Data Processing and Analysis

Rick Aster and Brian Borchers

September 9, 2013

Poles and Zeros

We showed that for any linear system relating two time functions, $x(t)$ and $y(t)$, the frequency-domain response (the transfer function) can be obtained from the governing linear differential equation with constant coefficients of a single variable

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \ . \tag{1}$$

Setting

$$x(t) = e^{\imath 2\pi f t} \ , \tag{2}$$

$$y(t) = \Phi(f) e^{\imath 2\pi f t} \ , \tag{3}$$

and solving (1) for $\Phi(f)$ gives the transfer function

$$\Phi(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{j=0}^{m} b_j (2\pi \imath f)^j}{\sum_{k=0}^{n} a_k (2\pi \imath f)^k} \equiv \frac{Z(f)}{P(f)} \tag{4}$$

where Z and P are complex polynomials in f. The values of f (or equivalently, of the angular frequency $\omega = 2\pi f$) where $Z(f) = 0$ are referred to as *zeros* of (4), as the response of the system will be zero at those frequencies, no matter what the amplitude of the input. Frequencies for which $P(f) = 0$ are referred to as *poles* of (4), as the response of the system will be infinite at those frequencies.

In general, the values of f where we have poles and zeros will be complex. If we express the polynomials in (4) in terms of $\imath f$ (or equivalently, $\imath 2\pi f$), then we have real coefficients, and the roots after this change of variables will be real or complex conjugate pairs. It is useful to express the input function (2) as

$$x(t) = e^{\imath 2\pi f t} \ = e^{\imath 2\pi (f_r + \imath f_i) t} = e^{\imath 2\pi f_r t} \cdot e^{-2\pi f_i t} \tag{5}$$

where $f = f_r + \imath f_i$ and f_r and f_i are real numbers.

This generalized input is:

- A constant for $f = 0$

- A sinusoid for $f_r \neq 0$ and $f_i = 0$.
- A growing exponential for $f_r = 0$ and $f_i < 0$
- A shrinking exponential for $f_r = 0$ and $f_i > 0$
- A growing exponentially weighted sinusoid for $f_r \neq 0$ and $f_i < 0$
- A shrinking exponentially weighted sinusoid for $f_r \neq 0$ and $f_i > 0$

Pole positions are usually displayed graphically in the complex plane using the Laplace transform convention

$$s = \imath 2\pi f = \sigma + \imath\omega = 2\pi(-f_i + \imath f_r) \ . \tag{6}$$

the positions of the poles in s in the complex plane are an especially useful and compact way to characterize the response of a linear system. Making the substitution (6, and normalizing the leading coefficients of the polynomials, gives us a transfer function expression

$$\Phi(s) = \frac{Y(s)}{X(s)} = (b_m/a_n)\frac{\sum_{j=0}^{m}(b_j/b_m)s^j}{\sum_{k=0}^{n}(a_k/a_n)s^k} = K\frac{Z(s)}{P(s)} \tag{7}$$

where $K = b_m/a_n$ is a scalar gain factor.

The complex roots in s of the numerator and denominator of (7) will be either real, or will be complex conjugate pairs if the coefficients are real (or equivalently, if the original impulse response is real-valued).

Systems where all pole frequencies have $\sigma < 0$ ($f_i > 0$), so that the poles lie on the left-hand side of the z plane, are stable. In this case the only way to get an infinite output is to drive the system with an exponentially increasing sinusoidal input. The impulse response of such a system will always decay back to zero.

On the other hand, systems where all pole frequencies have $\sigma > 0$ ($f_i < 0$), so that the poles lie on the right-hand side of the z plane, are unstable; we obtain an infinite output even when the input is exponentially decaying. The impulse response of such systems increases in amplitude with time.

Systems where $\sigma = 0$ and $\omega \neq 0$ ($f_i = 0$ and $f_r \neq 0$) have pole frequency responses that are sinudoisal. Such systems will oscillate forever once they get (even marginally) excited at their resonant frequencies.

Figure (1) shows $z-pole$ locations and cartoon impulse responses for various 2-pole systems.

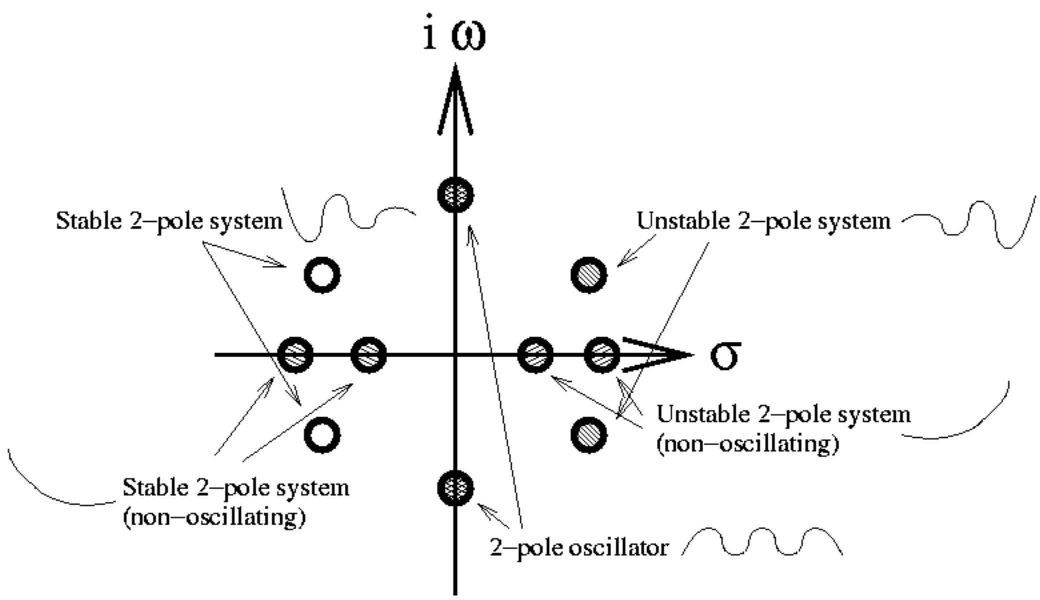

Figure 1: Pole locations and system stability for 2-pole systems, real-valued impulse response. Sketched time functions show oscillation and decay characteristics of the corresponding impulse responses.