The Wiener Filter

Brian Borchers and Rick Aster

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In this lecture we'll discuss the problem of optimally filtering noise from a signal. The Wiener filter was developed by Norbert Wiener in the 1940's. Although the filter can be derived in either continuous or discrete time, we'll derive a simple discrete time version of the filter.

Suppose that a noise sequence N_k is added to a signal S_k to produce a noisy signal Z_k.

$$Z_k = S_k + N_k. \tag{1}$$

We'll assume that both the signal and noise are covariance stationary, known means and covariances, and that the means are 0 (this is easy to arrange by subtracting out any nonzero mean.)

We want to filter out the noise and produce an estimate X_k such that $E[(S_k - X_k)^2]$ is minimized. We'll consider FIR filters of the form

$$X_k = \alpha_0 Z_k + \alpha_1 Z_{k-1} + \ldots + \alpha_m Z_{k-m}. \tag{2}$$

The minimization problem is

$$\min f(\alpha) = E[(S_k - N_k)^2]. \tag{3}$$

We can rewrite $f(\alpha)$ as

$$f(\alpha) = E[(S_k - (\alpha_0 Z_k + \alpha_1 Z_{k-1} + \ldots + \alpha_m Z_{k-m}))^2]. \tag{4}$$

$$f(\alpha) = E[S_k^2 - 2\sum_{i=0}^{m} \alpha_i S_k Z_{k-i} + \sum_{i=0}^{m}\sum_{j=0}^{m} \alpha_i \alpha_j Z_{k-i} Z_{k-j}]. \tag{5}$$

$$f(\alpha) = E[S_k^2] - 2\sum_{i=0}^{m} \alpha_i E[S_k Z_{k-i}] + \sum_{i=0}^{m}\sum_{j=0}^{m} \alpha_i \alpha_j E[Z_{k-i} Z_{k-j}]. \tag{6}$$

$$f(\alpha) = E[S_k^2] - 2\alpha^T v + \alpha^T T \alpha \tag{7}$$

where

$$v_i = E[S_k Z_{k-i}] \;\; i = 0, 1, 2, \ldots, m \tag{8}$$

and

$$T_{i,j} = E[Z_{k-i} Z_{k-j}] \;\; i, j = 0, 1, 2, \ldots, m. \tag{9}$$

To minimize $f(\alpha)$ and obtain the optimal filter weights α^*, we solve the equation

$$\nabla f(\alpha^*) = 0. \tag{10}$$

$$-2v + 2T\alpha^* = 0. \tag{11}$$

$$T\alpha^* = v. \tag{12}$$

Since

$$Z_k = S_k + N_k, \tag{13}$$

$$E[Z_{k-i}Z_{k-j}] = E[(S_{k-i} + N_{k-i})(S_{k-j} + N_{k-j})]. \tag{14}$$

$$E[Z_{k-i}Z_{k-j}] = E[S_{k-i}S_{k-j}] + E[S_{k-i}N_{k-j}] + E[N_{k-i}S_{k-j}] + E[N_{k-1}N_{k-j}]. \tag{15}$$

Since S and N are independent, the SN cross terms are 0, and

$$E[Z_{k-i}Z_{k-j}] = E[S_{k-i}S_{k-j}] + E[N_{k-i}N_{k-j}]. \tag{16}$$

Since S and N are covariance stationary and mean 0,

$$E[Z_{k-i}Z_{k-j}] = \gamma_{S,|i-j|} + \gamma_{N,|i-j|}. \tag{17}$$

Thus

$$T_{i,j} = \gamma_{S,|i-j|} + \gamma_{N,|i-j|}. \tag{18}$$

Similarly,

$$v_i = E[S_k Z_{k-i}]. \tag{19}$$

$$v_i = E[S_k(S_{k-i} + N_{k-i})]. \tag{20}$$

$$v_i = E[S_k S_{k-i}] + E[S_k N_{k-i}]. \tag{21}$$

$$v_i = \gamma_{S,i}. \tag{22}$$

Once we've found T and v, we can simply solve $T\alpha^* = v$ to obtain the optimal filter coefficients.

Next, we'll consider the expected value and variance of the filtered estimate X_k.

$$E[X_k] = E[\alpha_0^* Z_k + \ldots + \alpha_m^* Z_{k-m}]. \tag{23}$$

But $E[Z_{k-i} = 0$ for $i = 0, 1, \ldots, m$. So

$$E[X_k] = 0. \tag{24}$$

Thus X_k is an unbiased estimate of S_k.

The variance of the estimate is

$$E[(S_k - X_k)^2] = f(\alpha^*) = E[S_k^2] + \alpha^{*T} v + \alpha^{*T} T\alpha^*. \tag{25}$$

This allows us to put an uncertainty on our filtered estimate of S_k.

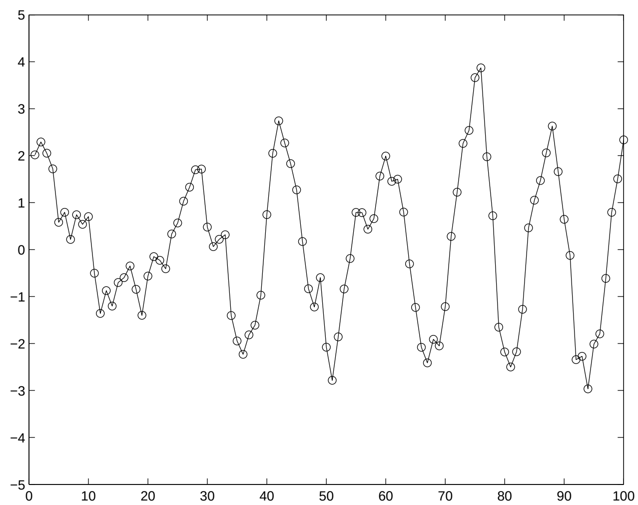

Figure 1: The original signal.

The basic idea can be extended further to predicting future values of S. Consider the problem of predicting S_{k+l}. This time, we minimize $E[(S_{k+l} - X_{k+l})^2]$, where

$$X_{k+l} = \alpha_0 Z_k + \ldots + \alpha_m Z_{k-m}. \tag{26}$$

By the same method as before, we find that the optimal filter coefficients α^* are the solution to $T\alpha^* = v$, where v has changed to

$$v_i = E[S_{k+l} Z_{k-i}]. \tag{27}$$

The special structure of the T matrix makes it possible to efficiently solve the $T\alpha = v$ systems of equations. Using an algorithm developed by Durbin and Levinson, this system can be solved in $O(m^2)$ time instead of the $O(m^3)$ time required by conventional Guassian elimination/factorization methods.

The following example shows how Wiener filtering can be used to substantially reduce the noise in a signal. In this example, we begin with a signal shown in Figure 1 that has most of its energy at low frequencies. The signal with added white noise is show in Figure 2. After constructing an optimal Wiener filter with $m = 10$, we filtered the noisy signal to get the result shown in Figure 3. Because the original signal had a spectrum with most of its energy at low frequencies and the noise has equal energy at all frequencies, the Wiener filter ends up filtering out high frequencies. The low frequency components of the signal are very well recovered. Some of small high frequency features in the original signal or smoothed away by the Wiener filter.

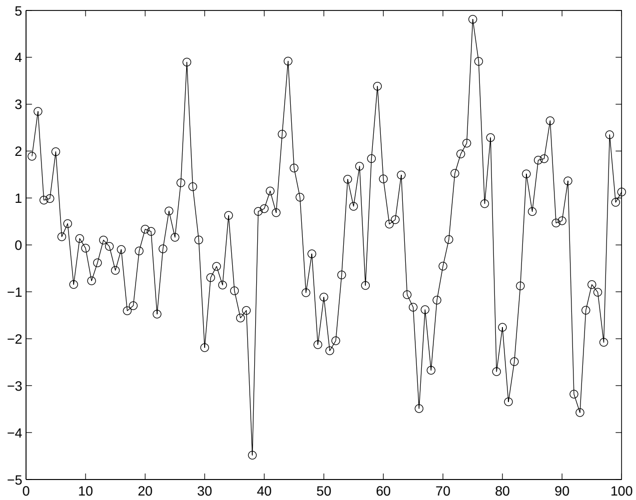

Figure 2: The original signal with white noise added.

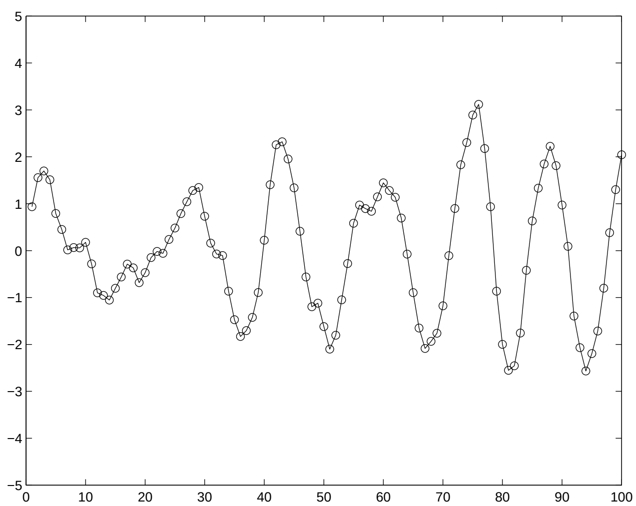

Figure 3: The filtered result.