# Notes on Random Processes

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## A Brief Review of Probability

In this section of the course, we will work with random variables which are denoted by capital letters, and which we will characterize by their **probability density functions** (pdf) and **cumulative density functions** (CDF.) We will use the notation $f_X(x)$ for the pdf and $F_X(a)$ for the CDF of $X$. Here, the subscript $X$ tells us which random variable's pdf or CDF we're working with. The relation between the pdf and CDF is

$$P(X \leq a) = F_X(a) = \int_{-\infty}^{a} f_X(x)dx. \tag{1}$$

Since probabilities are always between 0 and 1, the limit as $a$ goes to negative infinity of $F(a)$ is 0, and the limit as $a$ goes to positive infinity of $F(a)$ is 1. Also, $\int_{-\infty}^{\infty} f(x)dx = 1$. By the fundamental theorem of calculus, $F'(a) = f(a)$.

The most important distribution that we'll work with is the **normal distribution**.

$$P(X \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx. \tag{2}$$

Unfortunately, there's no simple formula for this integral. Instead, tables or numerical approximation routines are used to evaluate it. The normal distribution has a characteristic bell shaped pdf. The center of the bell is at $x = \mu$, and the parameter $\sigma^2$ controls the width of the bell. The particular case in which $\mu = 0$, and $\sigma^2 = 1$ is referred to as the **standard normal random variable**. The letter $Z$ is typically used for the standard normal random variable. Figure 1 shows the pdf of the standard normal.

The **expected value** of a random variable $X$ is

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx. \tag{3}$$

Note that this integral does not always converge!

For a normal random variable, it turns out (after a bit of work to evaluate the integral) that $E[X] = \mu$.

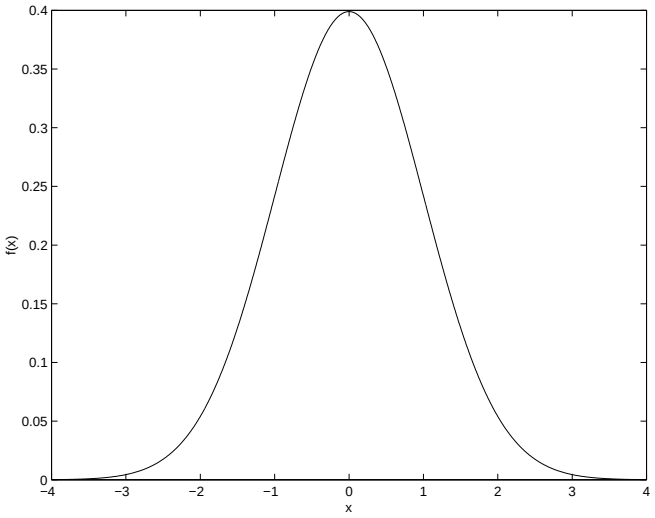


Figure 1: The standard normal pdf.

We'll often work with random variables that are functions of other random variables. If $X$ is a random variable with pdf $f_X(x)$ and $g()$ is a function, then $g(X)$ is also a random variable, and

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \tag{4}$$

Because integration is a linear operator,

$$E[X + Y] = E[X] + E[Y] \tag{5}$$

and

$$E[sX] = sE[X]. \tag{6}$$

The **variance** of a random variable $X$ is

$$Var(X) = E[(X - E[X])^2] \tag{7}$$

$$Var(X) = E[X^2 - 2XE[X] + E[X]^2] \tag{8}$$

Using the linearity of $E[]$ and the fact that the expected value of a constant is the constant, we get that

$$Var(X) = E[X^2] - 2E[X]E[X] + E[X]^2 \tag{9}$$

$$Var(X) = E[X^2] - E[X]^2. \tag{10}$$

For a normal random variable, it's relatively easy to show that $Var(X) = \sigma^2$.

If we have two random variables $X$ and $Y$, they *may* have a **joint probability density** $f(x, y)$ with

$$P(X \leq a \text{ and } Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y)\, dy\, dx \tag{11}$$

Two random variables $X$ and $Y$ are **independent** if they have a joint density and

$$f(x, y) = f_X(x) f_Y(y). \tag{12}$$

If $X$ and $Y$ have a joint density, then the **covariance** of $X$ and $Y$ is

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]. \tag{13}$$

It turns out that if $X$ and $Y$ are independent, then $E[XY] = E[X]E[Y]$, and $Cov(X, Y) = 0$. However, there are examples where $X$ and $Y$ are dependent, but $Cov(X, Y) = 0$. If $Cov(X, Y) = 0$, then we say that $X$ and $Y$ are **uncorrelated**.

The **correlation** of $X$ and $Y$ is

$$\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}. \tag{14}$$

The correlation is a sort of scaled version of the covariance that we will make frequent use of.

Some important properties of $Var$, $Cov$ and correlation include:

$$Var(X) \geq 0 \tag{15}$$

$$Var(sX) = s^2 Var(X) \tag{16}$$

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) \tag{17}$$

$$Cov(X, Y) = Cov(Y, X) \tag{18}$$

$$-1 \leq \rho_{XY} \leq 1 \tag{19}$$

The following example demonstrates the use of some of these properties.

**Example 1** Suppose that $Z$ is a standard normal random variable. Let

$$X = \mu + \sigma Z. \tag{20}$$

Then

$$E[X] = E[\mu] + \sigma E[Z] \tag{21}$$

so

$$E[X] = \mu. \tag{22}$$

Also,

$$Var(X) = Var(\mu) + \sigma^2 Var(Z) \tag{23}$$

$$Var(X) = \sigma^2. \tag{24}$$

Thus if we have a program to generate random numbers with the standard normal distribution, we can use it to generate random numbers with any desired normal distribution. The MATLAB command **randn** generates N(0,1) random numbers.

Suppose that $X_1$, $X_2$, ..., $X_n$ are independent realizations of a random variable $X$. How can we estimate $E[X]$ and $Var(X)$?

Let

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{25}$$

and

$$s^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1} \tag{26}$$

These estimates for $E[X]$ and $Var(X)$ are unbiased in the sense that

$$E[\bar{X}] = E[X] \tag{27}$$

and

$$E[s^2] = Var(X). \tag{28}$$

We can also estimate covariances with

$$\widehat{Cov}(X,Y) = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{n} \tag{29}$$

# Random Vectors

In digital signal processing we've been dealing with signals, represented in discrete time by vectors. Thus it's important to be able to work with random variables that are vectors. We'll consider random vectors of the form

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \tag{30}$$

where the individual random variables $X_i$ are assumed to have a joint probability density function.

The expected value of a random vector is

$$\mu = E[X] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}. \tag{31}$$

The covariance matrix of $X$ is

$$C = Cov(X) = E[XX^T] - E[X]E[X]^T. \tag{32}$$

Since

$$XX^T = \begin{bmatrix} X_1X_1 & X_1X_2 & X_1X_3 & \cdots & X_1X_n \\ X_2X_1 & X_2X_2 & X_2X_3 & \cdots & X_2X_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & X_nX_3 & \cdots & X_nX_n \end{bmatrix}, \tag{33}$$

$$C_{i,j} = E[X_iX_j] - E[X_i]E[X_j] = Cov(X_i, X_j). \tag{34}$$

We will also work with the correlation matrix

$$P_{i,j} = \frac{Cov(X_i, X_j)}{\sqrt{Cov(X_i, X_i)}\sqrt{Cov(X_j, X_j)}}. \tag{35}$$

Just as with scalar random variables, the expected value and covariance of a random vector have many useful properties. In deriving these properties we have to be somewhat careful, since matrix multiplication is not commutative. Thus

$$E[AX] = AE[X], \tag{36}$$

but

$$E[XA] = E[X]A, \tag{37}$$

An analogous result to $Var(sX) = s^2Var(X)$ is that

$$Cov(AX) = E[(AX)(AX)^T] - E[AX]E[AX]^T. \tag{38}$$

$$Cov(AX) = E[AXX^TA^T] - AE[X]E[X]^TA^T. \tag{39}$$

$$Cov(AX) = AE[XX^T]A^T - AE[X]E[X]^TA^T. \tag{40}$$

$$Cov(AX) = A(E[XX^T] - E[X]E[X]^T)A^T. \tag{41}$$

$$Cov(AX) = ACov(X)A^T. \tag{42}$$

Recall that a symmetric matrix $A$ is positive semidefinite (PSD) if $x^TAx \geq 0$, for all $x$. Also $A$ is positive definite (PD) if $x^TAx \geq 0$, for all nonzero $x$.

Corresponding to the property that $Var(X) \geq 0$, we find that the covariance matrix $C$ of a random variable is always positive semidefinite. To show this, let

$$W = \alpha_1X_1 + \ldots \alpha_nX_n = \alpha^TX. \tag{43}$$

Then

$$Var(W) = E[(W - E[W])(W - E[W])^T]. \tag{44}$$

$$Var(W) = E[(W - E[W])(W - E[W])^T]. \tag{45}$$

Since

$$W - E[W] = \alpha^Tx - \alpha^T\mu, \tag{46}$$

$$Var(W) = E[\alpha^T(x - \mu)(x - \mu)^T\alpha]. \tag{47}$$

$$Var(W) = \alpha^TE[(x - \mu)(x - \mu)^T]\alpha. \tag{48}$$

$$Var(W) = \alpha^TC\alpha. \tag{49}$$

But $Var(W) \geq 0$. Thus $\alpha^T C \alpha \geq 0$, for every vector $\alpha$, and $C$ is positive semidefinite.

We can estimate $E[X]$ and $Cov(X)$ from a sample of random vectors drawn from the distribution. Suppose that the columns of an $n$ by $m$ matrix $X$ are $m$ random vectors drawn from the distribution. Then we can estimate

$$E[X_j] \approx \frac{\sum_{j=1}^{n} X_{1,j}}{m} \tag{50}$$

or

$$E[X] \approx \frac{Xe}{m}, \tag{51}$$

where $e$ is the vector of all ones. We can also estimate that

$$Cov(X) \approx \frac{XX^T}{m} - \mu\mu^T. \tag{52}$$

# The Multivariate Normal (MVN) Distribution

The **multivariate normal distribution** (MVN) is an important joint probability distribution. If the random variables $X_1$, ..., $X_n$ have an MVN, then the probability density is

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{|C|}} e^{-(x-\mu)^T C^{-1} (X-\mu)/2}. \tag{53}$$

Here $\mu$ is a vector of the mean values of $X_1$, ..., $X_n$, and $C$ is a matrix of covariances with

$$C_{i,j} = Cov(X_i, X_j). \tag{54}$$

The multivariate normal distribution is one of a very few multivariate distributions with useful properties. Notice that the vector $\mu$ and the matrix $C$ completely characterize the distribution.

We can generate vectors of random numbers according to an MVN distribution by using the following process, which is very similar to the process for generating random normal scalars.

1. Find the Cholesky factorization $C = LL^T$.
2. Let $Z$ be a vector of $n$ independent N(0,1) random numbers.
3. Let $X = \mu + LZ$.

To see that $X$ has the appropriate mean and covariance matrix, we'll compute them.

$$E[X] = E[\mu + LZ] = \mu + E[LZ] = \mu + LE[Z] = \mu. \tag{55}$$

$$Cov[X] = E[(X-\mu)(X-\mu)^T] = E[(LZ)(LZ)^T]. \tag{56}$$

$$Cov[X] = LE[ZZ^T]L^T = LIL^T = LL^T = C. \tag{57}$$

## Covariance Stationary processes

A **discrete time stochastic process** is a sequence of random variables $Z_1$, $Z_2$, .... In practice we will typically analyze a single realization $z_1$, $z_2$, ..., $z_n$ of the stochastic process and attempt to estimate the statistical properties of the stochastic process from the realization. We will also consider the problem of predicting $z_{n+1}$ from the previous elements of the sequence.

We will begin by focusing on the very important class of **stationary** stochastic processes. A stochastic process is **strictly stationary** if its statistical properties are unaffected by shifting the stochastic process in time. In particular, this means that if we take a subsequence $Z_{k+1}, \ldots, Z_{k+m}$, then the joint distribution of the $m$ random variables will be the same no matter what $k$ is.

In practice, we're often only interested in the means and covariances of the elements of a time series. A time series is **covariance stationary**, or **second order stationary** if its mean and its autocovariances (or autocorrelations) at all lags are finite and constant. For a covariance stationary process, the **autocovariance at lag** $m$ is $\gamma_m = Cov(Z_k, Z_{k+m})$. Since covariance is symmetric, $\gamma_{-m} = \gamma_m$. The correlation of $Z_k$ and $Z_{k+m}$ is the **autocorrelation at lag** $m$. We will use the notation $\rho_m$ for the autocorrelation. It is easy to show that

$$\rho_k = \frac{\gamma_k}{\gamma_0}. \tag{58}$$

## The autocovariance and autocorrelation matrices

The covariance matrix for the random variables $Z_1$, ..., $Z_n$ is called an **autocovariance matrix**.

$$\Gamma_n = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix} \tag{59}$$

Similarly, we can form an **autocorrelation matrix**

$$P_n = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1} & \rho_{n-2} & \cdots & \rho_1 & 1 \end{bmatrix}. \tag{60}$$

Note that

$$\Gamma_n = \sigma_Z^2 P_n. \tag{61}$$

Since the autocovariance matrix is a covariance matrix, it is positive semidefinite. It's easy to show that the autocorrelation matrix is also positive semidefinite.

An important example of a stationary process that we will work with occurs when the joint distribution of $Z_k$, ..., $Z_{k+n}$ is multivariate normal. In this situation, the autocovariance matrix $\Gamma_n$ is precisely the covariance matrix $C$ for the multivariate normal distribution.

## Estimating the mean, autocovariance, and autocorrelation

Given a realization $z_0$, $z_2$, ..., $z_{N-1}$, of a stochastic process, how can we estimate the mean, variance, autocovariance and autocorrelation?

We will estimate the mean by

$$\bar{z} = \frac{\sum_{j=0}^{N-1} z_j}{N}. \tag{62}$$

We will estimate the autocovariance at lag $k$ with

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} (z_j - \bar{z})(z_{j+k} - \bar{z}). \tag{63}$$

Here we have used the convention that $z_k$ is a periodic sequence to get $z_{j+k}$ in cases where $j + k > N - 1$.

Note that $c_0$ is an estimate of the variance, but it is not the same unbiased estimate that we used in the last lecture. The problem here is that the $z_i$ are correlated, so that the formula from the last lecture no longer provides an unbiased estimator. The formula given here is also biased, but is considered to work better in practice.

We will estimate the autocorrelation at lag $k$ with

$$r_k = \frac{c_k}{c_0}. \tag{64}$$

The following example demonstrates the computation of autocorrelation and autocovariance estimates.

**Example 2** Consider the time series of yields from a batch chemical process given in Table 1. The data is plotted in Figure 2. These data are taken from p 31 of Box, Jenkins, and Reinsel. Read the table by rows. Figure 3 shows the

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 47 | 64 | 23 | 71 | 38 | 64 | 55 | 41 | 59 | 48 |
| 71 | 35 | 57 | 40 | 58 | 44 | 80 | 55 | 37 | 74 |
| 51 | 57 | 50 | 60 | 45 | 57 | 50 | 45 | 25 | 59 |
| 50 | 71 | 56 | 74 | 50 | 58 | 45 | 54 | 36 | 54 |
| 48 | 55 | 45 | 57 | 50 | 62 | 44 | 64 | 43 | 52 |
| 38 | 59 | 55 | 41 | 53 | 49 | 34 | 35 | 54 | 45 |
| 68 | 38 | 50 | 60 | 39 | 59 | 40 | 57 | 54 | 23 |

Table 1: An example time series.

estimated autocorrelation for this data set. The fact that $r_1$ is about -0.4 tells us that whenever there is a sample in the data that is well above the mean, it is likely to be followed by a sample that is well below the mean, and vice

versa. Notice that the autocorrelation tends to alternate between positive and negative values and decays rapidly towards a noise level. After about $k = 6$, the autocorrelation seems to have died out.

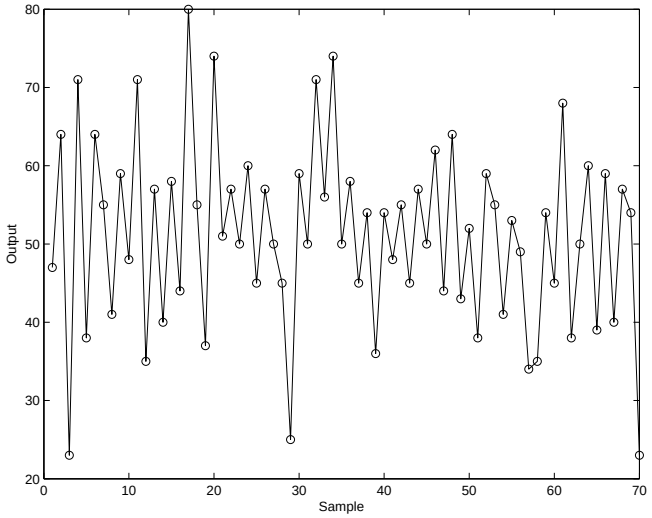


Figure 2: An example time series.

Just as with the sample mean, the autocorrelation estimate $r_k$ is a random quantity with its own standard deviation. It can be shown that

$$Var(r_k) \approx \frac{1}{n} \sum_{v=-\infty}^{\infty} (\rho_v^2 + \rho_{v+k}\rho_{v-k} - 4\rho_k\rho_v\rho_{v-k} + 2\rho_v^2\rho_k^2). \tag{65}$$

The autocorrelation function typically decays rapidly, so that we can identify a lag $q$ beyond which $r_k$ is effectively 0. Under these circumstances, the formula simplifies to

$$Var(r_k) \approx \frac{1}{n}(1 + 2\sum_{v=1}^{q} \rho_v^2), \quad k > q. \tag{66}$$

In practice we don't know $\rho_v$, but we can use the estimates $r_v$ in the above formula. This provides a statistical test to determine whether or not an autocorrelation $r_k$ is statistically different from 0. An approximate 95% confidence interval for $r_k$ is $r_k \pm 1.96 * \sqrt{Var(r_k)}$. If this confidence interval includes 0, then we can't rule out the possibility that $r_k$ really is 0 and that there is no correlation at lag $k$.

**Example 3** Returning to our earlier data set, consider the variance of our estimate of $r_6$. Using $q = 5$, we estimate that $Var(r_6) = .0225$ and that the standard deviation is about 0.14. Since $r_6 = -0.0471$ is considerably smaller than the standard deviation, we will decide to treat $r_k$ as essentially 0 for $k \geq 6$.

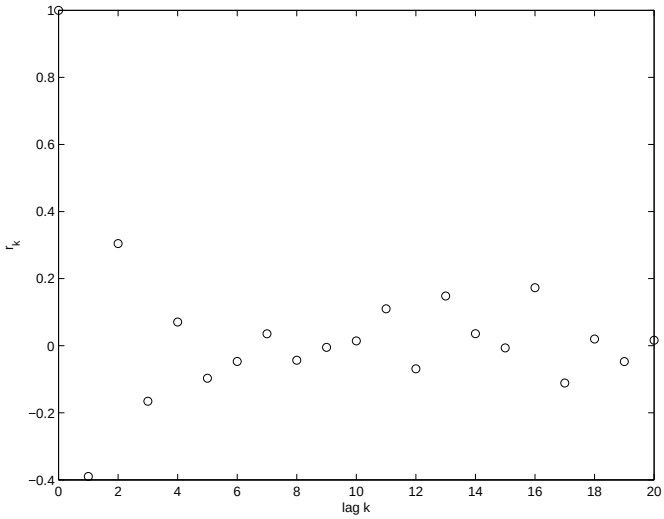


Figure 3: Estimated autocorrelation for the example data.

## The spectrum and autocorrelation

In continuous time, the spectrum of a signal $\phi(t)$ is given by

$$PSD(f) = |\Phi(f)|^2 = \Phi(f)\Phi(f)^*. \tag{67}$$

Since

$$\Phi(f) = \int_{t=-\infty}^{\infty} \phi(t) e^{-2\pi i f t} dt, \tag{68}$$

$$\Phi(f)^* = \int_{t=-\infty}^{\infty} \phi(t)^* e^{+2\pi i f t} dt. \tag{69}$$

Let $\tau = -t$. Then $d\tau = -dt$, and

$$\Phi(f)^* = \int_{\tau=-\infty}^{\infty} \phi(-\tau)^* e^{-2\pi i f \tau} d\tau. \tag{70}$$

$$\Phi(f)^* = F\left[\phi(-t)^*\right]. \tag{71}$$

Thus

$$PSD(f) = F\left[\phi(t)\right] F\left[\phi(-t)^*\right], \tag{72}$$

or by the convolution theorem,

$$PSD(f) = F\left[\phi(t) * \phi(-t)^*\right] = F\left[\text{autocorr } \phi(t)\right]. \tag{73}$$

We can derive a similar connection in discrete time between the periodogram and the autocovariance. Given a $N$-periodic sequence $z_n$, the autocovariance is

$$c_n = \frac{1}{N}\sum_{j=0}^{N-1}(z_j - \bar{z})(z_{j+n} - \bar{z}). \tag{74}$$

$$c_n = \frac{1}{N}\left(\sum_{j=0}^{N-1} z_j z_{j+n} - 2\sum_{j=0}^{N-1} z_j \bar{z} + \sum_{j=0}^{N-1} \bar{z}^2\right). \tag{75}$$

Since

$$\bar{z} = \frac{1}{N}\sum_{j=0}^{N-1} z_j, \tag{76}$$

$$c_n = \frac{1}{N}\left(\sum_{j=0}^{N-1} z_j z_{j+n} - N\bar{z}^2\right). \tag{77}$$

Now, we'll compute the DFT of $c_n$.

$$C_m = \sum_{n=0}^{N-1} c_n e^{-2\pi i n m/N}. \tag{78}$$

$$C_m = \sum_{n=0}^{N-1}\left(\sum_{j=0}^{N-1}\frac{z_j z_{j+n}}{N} - \bar{z}^2\right)e^{-2\pi i n m/N}. \tag{79}$$

By our "technical result",

$$\sum_{n=0}^{N-1} -\bar{z}^2 e^{-2\pi i n m/N} = -N\bar{z}^2\delta_m. \tag{80}$$

When $m = 0$, $e^{-2\pi i m n/N} = 1$, so we get

$$C_0 = \left(\sum_{n=0}^{N-1}\sum_{j=0}^{N-1}\frac{z_j z_{j+n}}{N}\right) - N\bar{z}^2. \tag{81}$$

Since

$$\sum_{n=0}^{N-1}\sum_{j=0}^{N-1}\frac{z_j z_{j+n}}{N} = N\bar{z}^2, \tag{82}$$

$$C_0 = 0. \tag{83}$$

When $m \neq 0$, things are more interesting. In this case,

$$C_m = \sum_{n=0}^{N-1}\sum_{j=0}^{N-1}\frac{z_j z_{j+n}}{N}e^{-2\pi i n m/N}. \tag{84}$$

$$C_m = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} z_j z_{j+n} e^{-2\pi i n m/N}. \tag{85}$$

$$C_m = \frac{1}{N} \sum_{j=0}^{N-1} z_j \sum_{n=0}^{N-1} z_{j+n} e^{-2\pi i n m/N}. \tag{86}$$

$$C_m = \frac{1}{N} \sum_{j=0}^{N-1} z_j e^{+2\pi i j m/N} \sum_{n=0}^{N-1} z_{j+n} e^{-2\pi i (j+n) m/N}. \tag{87}$$

Using the fact that $z$ is real we get,

$$C_m = \frac{1}{N} \sum_{j=0}^{N-1} z_j^* e^{+2\pi i j m/N} \sum_{n=0}^{N-1} z_{j+n} e^{-2\pi i (j+n) m/N}. \tag{88}$$

$$C_m = \frac{1}{N} Z_m^* \sum_{n=0}^{N-1} z_{j+n} e^{-2\pi i (j+n) m/N}. \tag{89}$$

Using the fact that $z$ is $N$-periodic, we get

$$C_m = \frac{1}{N} Z_m^* Z_m. \tag{90}$$

Note that because $c_n$ is symmetric, $C_m$ is real. Also note that the right hand side of this equation is always nonnegative. This means that $C_m \geq 0$. It turns out that $C_m \geq 0$ is equivalent to the autocovariance matrix being positive semidefinite.

Thus knowing the spectrum of $z$ is really equivalent to knowing the autocovariance, $c$, or its DFT, $C$. In practice, the sample spectrum from a short time series is extremely noisy, so it's extremely difficult to make sense of the spectrum. On the other hand, it is much easier to make sense of the autocorrelation function of a short time series. For this reason, the autocorrelation is more often used in analyzing shorter time series.

**Example 4** Figure 4 shows the periodogram for our example data. It's very difficult to detect any real features in this spectrum. The problem is that with a short time series you get little frequency resolution, and lots of noise. Longer time series make it possible to obtain both better frequency resolution (by using a longer window) and reduced noise (by averaging over many windows.) However, if you're stuck with only a short time series, the first few autocorrelations may be more informative than the periodogram.

## Generating Correlated Gaussian Noise

The connection between the autocovariance and spectrum also provides us with another way to generate random Gaussian noise with specified autocovariance. In this approach, introduced by Shinozuka and Jan (1972), we start with a

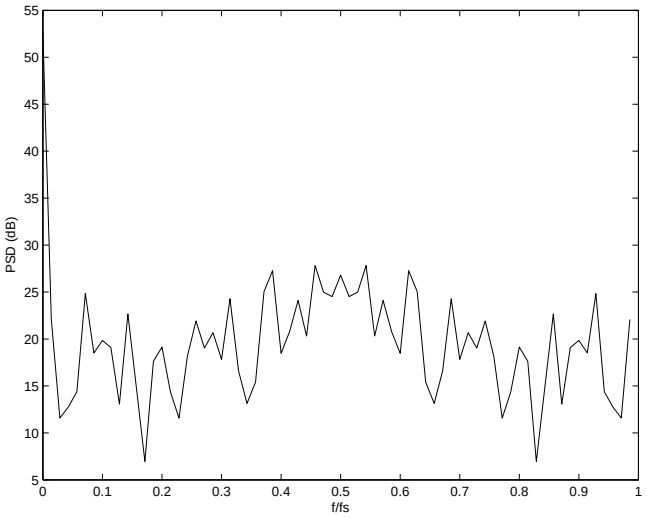


Figure 4: Periodogram of the sample time series.

desired autocovariance $c_n$, compute the DFT $C_m$, and then use (90) to get a real, nonnegative square root, $Z_m$. We could simply invert this to obtain a real sequence $z_n$. However, this sequence wouldn't be random. Shinozuka's clever idea was to compute $Z_m$, and then apply random phases to each of the $Z_m$ components, while keeping the sequence $Z_m$ Hermitian. To do this, we multiply $Z_k$ by $e^{\theta_k i}$, and multiply $Z_{-k}$ by $e^{-\theta_k i}$, where $\theta_k$ is uniformly distributed between 0 and $2\pi$. We can then invert the discrete Fourier transform to get a random sequence $z_n$ with exactly the required autocovariances.

An important advantage of this spectral method for generating correlated Gaussian noise is that it does not require computing and storing the Cholesky factorization of the autocovariance matrix. This makes the generation of long (millions of points) sequences or 2-D or 3-D random fields computationally tractable.