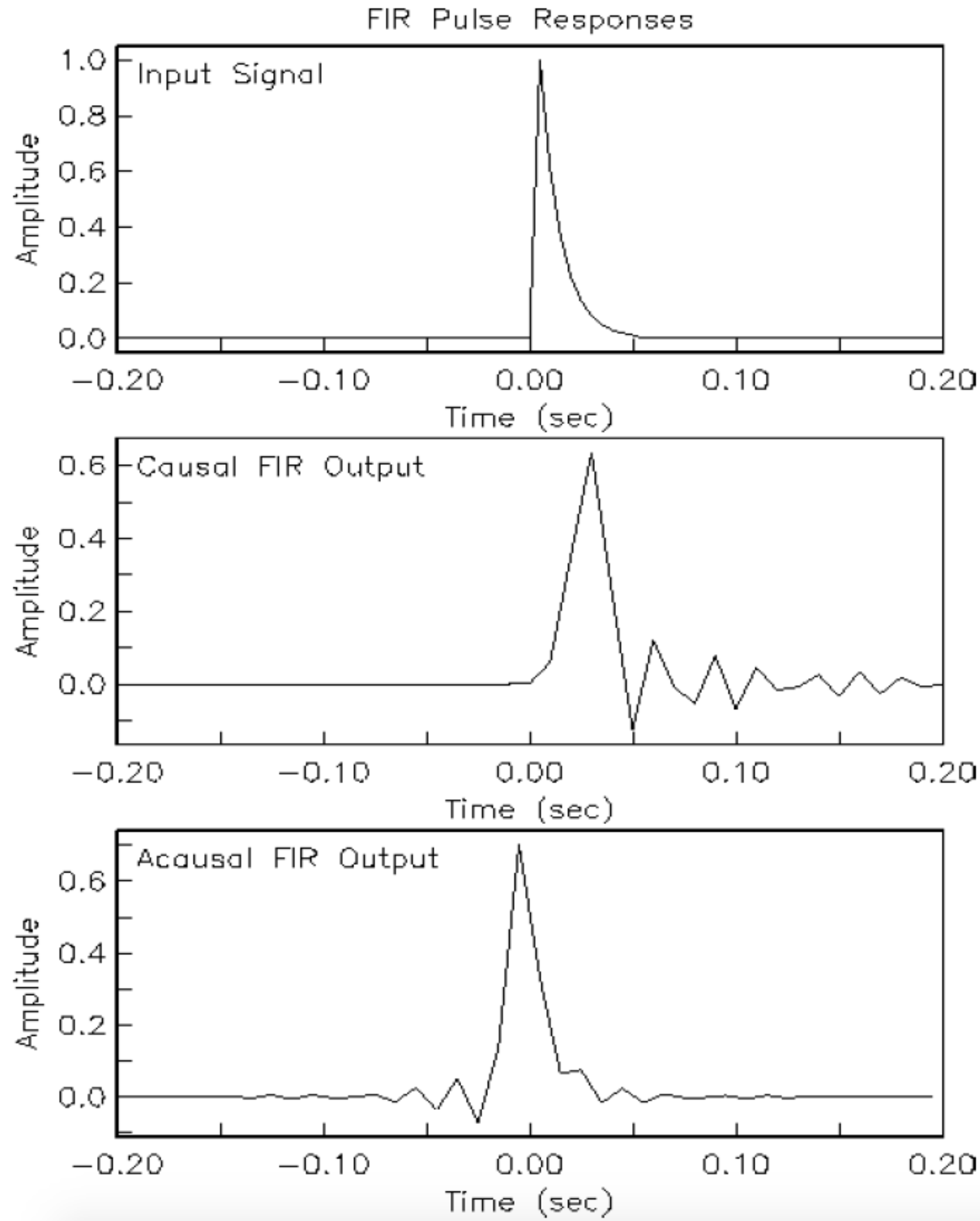
Many of the systems we study are also time invariant. That is, the response of the system, ϕ, is constant. These are linear time invariant systems, or **LTI**.

In some cases we may search for small variations in ϕ (searching for changes in seismic anisotropy for example) though normally ϕ is required, or assumed, to be stable over the time period of our study.

Causality

In most cases, our processes are also causal: there is no output prior to a non-zero input. The the swing for example, does not oscillate before we push it.

It is possible to design acausal digital filters. Some seismic digitizers have acausal responses which produce precursors that are artifacts of the digital filter.


FIR Pulse Responses
Input Signal
Amplitude
1.0
0.8
0.6
0.4
0.2
0.0
−0.20
−0.10
0.00
0.10
0.20
Time (sec)
Causal FIR Output
Amplitude
0.6
0.4
0.2
0.0
−0.20
−0.10
0.00
0.10
0.20
Time (sec)
Acausal FIR Output
Amplitude
0.6
0.4
0.2
0.0
−0.20
−0.10
0.00
0.10
0.20
Time (sec)

Stability

We also require our systems to be stable. That is,

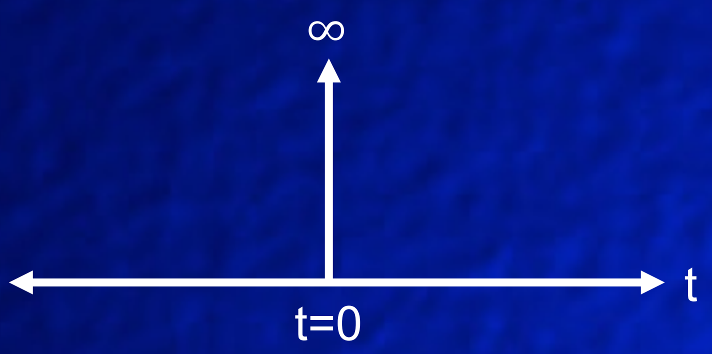
$$y(t) = \phi[x(t)] < \infty \; for \; x(t) < \infty$$

An idealized mass on a spring with no friction is not stable for a sinusoidal input. In natural systems, for example a building excited at it's natural period, the response becomes nonlinear rather than infinite. That is the building will deform before reaching oscillations of infinite amplitude.

Before diving into signal processing, we must first review a few handy mathematical functions

A delta function, $\delta(t)$, is zero everywhere except where its argument is 0 at which point it is ∞.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



This definition of the delta function is not particularly well defined (note the infinity). Instead, let's use a boxcar.

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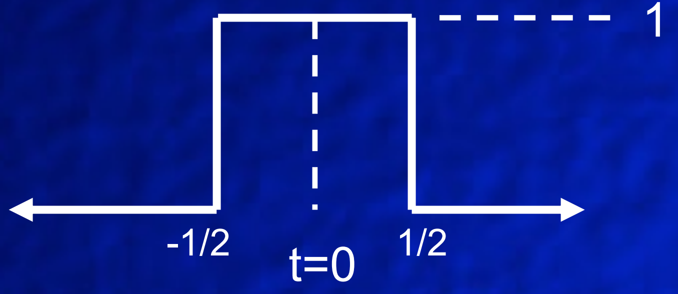
The boxcar or rectangular function is defined as

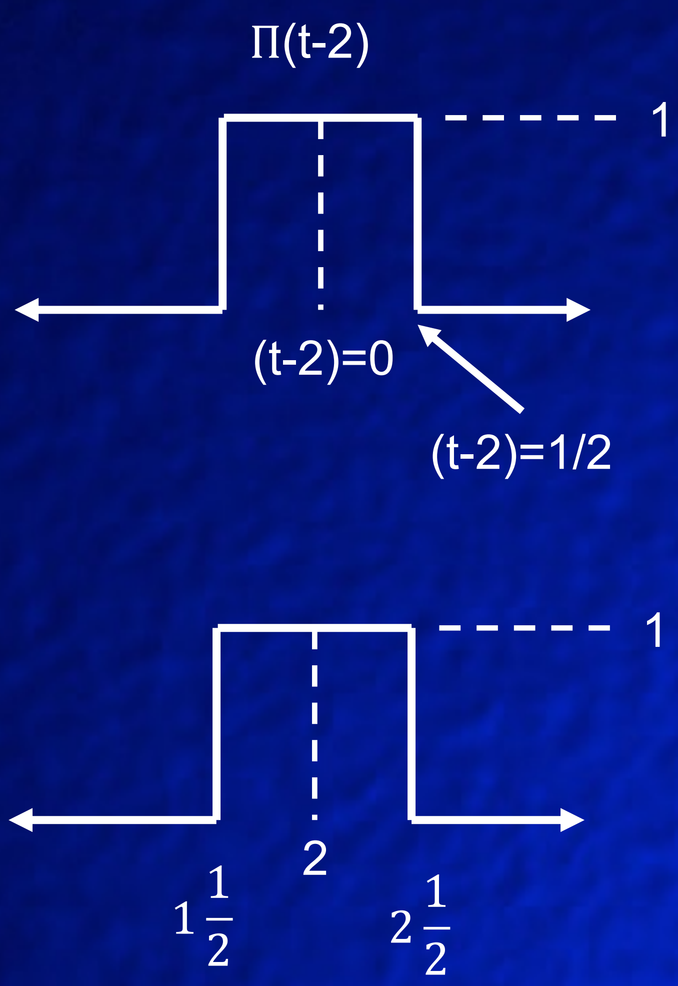
$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$

And we can let the mathematicians debate its value at 1/2

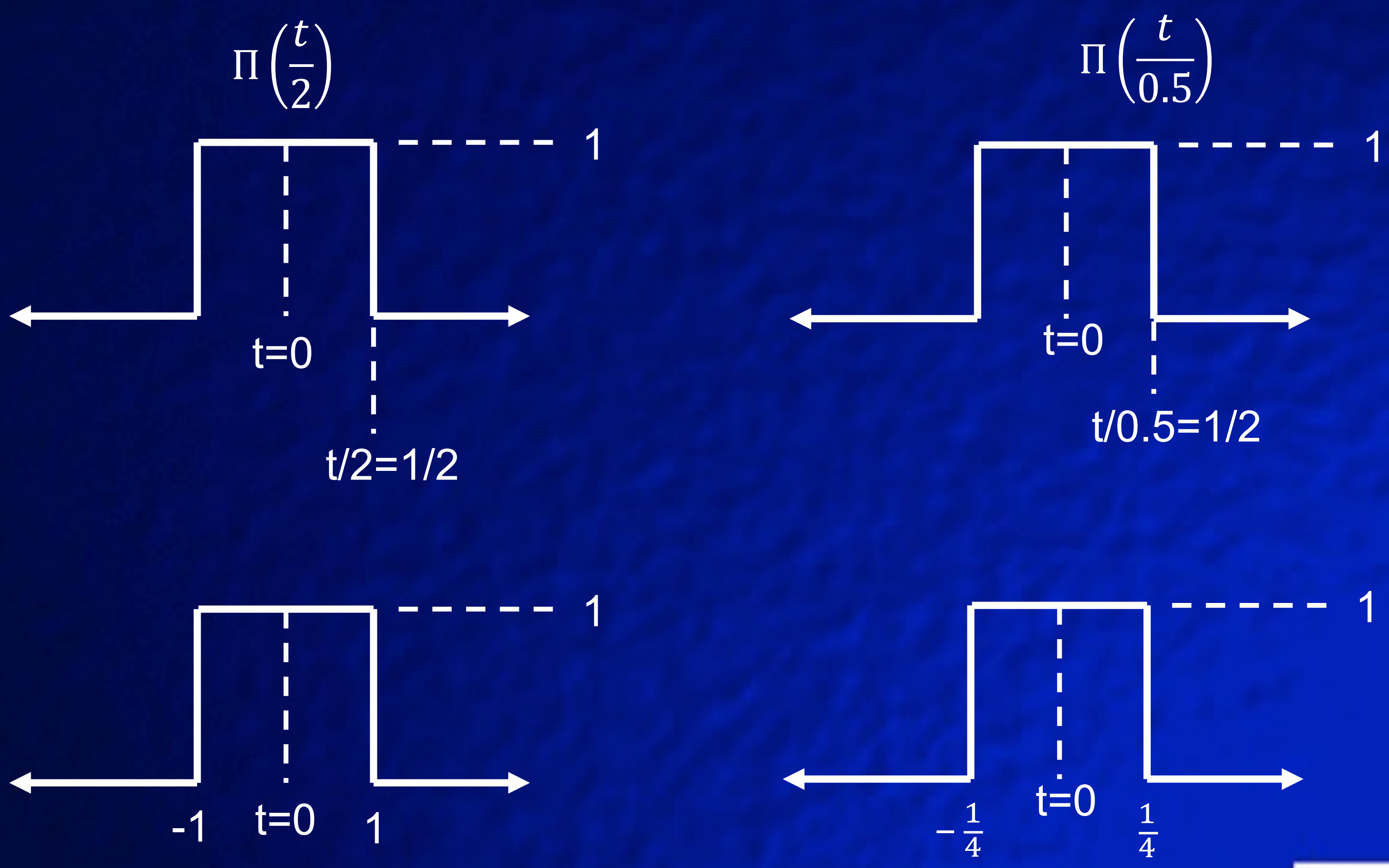


What is the area under the curve of a boxcar?

We can translate the rectangle along the t-axis



We can modify the width.



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And we can also modify the height so that in general.

$$f(x) = h\Pi\left(\frac{x-c}{b}\right)$$

This boxcar is a function of *x*. It has height *h*, width *b*, and is centered at *c*.

Look for the transition points at $\frac{x-c}{b} = \frac{1}{2} \rightarrow x - c = \frac{b}{2} \rightarrow x = \frac{b}{2} + c$

We can now more rigorously define the delta function using a boxcar.

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{t}{\tau}\right)$$

transitions at $\left|\frac{t}{\tau}\right| = \frac{1}{2}$

So that $\frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right)$ has height $\frac{1}{\tau}$ and full width of τ so it becomes $\delta(t)$ in the limit as $\tau \to 0$

In the limit as $\tau \to 0$ the box car get's infinitely narrow and infinitely tall. And the area under the curve remains 1. This leads to the useful property of the delta function,

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

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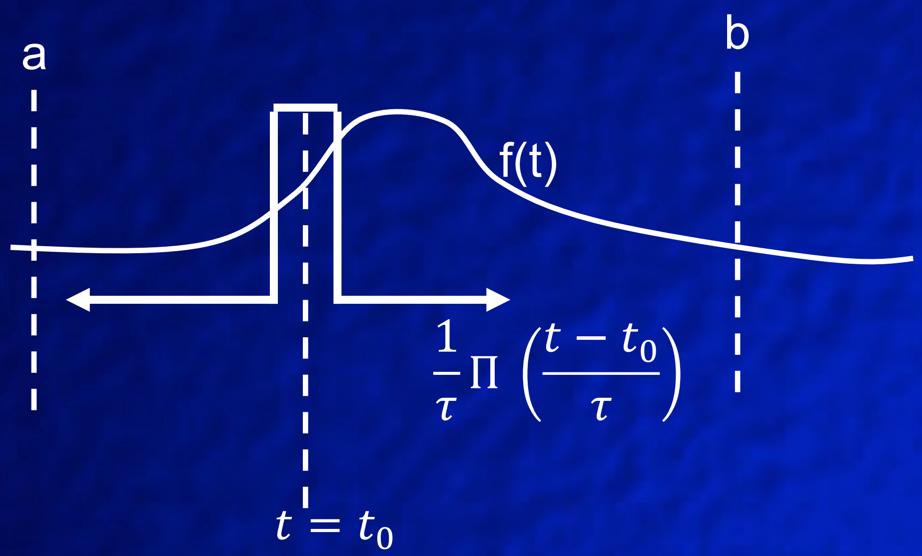
The sifting property of the delta function.

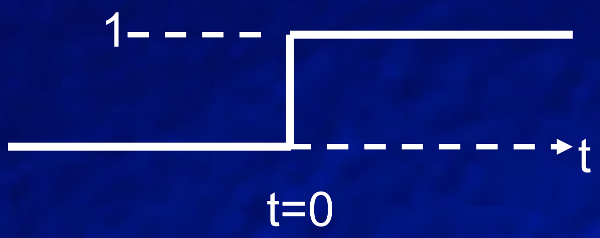
$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0), \qquad where\ a \le t_0 \le b$$

It is straight forward to prove this using the boxcar definition of the delta function.



The step function

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



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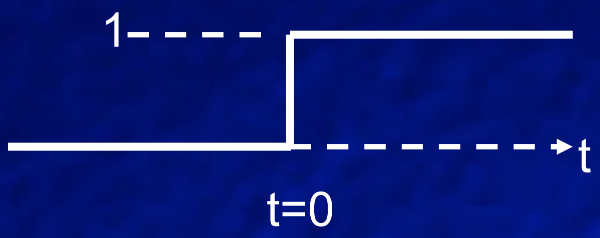

Mathematically, we can define the step function using the delta function,

$$H(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

- Integrate over τ up to t for all values of t
- For values of t < 0, the integral is 0
- For values of t > 0, the integral is 1
- This results in a function in t, that transitions from 0 to 1 at t=0

We can shift the step function by t_0

$$H(t - t_0) = \int_{-\infty}^{t} \delta(\tau - t_0) d\tau$$

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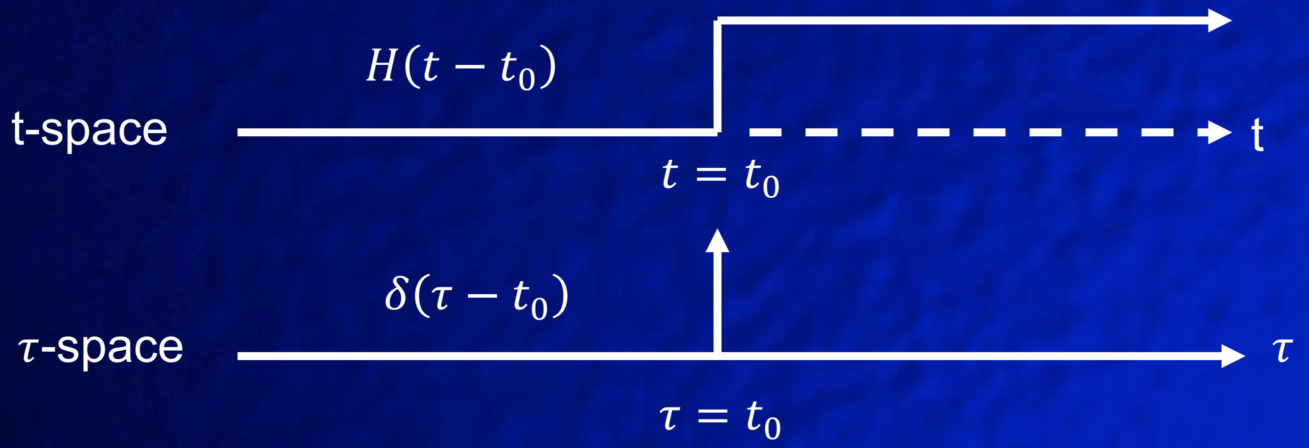
$$H(t - t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)\,d\tau$$

- Again, integrate in τ space from $-\infty$ to *t* for all values of t
- For values of t where $\tau - t_0 < 0$, the integral is 0
- Transitions at values of t where $\tau - t_0 = 0, \tau = t_0$



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An **Impulse Response** is the response of a linear system to a delta function aka impulse function.

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$h(t) = \phi[\delta(t)]$ the impulse response of ϕ

Recall the sifting property,

$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt$

Similarly for any t,

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau$$

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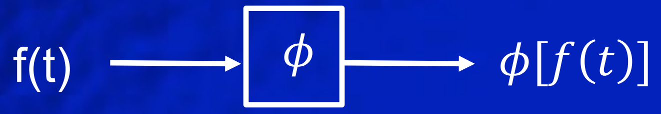
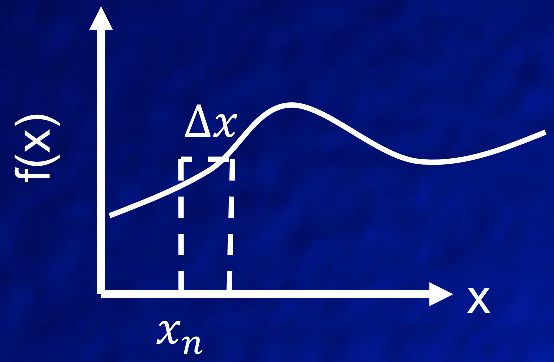
If f(t) is an arbitrary input function to ϕ, then

$$\phi[f(t)] = \phi\left[\int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau\right]$$



Recall the definition of the integral

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{n=-\infty}^{\infty} f(x_n)\Delta x$$



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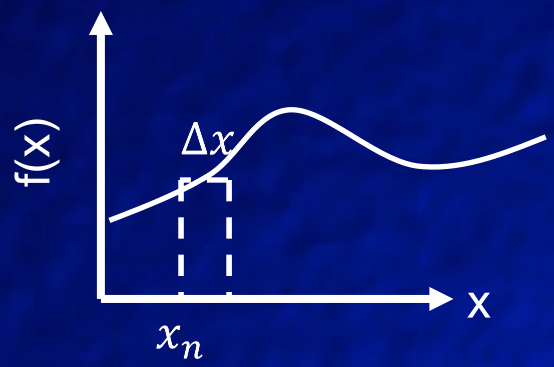

Let $g(t) = \phi[f(t)]$, then

$$g(t) = \phi\left[\lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)\delta(t-\tau_n)\Delta\tau\right]$$

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$$g(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)\phi[\delta(t-\tau_n)]\Delta\tau$$

$\phi[\delta(t-\tau_n)]$ is the impulse response of the linear system ϕ

$$g(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)h(t-\tau_n)\Delta\tau$$

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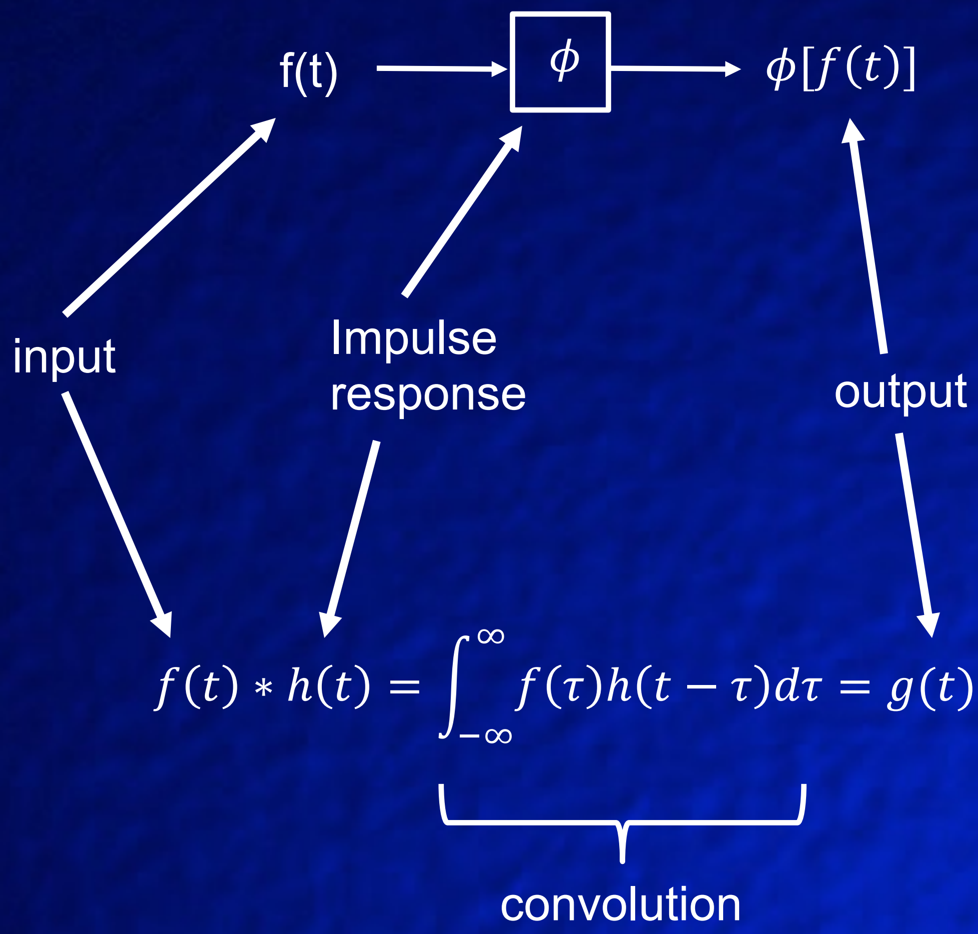
Two important points

1. Convolution is an integral $g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$
2. And, if we can model any physical system as a linear system, then we can use the impulse response to find the output of the system to any arbitrary input by convolving the input with the impulse response


f(t)
ϕ
$\phi[f(t)]$
input
Impulse response
output
$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = g(t)$
convolution

Recap (because this is important)

The convolution of two functions is defined by the integral

$$c(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

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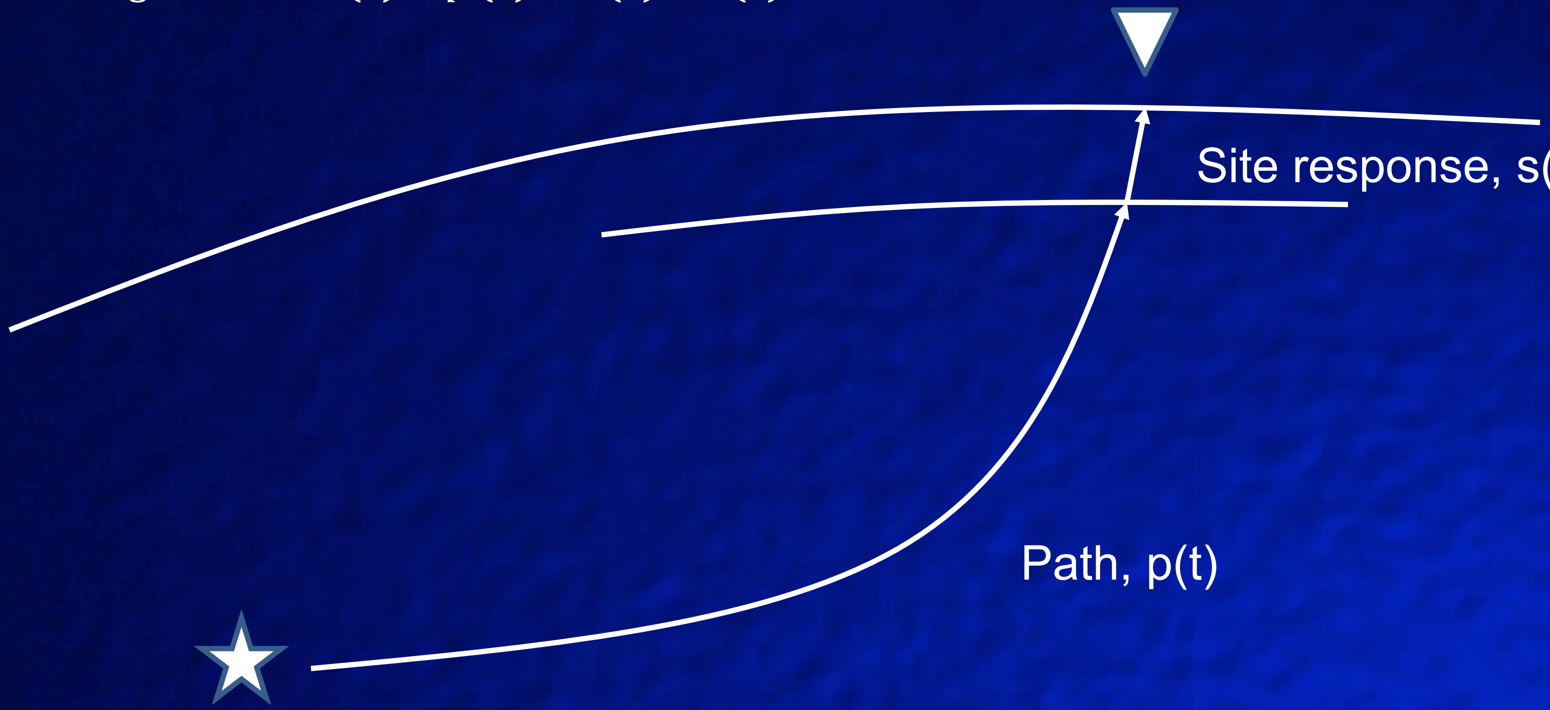
It is common to work with the impulse response, $h(t)$, rather than the linear system itself.
(more on that later)

$$Seismogram = F(t) * p(t) * s(t) * I(t)$$



But what exactly is convolution? Let's try an example.

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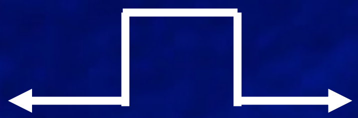
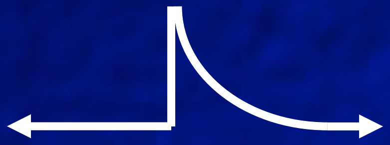


A boxcar function

and $f_2(t) = H(t)e^{-t}$



A decaying exponential

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A boxcar function

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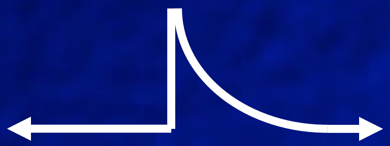

A decaying exponential

$$g(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau$$

$$g(t) = \int_{-\infty}^{\infty} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau$$

We're integrating over τ and in $\tau - space$ t is a constant, so let's consider $f_2(-\tau)$

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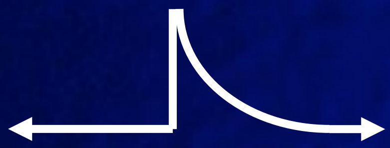
The original f_2 was a decaying exponential so $f_2(-\tau)$ is the same exponential, in τ, flipped left-to-right.



$f_2(\tau)$

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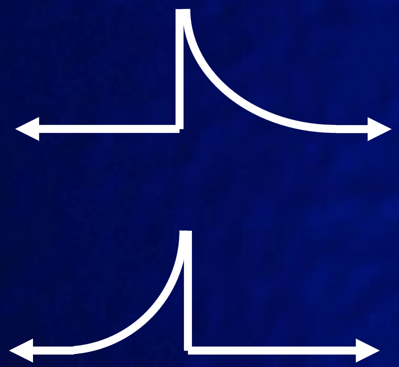

$f_2(\tau)$

$f_2(-\tau)$

Then $f_2(t-\tau)$ is f_2 flipped left-to-right and shifted by t. So we must consider the value of the integral (the area under the curve) for all values of t (or shifts) from $-\infty$ to ∞.

Steps in convolving f_1 with f_2:

1. Flip f_2 left-to-right

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4. Integrate

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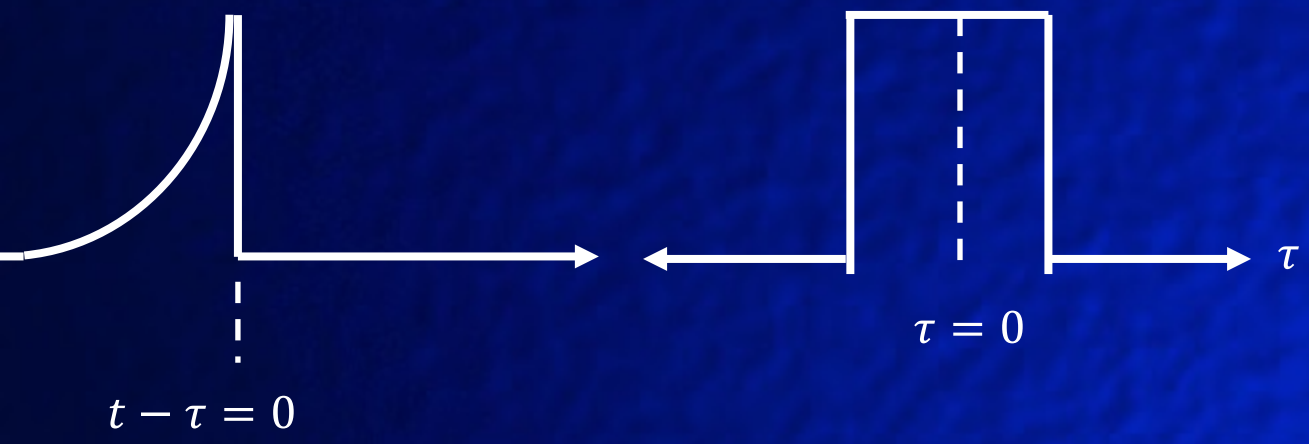
4. Integrate
5. Repeat from step 2 for all values of t



Large negative t

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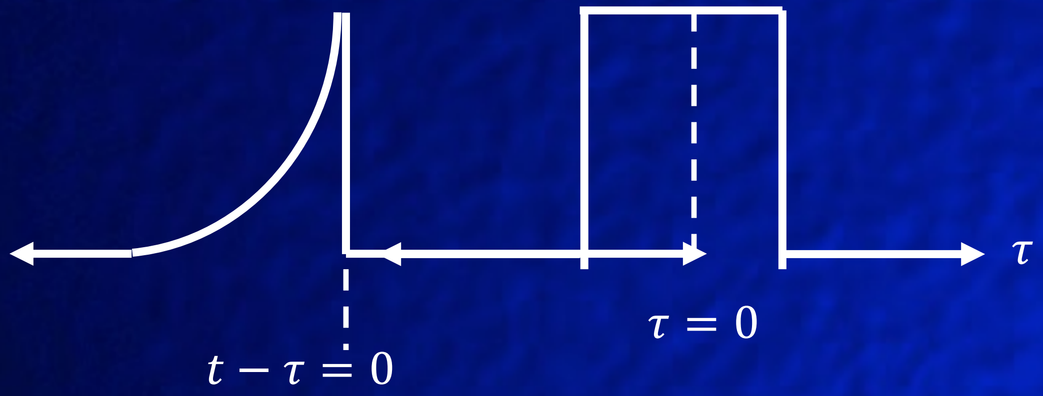

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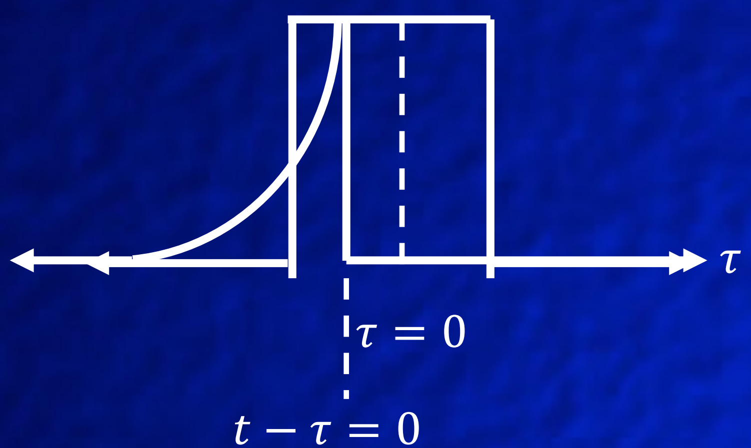

Non-zero overlap, integral is area under the curve

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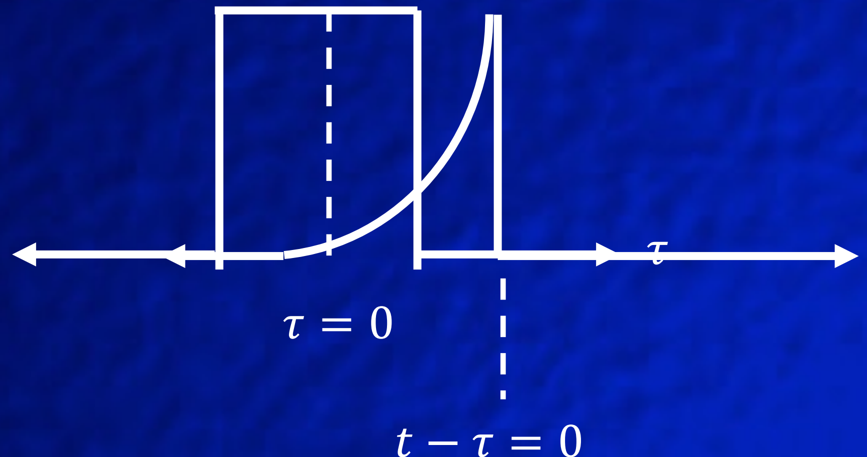
5. Repeat from step 2 for all values of t



positive values of t

Steps in convolving f_1 with f_2:

1. Flip f_2 left-to-right
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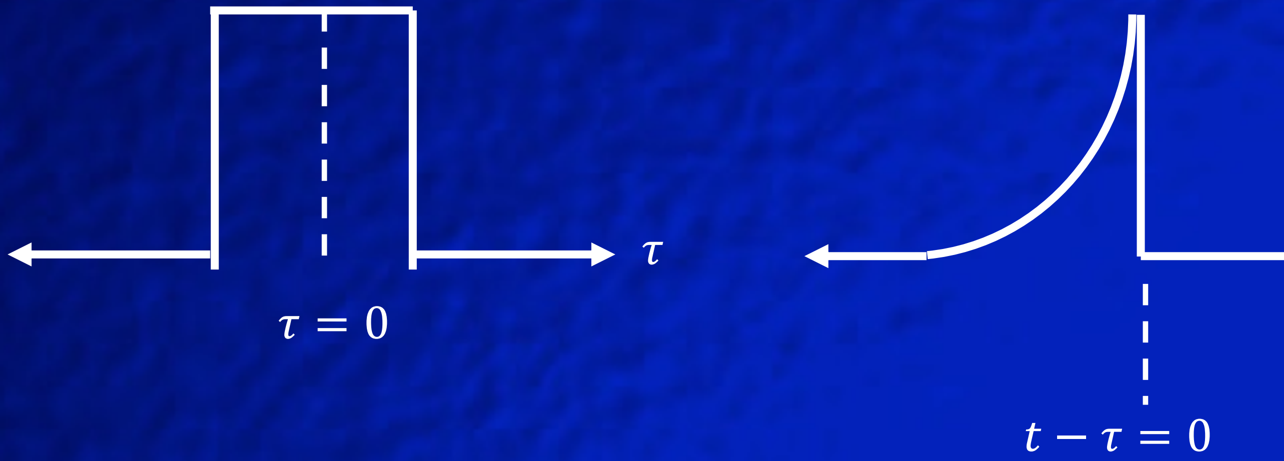
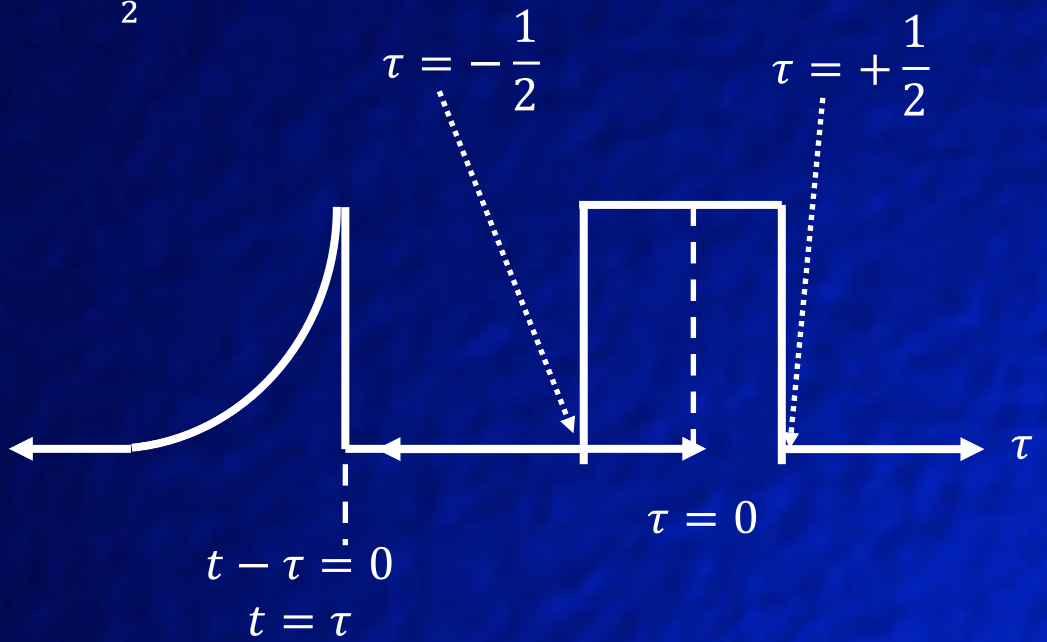


Large positive values of t

Our example functions have discontinuities so we need to split the integration into three cases.

Where is $t = -\frac{1}{2}$?

Case I) $t < -\frac{1}{2}$



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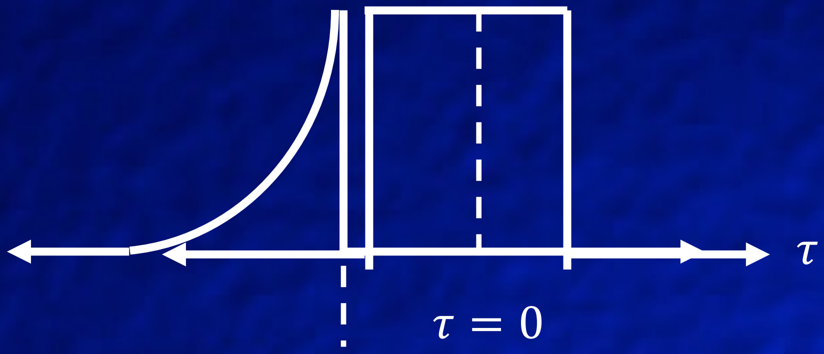

Integrating over the non-zero portion of the step function

$$g_I(t) = \int_{-\infty}^{-\frac{1}{2}} \Pi(\tau)H(t-\tau)e^{-(t-\tau)}d\tau$$

$$= \int_{-\infty}^{-\frac{1}{2}} 0 \cdot 1 \cdot e^{-(t-\tau)}d\tau = 0$$

Technically, we would need to piecewise integrate from $-\infty$ to $+\infty$ but one or the other function is 0 over the entire interval for shifts of $t < -\frac{1}{2}$.

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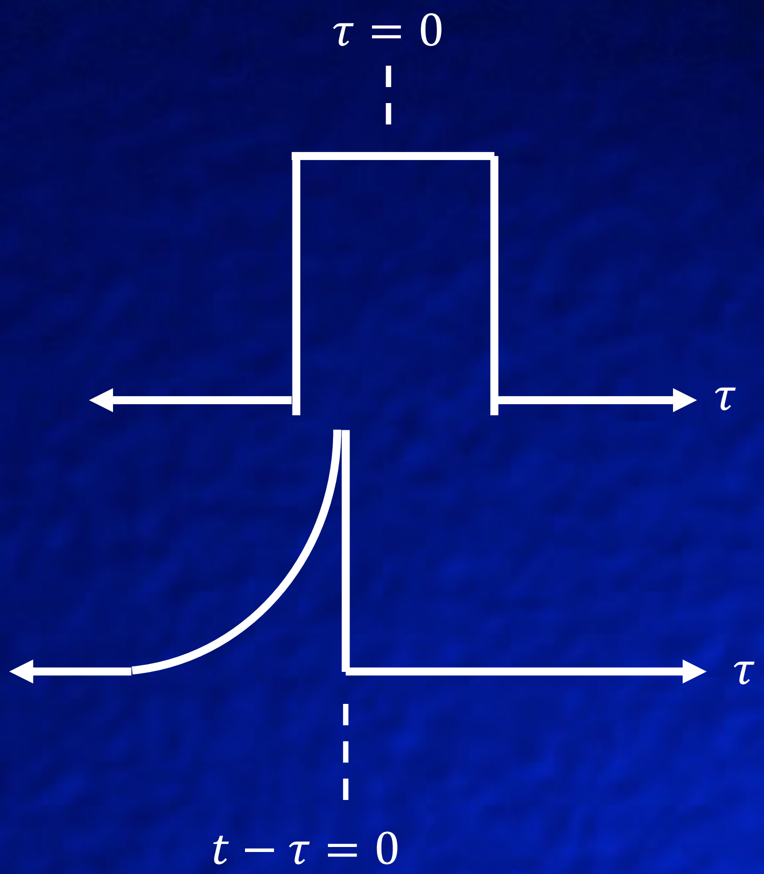
Case II) $-\frac{1}{2} < t < \frac{1}{2}$

Again, we technically need to integrate over the entire interval from $-\infty$ to $+\infty$, but we'll get the same result integrating over just the portion with non-zero overlap.



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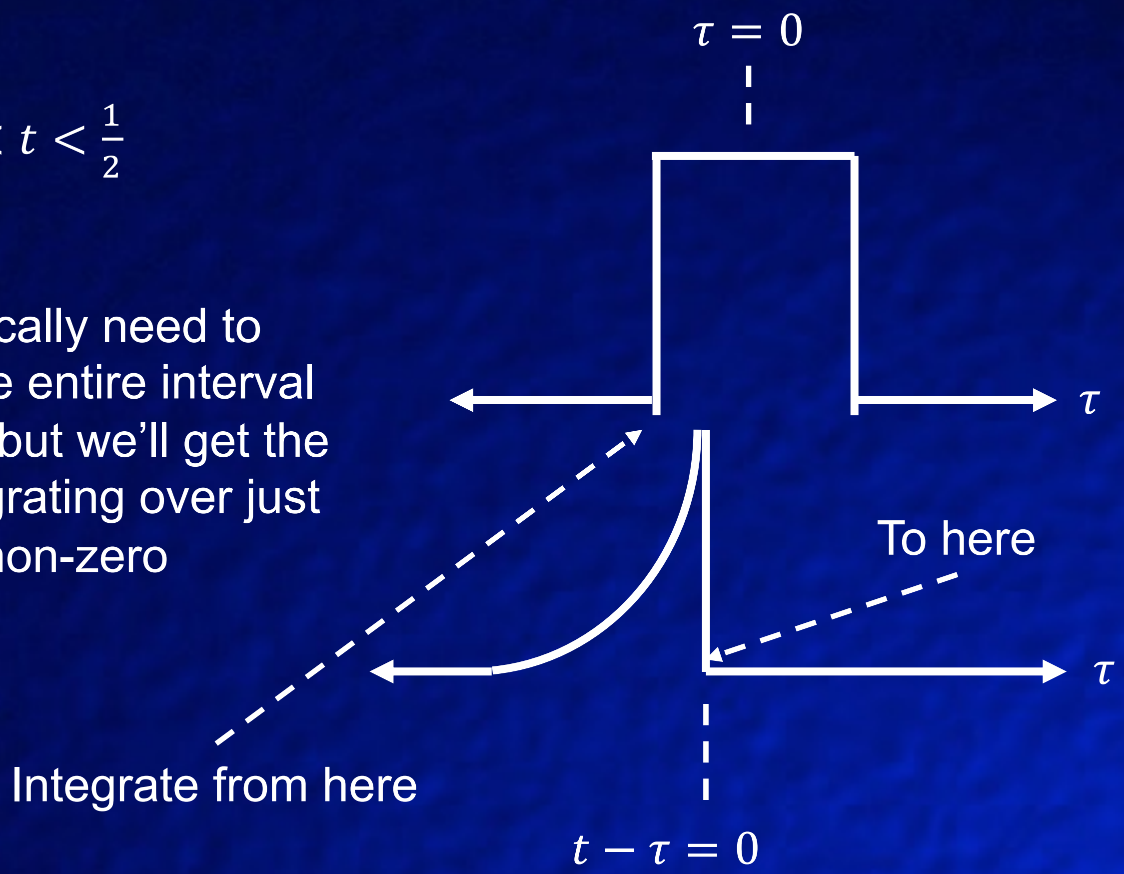

$$g_{II}(t) = \int_{?}^{?} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau$$

What are the limits of the integration?

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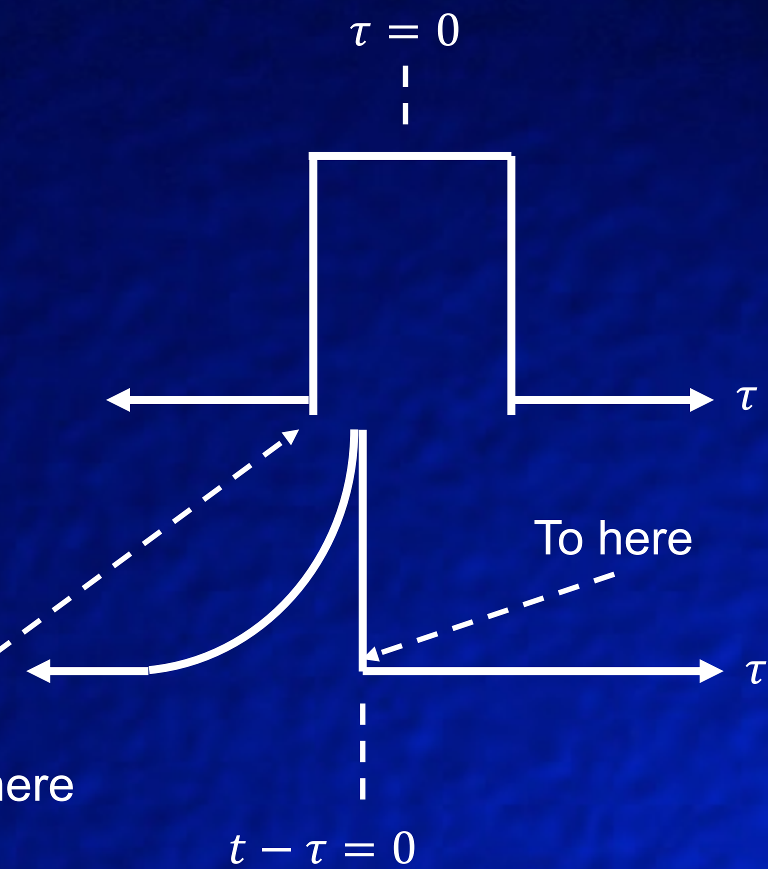

$$g_{II}(t) = \int_{-\frac{1}{2}}^{t} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau = \int_{-\frac{1}{2}}^{t} e^{-(t-\tau)} d\tau = e^{-t} \int_{-\frac{1}{2}}^{t} e^{\tau} d\tau$$

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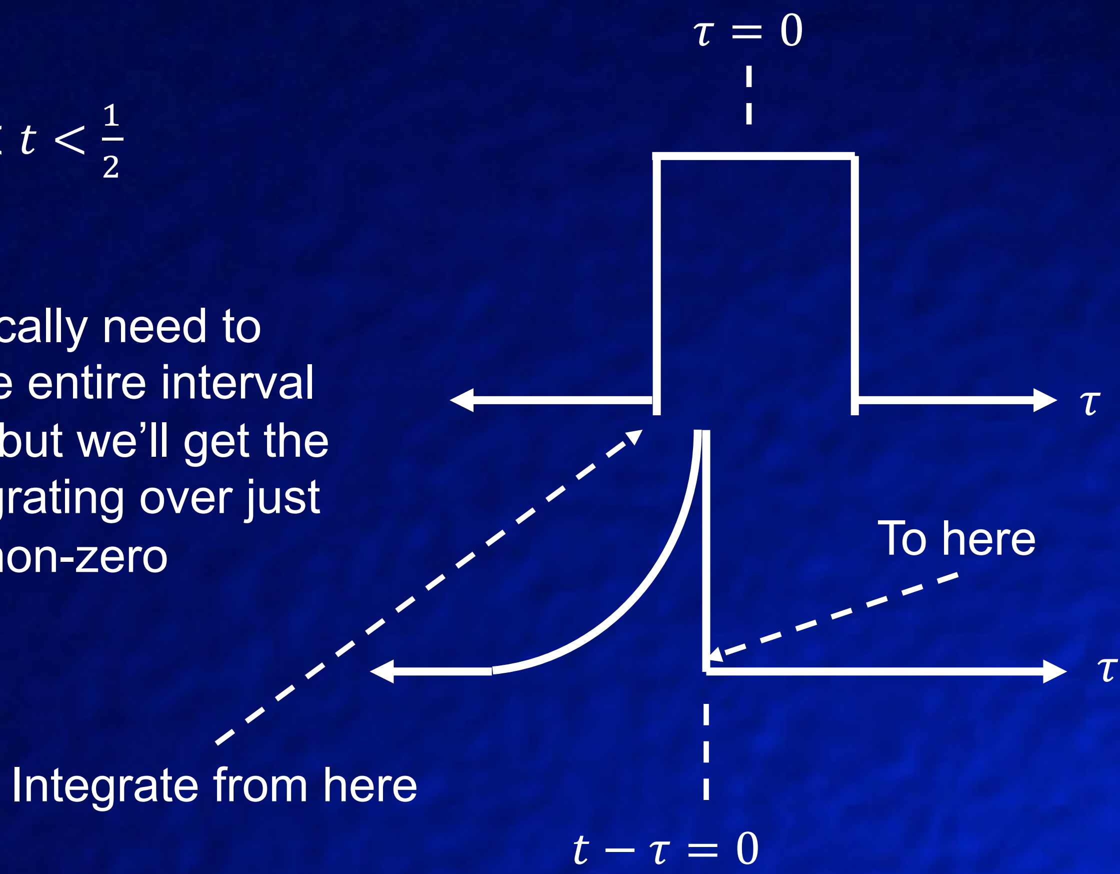

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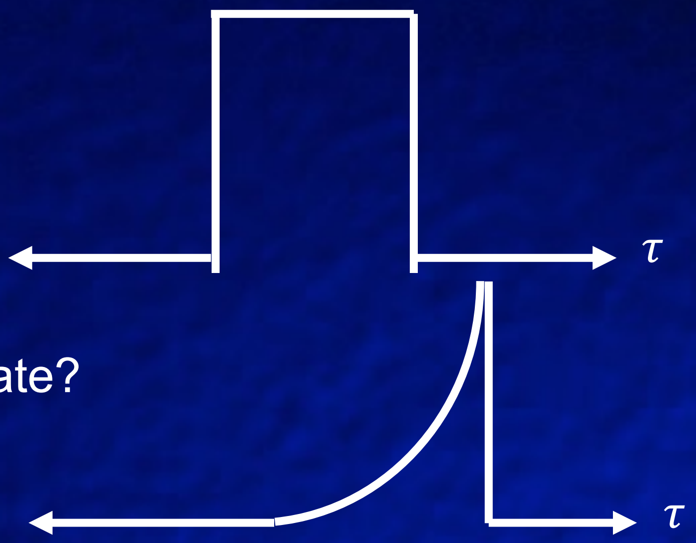
$$= e^{-t} e^{\tau} \Big|_{\tau=-\frac{1}{2}}^{t} = e^{-t}\left(e^{t} - e^{-\frac{1}{2}}\right) = e^{-t+t} - e^{-\left(t+\frac{1}{2}\right)} = e^{0} - e^{-\left(t+\frac{1}{2}\right)} = 1 - e^{-\left(t+\frac{1}{2}\right)}$$

Case III) $t > \frac{1}{2}$



Over what interval do we need to integrate?
From where to where?

$$g_{III}(t) = \int_{?}^{?} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau$$

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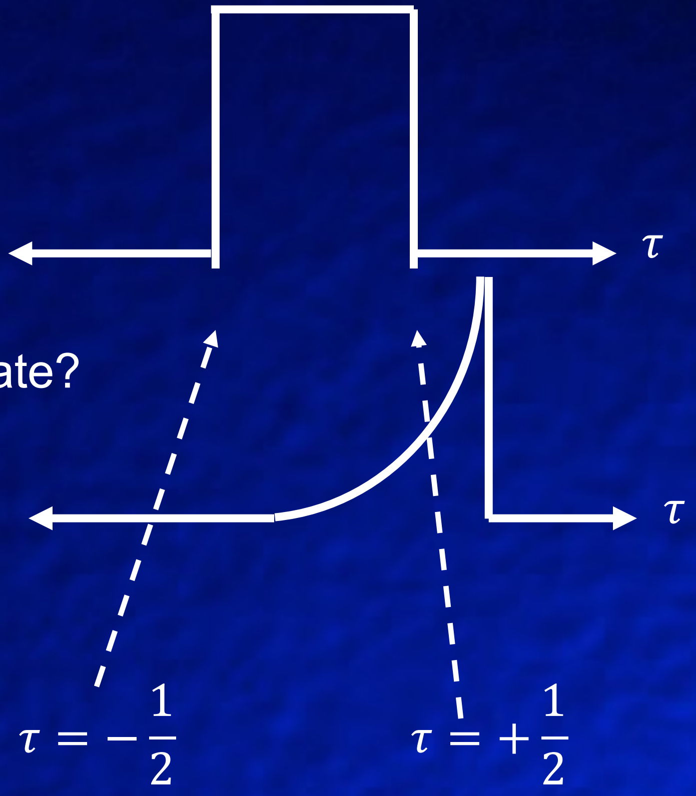
Case III) $t > \frac{1}{2}$



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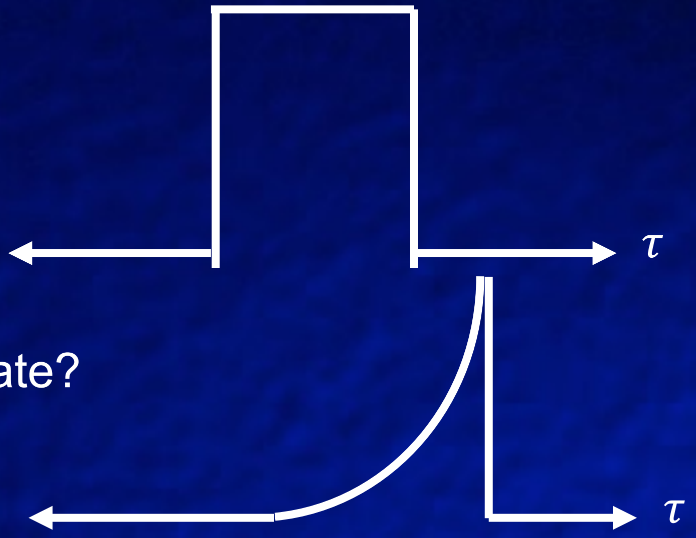


Over what interval do we need to integrate?
From where to where?

$$g_{III}(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(\tau) H(t-\tau) e^{-(t-\tau)} d\tau = e^{-t} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\tau}\, d\tau = e^{-t}\left(e^{1/2} - e^{-1/2}\right)$$

So the full answer requires all three cases and is an incomplete answer without the interval over which each piece is valid

Case I) $g_I(t) = 0, \quad t < -\frac{1}{2}$

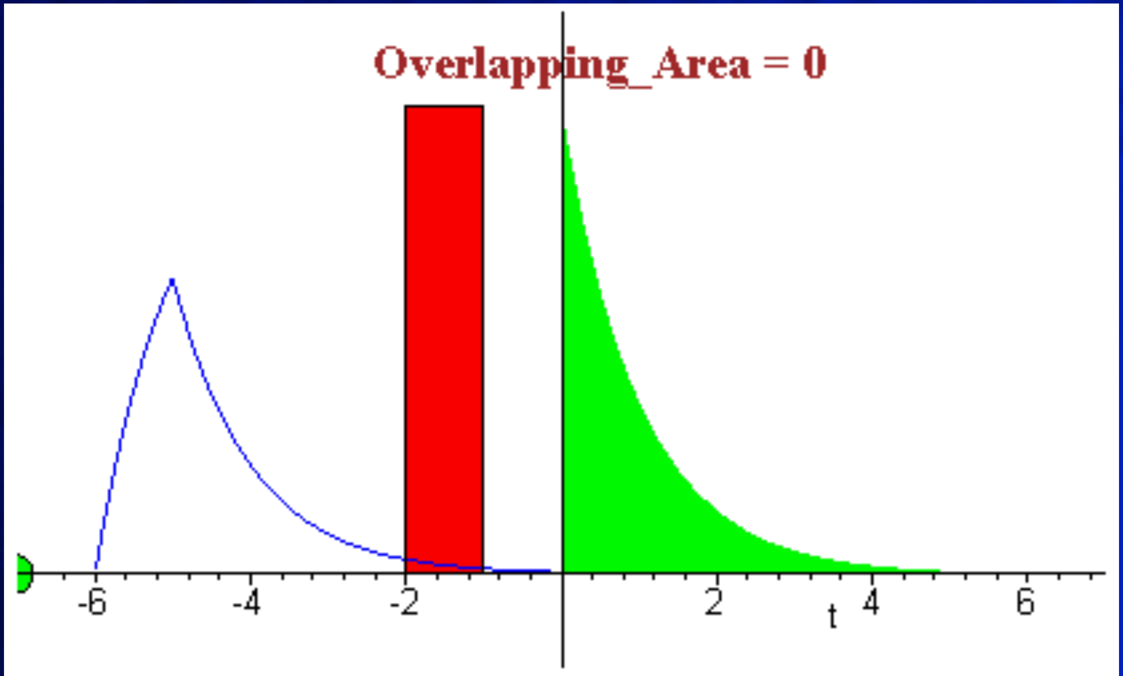
Case II) $g_{II}(t) = 1 - e^{-(t+{}^1/_2)}, \quad -\frac{1}{2} < t < \frac{1}{2}$

Case III) $g_{III}(t) = e^{-t}\left(e^{{}^1/_2} - e^{-{}^1/_2}\right), \quad t > \frac{1}{2}$

Recap of steps in convolving f_1 with f_2:

1. Flip f_2 left-to-right
2. Shift f_2 to $-\infty$
3. Multiply f_1 with the flipped and shifted f_2
4. Integrate
5. Repeat from step 2 for all values of t

A convolution animation (c/o Harald Pleym: https://www.hpleym.no/



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Recall convolution from last time

$$c(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

1. Flip f_2
2. Advance f_2 for all values of t from $-\infty$ to $+\infty$
3. For every t, multiply and integrate over τ

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Correlation is mathematically similar to convolution w/o the flip.

$$c(t) = f_1(t) \star f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t) d\tau$$

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Correlation is mathematically similar to convolution w/o the flip.

$$c(t) = f_1(t) \star f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t) d\tau$$

Note that the argument for f_2 needs to be $\tau - t$ not $t + \tau$ to preserve the direction of the shift from left to right, where $\tau - t = 0$.

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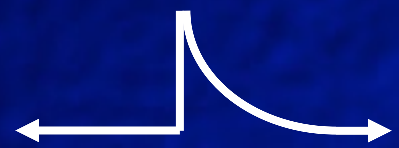
Using our previous example but with correlation.

Let $f_1(t) = \Pi(t)$

A boxcar function

and $f_2(t) = H(t)e^{-t}$



A decaying exponential

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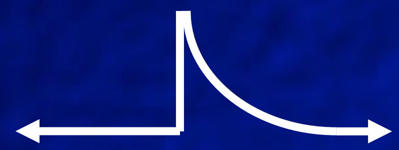

A decaying exponential

$$c(t) = f_1(t) \star f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t)\, d\tau$$

$$= \int_{-\infty}^{\infty} \Pi(\tau) H(\tau - t) e^{-(\tau - t)}\, d\tau$$

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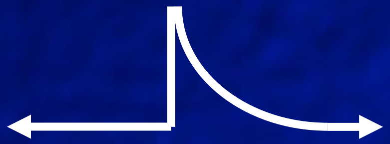

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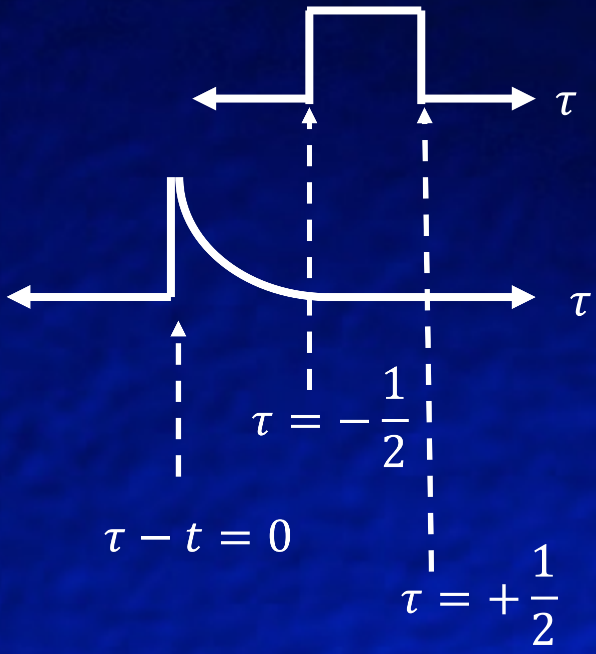
Again have to split the calculation into three parts because of the discontinuities.

Case I) Condition for t for this case?



$$c_I(t) = \int_{?}^{?} \Pi(\tau) H(\tau - t) e^{-(\tau - t)} d\tau$$

What are the limits of integration?

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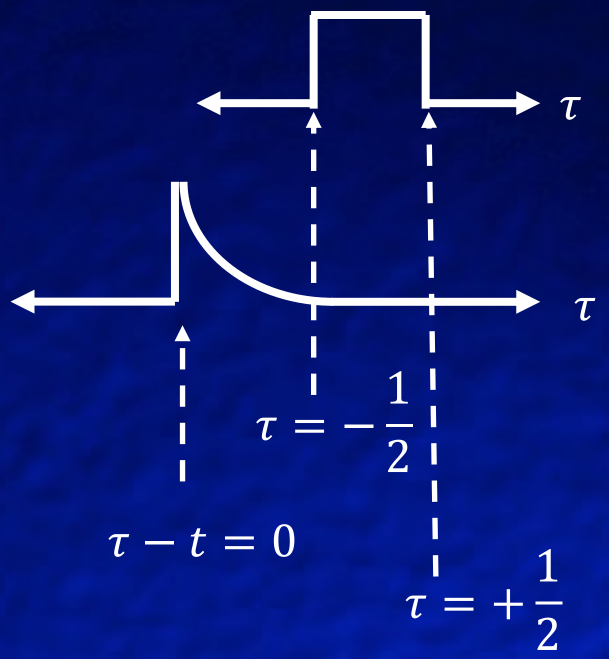
Case I) $t < -\frac{1}{2}$



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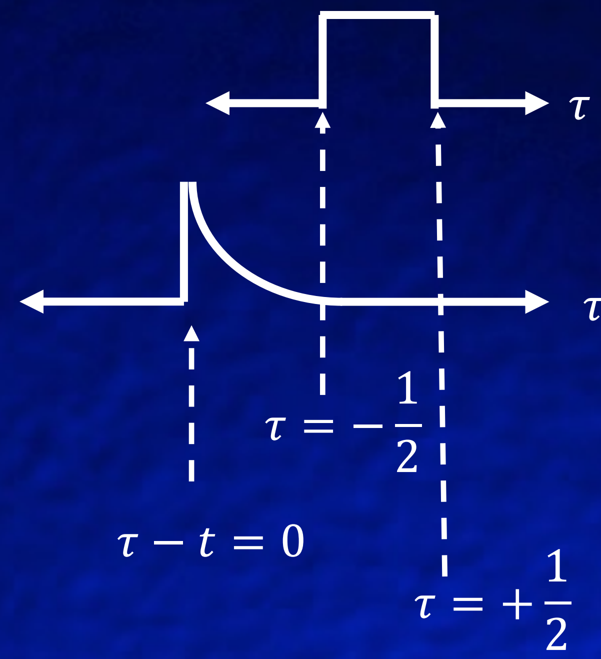

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$$= \int_{-1/2}^{1/2} e^{-(\tau - t)}\, d\tau = e^t \int_{-1/2}^{1/2} e^{-\tau}\, d\tau = -e^t\, e^{-\tau}\Big|_{\tau = -1/2}^{1/2}$$

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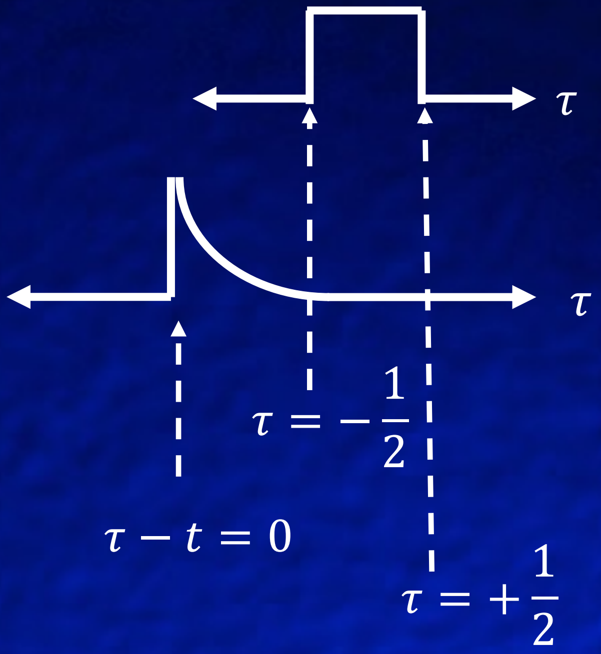

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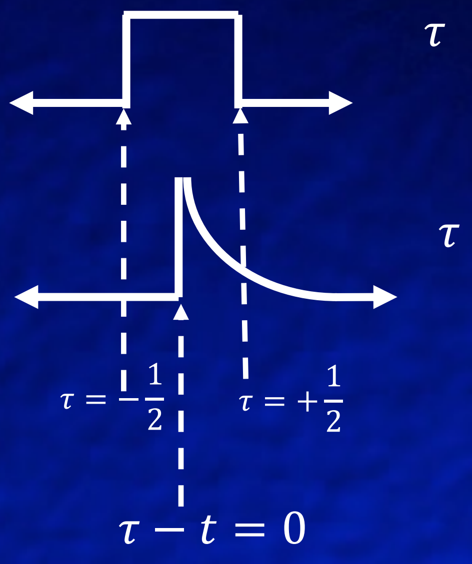
$$c_I(t) = -e^t e^{-1/2} + e^t e^{1/2} = e^{(t+1/2)} - e^{(t-1/2)}, \qquad where\ t < -\frac{1}{2}$$

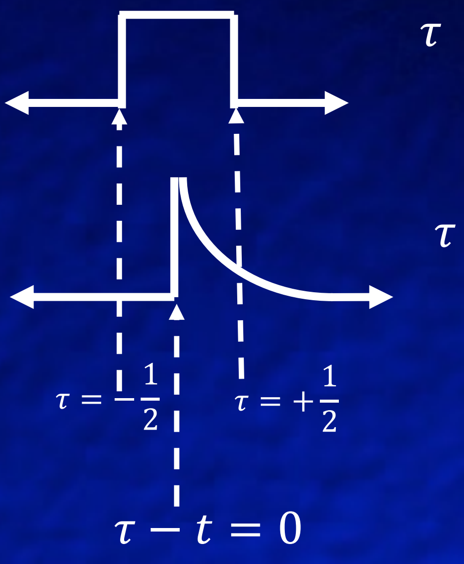
Case II) Conditions for t?

$$c_I(t) = \int_{?}^{?} \Pi(\tau) H(\tau - t) e^{-(\tau - t)} d\tau$$



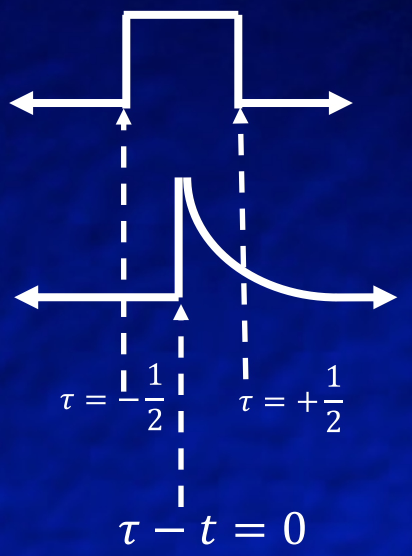
What are the limits of integration over τ?

Case II) $-\frac{1}{2} < t < \frac{1}{2}$



$$c_I(t) = \int_t^{1/2} \Pi(\tau) H(\tau - t) e^{-(\tau - t)} d\tau$$

Case II) $-\frac{1}{2} < t < \frac{1}{2}$



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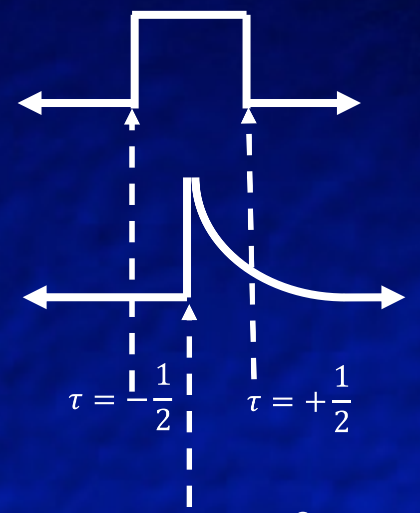

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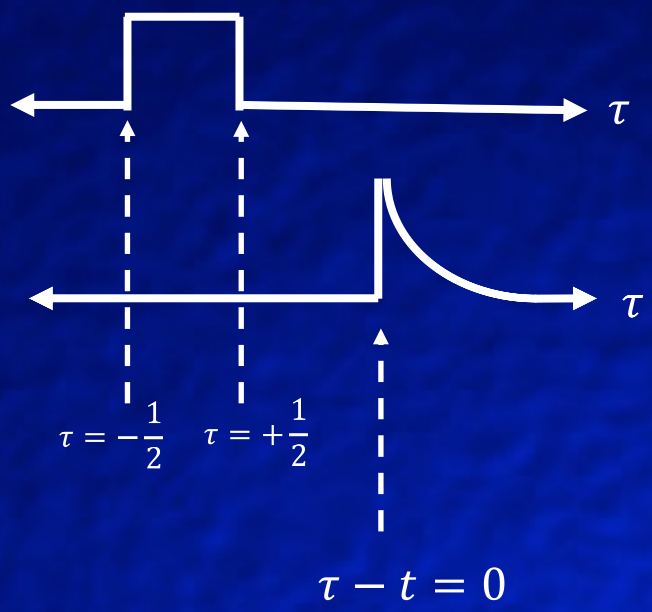
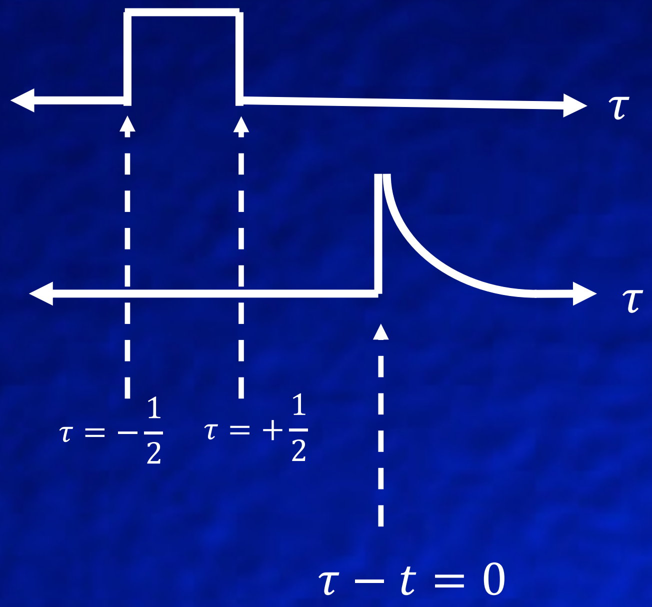
$$c_{II}(t) = -e^t e^{-1/2} + e^t e^{-t} = 1 - e^{(t - 1/2)}, \qquad where\ -\frac{1}{2} < t < \frac{1}{2}$$

Case III) Conditions for t?



Case III) $t > \frac{1}{2}$

$$c_{III}(t) = 0, \qquad where\ t > \frac{1}{2}$$



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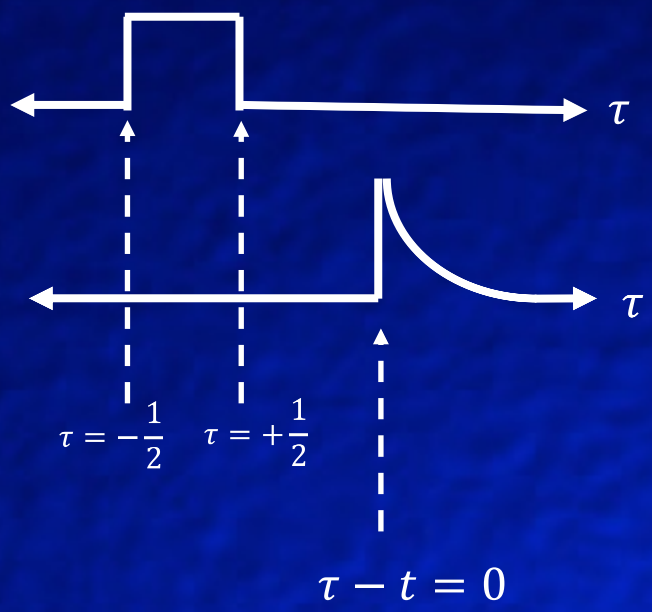

$$c_{III}(t) = 0, \qquad where\ t > \frac{1}{2}$$

Again, the complete answer has three parts each of which has conditions of t over which it is valid. Without the discontinuities, we could do the correlation in a single integration.

Autocorrelation is correlation of a function with itself

$$\mathrm{c}(t) = f_1(t) \star f_1(t) = \int_{-\infty}^{\infty} f_1(\tau) f_1(\tau - t) d\tau$$

- Convolution is used to “run” input signals through a linear system using the impulse response (or transfer function).
- Correlation is often used to test similarity between two functions (e.g. pattern matching).

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- Correlation is often used to test similarity between two functions (e.g. pattern matching).

When testing the similarity between two functions using correlation (or autocorrelation) it is common to normalize so that the result is $-1 < a(t) < 1$.

$$A(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t) d\tau$$

Total energy in denominator (more on that later)

$$a(t) = \frac{A(t)}{\int_{-\infty}^{\infty} f_1(\tau) f_2(\tau) d\tau}$$

- A(t=0) is said to be the correlation coefficient at 0 lag.
- Or the max A(t) is given for t-lag (or lead depending on whether max A(t) is + or -).
- A(t) max is at t=0 for autocorrelation (though there may be multiple peaks or maxima depending on the function).

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If $f_1(t)$ is a purely random function,

$$f_1(t) \star f_1(t) = \delta(t)$$

Can you explain why?

Does correlation obey the commutative property?

$$f(t) \star g(t) \stackrel{?}{=} g(t) \star f(t)$$

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Let $u = \tau - t$, then $du = d\tau$

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Let $u = \tau - t$, then $du = d\tau$

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(u + t) g(u) du$$

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$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(u + t) g(u) du$$

$$= \int_{-\infty}^{\infty} g(u) f(u + t) du$$

$$\neq \int_{-\infty}^{\infty} g(u) f(u - t) du = g(t) \star f(t)$$

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$$\neq \int_{-\infty}^{\infty} g(u) f(u - t) du = g(t) \star f(t)$$

No, in general correlation does not obey the commutative property.

$f(t) \star g(t) \neq g(t) \star f(t)$, so be careful

Cummutative: $f(t) \star g(t) \neq g(t) \star f(t)$

It can similarly be shown that

Distributive: $f(t) \star [g(t) + h(t)] = f(t) \star g(t) + f(t) \star h(t)$

Associative: $f(t) \star [g(t) \star h(t)] \neq [f(t) \star g(t)] \star h(t)$

Though these may all be true in some cases with appropriate symmetry.
e.g. if $f(u + t) = f(u - t)$.