Data Processing and Analysis

Rick Aster and Brian Borchers

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Introduction to Multidimensional and Multichannel Processing

We have now covered most of the basic tools in analyzing one-dimensional time or spatial series. Many data sets in geophysics and other fields, however, are inherently multi-dimensional, either because the independent variable is multidimensional (e.g., a 2-dimensional survey or a 3-dimensional structure) or because the data itself is a vector quantity (e.g., three-component seismic or electromagnetic data).

Two or higher dimensional data sets require a multidimensional analysis technique. Some examples include photographic records, remote sensing data, or other 2-d images, seismic records from a 2-dimensional array, and gravity and magnetic surveys. Other signals may be considered multidimensional, with the two axes being physically different, such as a linear array of seismometers, where one dimension is temporal and the other is spatial or a two-dimensional array with a third time dimension. In general, much of one's intuition developed from analyzing 1-dimensional systems may be applied, although there are some very important concepts of 1-dimensional systems which do not apply in more dimensions.

Let $x(n_1, n_2)$ be a two-dimensional sequence defined for integer n_1 and n_2. Such a 2-d sequence is usually obtained from sampling a continuous 2-dimensional function. Some examples of 2-d sequences would be the unit impulse:

$$\delta(n_1, n_2) = \begin{cases} 1 \text{ for } n_1 = n_2 = 0 \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

the step function

$$H(n_1, n_2) = \begin{cases} 1 \text{ for } n_1, n_2 \geq 0 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

the exponential

$$x(n_1, n_2) = \begin{cases} \alpha_1^{n_1} \alpha_2^{n_2} \text{ for } n_1, n_2 \geq 0 \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

and the sinusoid

$$x(n_1, n_2) = e^{\imath 2\pi (f_1 n_1 + f_2 n_2)} . \tag{4}$$

If a system is linear and time invariant, then convolution is a valid concept in dimensions higher than 1, thus if $x(n_1, n_2)$ is an input to a two dimensional system which has an impulse response of $\phi(n_1, n_2)$, then the output is

$$y(n_1,n_2) = x(n_1,n_2) * \phi(n_1,n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(m_1,m_2)x(n_1-m_1,n_2-m_2) \tag{5}$$

$$= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(n_1-m_1,n_2-m_2)x(m_1,m_2) \tag{6}$$

(6) is usually difficult to apply, however, consider a simple case given by (*Rabiner and Gold*, 1975), where

$$\phi(n_1,n_2) = \alpha^{n_1 n_2} \tag{7}$$

and

$$x(n_1,n_2) = \begin{cases} 1 \text{ for } 0 \le n_1, n_2 \le 2 \\ 0 \text{ otherwise} \end{cases} \tag{8}$$

the response, $\phi(n_1,n_2) * x(n_1,n_2)$ is thus

$$y(n_1,n_2) = \sum_{m_1=0}^{2} \sum_{m_2=0}^{2} \alpha^{(n_1-m_1)(n_2-m_2)} \tag{9}$$

which, in general must be evaluated term by term for each (n_1, n_2) where each term requires $3^2 = 9$ operations. If $\phi(n_1, n_2)$ is *separable* , i.e., it can be written as

$$\phi(n_1,n_2) = g(n_1) \cdot f(n_2) \tag{10}$$

then the response can be calculated in terms of consecutive 1-dimensional convolutions, as (6) now becomes

$$y(n_1,n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} g(m_1)f(m_2)x(n_1-m_1,n_2-m_2) \tag{11}$$

$$= \sum_{m_1=-\infty}^{\infty} g(m_1) \left(\sum_{m_2=-\infty}^{\infty} f(m_2)x(n_1-m_1,n_2-m_2) \right) \tag{12}$$

where the term inside of the parentheses is a sequence of 1-d convolutions where m_1 is allowed to range from $-\infty$ to ∞. If the input sequence is also separable, so that $x(n_1, n_2) = a(n_1) \cdot b(n_2)$, then

$$y(n_1,n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} g(m_1)f(m_2)a(n_1-m_1)b(n_2-m_2) \tag{13}$$

$$= \left(\sum_{m_1=-\infty}^{\infty} g(m_1)a(n_1-m_1) \right) \left(\sum_{m_2=-\infty}^{\infty} f(m_2)b(n_2-m_2) \right) \tag{14}$$

which is also separable, i.e.,

$$y(n_1, n_2) = \alpha(n_1) \cdot \beta(n_2) \tag{15}$$

where $\alpha(n_1)$ and $\beta(n_2)$ are 1-dimensional convolutions (14).

As in 1-d systems, sinusoidal inputs play the fundamental functional role in the Fourier analysis of 2-d systems. This is because 2-dimensional sinusoidal functions

$$x(n_1, n_2) = e^{\imath 2\pi f_1 n_1} e^{\imath 2\pi f_2 n_2} \tag{16}$$

are eigenfunctions of the 2-d convolution operation. Consider the output of a system with impulse response $\phi(n_1, n_2)$ to a complex exponential input

$$y(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(m_1, m_2) e^{\imath 2\pi f_1 (n_1 - m_1)} e^{\imath 2\pi f_2 (n_2 - m_2)} \tag{17}$$

$$= e^{\imath 2\pi f_1 n_1} e^{\imath 2\pi f_2 n_2} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(m_1, m_2) e^{-\imath 2\pi f_1 m_1} e^{-\imath 2\pi f_2 m_2} = x(n_1, n_2)\Phi(f_1, f_2) \tag{18}$$

where $\Phi(f_1, f_2)$ is the frequency response of the system in two dimensions and hence defines a 2-d Fourier transform of a 2-d sampled function. The corresponding inverse transformation (see table below) is just

$$\phi(n_1, n_2) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \Phi(f_1, f_2) e^{\imath 2\pi f_1 n_1} e^{\imath 2\pi f_2 n_2} \, df_1 \, df_2 \ . \tag{19}$$

Note that $\Phi(f_1, f_2)$ is periodic in frequency with unit period for both f_1 and f_2, as we'd expect for a sampled function

$$\Phi(f_1, f_2) = \phi(f_1 + l, f_2 + m) \ (l, m \text{ integers}) \tag{20}$$

which is two-dimensional aliasing. If $\phi(n_1, n_2)$ is real, then

$$\Phi(f_1, f_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(m_1, m_2) e^{-\imath 2\pi f_1 m_1} e^{-\imath 2\pi f_2 m_2} = \Phi^*(-f_1, -f_2) \tag{21}$$

so that $\Phi(f_1, f_2)$ is Hermitian in a 2-d sense.

We now have the tools for performing windowed filter design in 2-dimensions in a manner entirely analogous to that which we previously examined for 1-d FIR filters. Consider the perfect low pass filter with a response given by

$$\Phi(f_1, f_2) = \begin{cases} 1 \text{ for } -\alpha \leq f_1 \leq \alpha, -\beta \leq f_2 \leq \beta \\ 0 \text{ otherwise} \end{cases} \tag{22}$$

taking the inverse Fourier transform gives the n domain series

$$\phi(n_1, n_2) = \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} e^{\imath 2\pi f_2 n_2} e^{\imath 2\pi f_1 n_1} df_2 df_1 \tag{23}$$

if the frequency response is separable, so is the n domain response, so

$$\phi(n_1, n_2) = \left(\int_{-\alpha}^{\alpha} e^{\imath 2\pi f_1 n_1} df_1\right)\left(\int_{-\beta}^{\beta} e^{\imath 2\pi f_2 n_2} df_2\right) \tag{24}$$

$$= \left(\frac{e^{\imath 2\pi f_1 n_1}}{\imath 2\pi n_1}\right)\Bigg|_{f_1=-\alpha}^{\alpha} \left(\frac{e^{\imath 2\pi f_2 n_2}}{\imath 2\pi n_2}\right)\Bigg|_{f_2=-\beta}^{\beta} = \left(\frac{\sin(2\pi\alpha n_1)}{\pi n_1}\right)\left(\frac{\sin(2\pi\beta n_2)}{\pi n_2}\right). \tag{25}$$

This frequency response and a plot of its corresponding filter weights is shown on the following page.

Unless we have a physical reason for wishing to treat the n_1 and n_2 directions unequally, we would generally want to have a response which is circularly symmetric in the time and frequency domains. Such a filter is specified by

$$\Phi(f_1, f_2) = \begin{cases} 1 \ f_1^2 + f_2^2 \leq f_{\max}^2 \\ 0 \text{ otherwise} \end{cases} \tag{26}$$

and the corresponding filter weights are obtainable as

$$\phi(n_1, n_2) = \int_{-f_{\max}}^{f_{\max}} \int_{-(f_{\max}^2 - f_1^2)^{1/2}}^{(f_{\max}^2 - f_1^2)^{1/2}} e^{\imath 2\pi f_1 n_1} e^{\imath 2\pi f_2 n_2} \ df_2 \ df_1 \ . \tag{27}$$

An easy way to evaluate this integral is to note that both the n and frequency response of the system are circularly symmetric, thus, we can obtain the general solution by finding $\phi(n_1, 0)$ and then substituting $(n_1^2 + n_2^2)^{1/2}$ for n_1.

$$\phi(n_1, 0) = \int_{-f_{\max}}^{f_{\max}} \int_{-(f_{\max}^2 - f_1^2)^{1/2}}^{(f_{\max}^2 - f_1^2)^{1/2}} e^{\imath 2\pi f_1 n_1} \ df_2 \ df_1 \tag{28}$$

$$= \int_{-f_{\max}}^{f_{\max}} e^{\imath 2\pi f_1 n_1} \cdot 2(f_{\max}^2 - f_1^2)^{1/2} \ df_1 \tag{29}$$

using the polar substitution $f_1 = f_{\max} \sin\theta$ gives

$$= \int_{-\pi/2}^{\pi/2} 2(f_{\max}^2 - f_{\max}^2 \sin^2\theta)^{1/2} e^{\imath 2\pi f_{\max} n_1 \sin\theta} \cdot f_{\max} \cos\theta \ d\theta \tag{30}$$

$$= 2 f_{\max}^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta e^{\imath 2\pi f_{\max} n_1 \sin\theta} d\theta \tag{31}$$

$$= \frac{2\pi f_{\max} J_1(2\pi f_{\max} n_1)}{n_1} \tag{32}$$

where J_1 is the first-order Bessel function. Thus

$$\phi(n_1, n_2) = \frac{2\pi f_{\max} J_1(2\pi f_{\max}(n_1^2 + n_2^2)^{1/2})}{(n_1^2 + n_2^2)^{1/2}} \ . \tag{33}$$

Before proceeding further with the topic of 2-d filtering, we must define a 2-d DFT. The utility of the multidimensional DFT arises for the same reasons as for 1-d series; it enables us to deal with limited time series (with the added implication that our sampled signals are now periodic), and it is implementable with highly efficient FFT routines.

A periodic signal in two dimensions satisfies

$$x(n_1, n_2) = x(n_1 + m_1 N_1, n_2 + m_2 N_2) \tag{34}$$

where (N_1, N_2) are the periods of the 2-d signal (in samples) along the two grid axes and

$$(m_1, m_2) \text{ integers} \tag{35}$$

As in one dimension, such 2-d signals can be decomposed into a linear combination of a finite number of exponential basis functions which have periods which are submultiples of (N_1, N_2). Thus,

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) e^{\imath 2\pi n_1 k_1 / N_1} e^{\imath 2\pi n_2 k_2 / N_2} \tag{36}$$

where $X(k_1, k_2)$ is the 2-d DFT of $x(n_1, n_2)$. The corresponding DFT is therefore

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-\imath 2\pi n_1 k_1 / N_1} e^{-\imath 2\pi n_2 k_2 / N_2}. \tag{37}$$

Noe that we could also define a 2-d z transform

$$x(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{38}$$

and a corresponding inverse z transform (with contours c_1 and c_2)

$$x(n_1, n_2) = \frac{1}{(\imath 2\pi)^2} \int_{c_1} \int_{c_2} X(z_1, z_2) z_1^{n_1 - 1} z_2^{n_2 - 1} \, dz_1 \, dz_2 \ . \tag{39}$$

A general 2-d digital filter is thus characterizable by a difference equation

$$y(n_1, n_2) = \sum_{i=-p}^{p} \sum_{j=-q}^{q} \alpha_{ij} x(n_1 - i, n_2 - j) - \sum_{i=-r}^{r} \sum_{j=-s}^{s} \beta_{ij} y(n_1 - i, n_2 - j) \tag{40}$$

where i and j are not both zero in the second summation, and we have made the constant coefficients symmetric about $y(0, 0)$. This has a z transform given by

$$Y(z_1, z_2) = \frac{\sum_{i=-p}^{p} \sum_{j=-q}^{q} \beta_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=-r}^{r} \sum_{j=-s}^{s} \alpha_{ij} z_1^{-i} z_2^{-j}} \tag{41}$$

where $\alpha_{00} = 1$ has poles and zeros in a 4-dimensional space defined by the real and imaginary parts of z_1 and z_2. Evaluating stability for such filters is difficult,

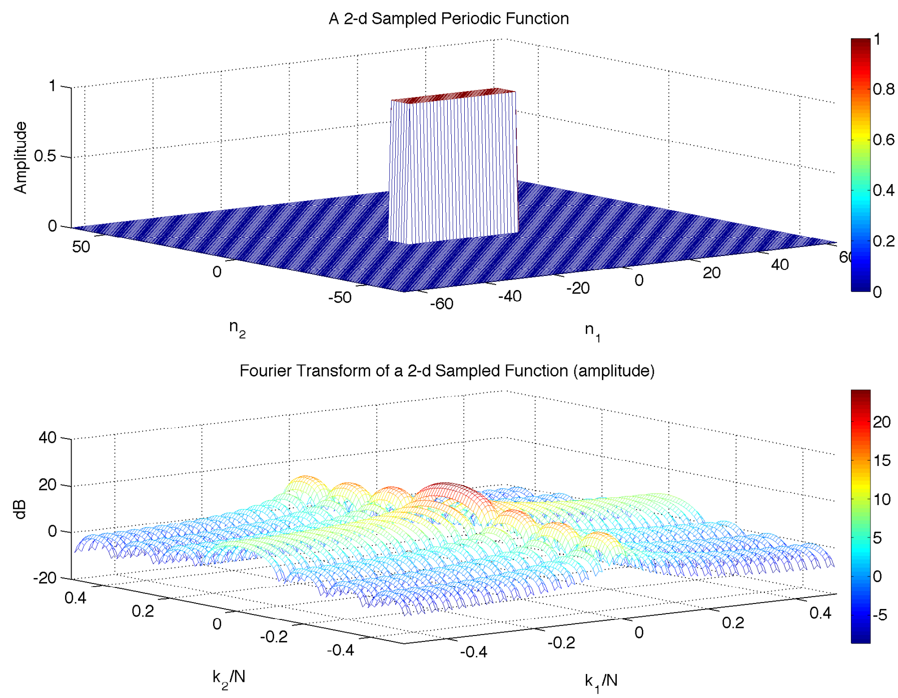


Figure 1: A 2-dimensional sampled function and its DFT

primarily because one cannot, in general, factor the 2-dimensional numerator and denominator to obtain a simple view of the zero and pole frequencies. As a result of this property (or non-property) of higher-dimensional polynomials, the cascade of two stable IIR filters may not even be stable! (this issue is still a current research topic). Because of these difficulties, we will primarily concern ourselves with FIR higher-dimensional filters here (this is the case where there are no poles and thus no potential stability problems).

The two dimensional function shown in Figure (1) could be applied as an FIR filter to effect low-pass filtering by the use of the convolution theorem (direct manipulation of the DFT) or via convolution with a corresponding kernel in two dimensions. However, this filter is anisotropic in the (k_1, k_2) plane, in the sense the wavenumber components along the diagonals will experience different filtering than along the k_1 or k_2 directions, and the filtering in k_1 and k_2 directions has different cutoff wavenumbers. Consider, instead, a circularly symmetric, low-pass filter case defined by the ideal response (Figure 2), which has the transfer function

$$\Phi(f_1, f_2) = \begin{cases} 1 \; f_1^2 + f_2^2 \leq 1/4 \\ 0 \text{ otherwise} \end{cases} \tag{42}$$

from (33), we know that the corresponding filter weights are given by

$$w(n_1, n_2) = \begin{cases} \pi f_{\max} J_1((\pi/2)(n_1^2 + n_2^2)^{1/2}) \\ 2(n_1^2 + n_2^2)^{1/2} \end{cases} \tag{43}$$

Taking a N by N -point rectangular window (a simple truncation of the 2-d series) produces a filter with a frequency response

$$W(f_1, f_2) = \sum_{n_1=-(N-1)/2}^{(N-1)/2} \sum_{n_2=-(N-1)/2}^{(N-1)/2} w(n_1, n_2) e^{-\imath 2\pi n_1 f_1} e^{-\imath 2\pi n_2 f_2} \tag{44}$$

where (f_1, f_2) is normalized to the Nyquist interval, so that both frequencies span $(-1/2, 1/2)$.

As in the 1-d case, we can improve the ripple features of the filtering by applying a windowing function with better spectral leakage characteristics than the 2-dimensional rectangular window implied by simply convolving with a truncated $w(n_1, n_2)$. As we usually wish our window to be circularly symmetric in the (f_1, f_2) and (n_1, n_2) planes, we can take a window function, $\hat{w}$, from 1-dimensional analysis and substitute the radius in (n_1, n_2)-space for n to obtain a circularly symmetric 2-dimensional window

$$w(n_1, n_2) = \hat{w}(n_1^2 + n_2^2)^{1/2} \; . \tag{45}$$

As in 1-d processing, the Kaiser-Bessel window is a good candidate for a windowing function due to its low spectral leakage. An N by N, 2-d Kaiser-Bessel window is

$$w(n_1, n_2) = \frac{I_0\left[2\pi\sqrt{1 - (n_1^2 + n_2^2)/N^2)}\right]}{I_0(2\pi)} \tag{46}$$

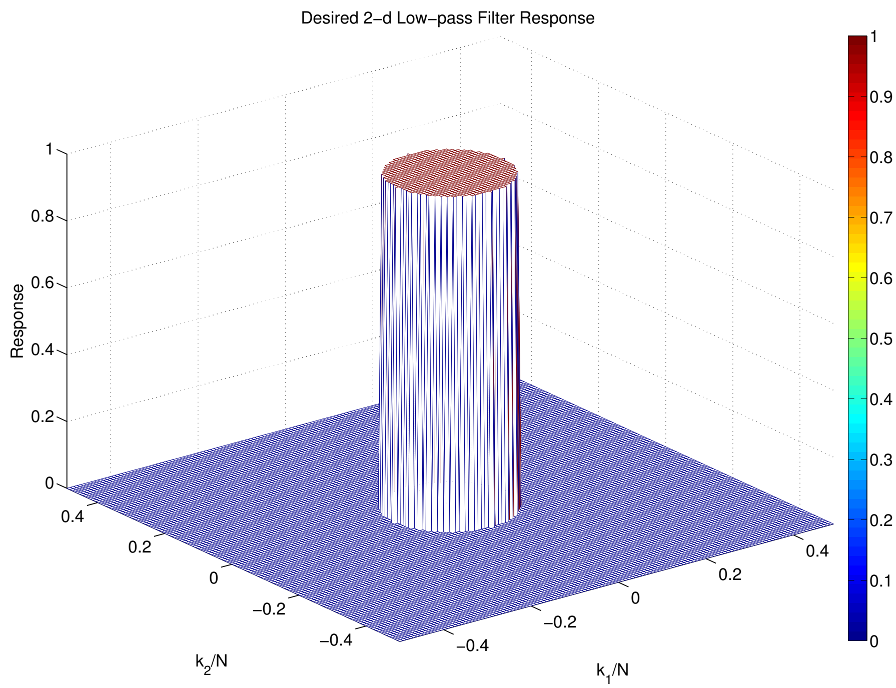


Figure 2: A 2-dimensional ideal lowpass filter response.

for $n_1^2 + n_2^2 \leq N^2$ and

$$w(n_1, n_2) = 0 \tag{47}$$

for $n_1^2 + n_2^2 > N^2$, where $I_0(x)$ is the modified Bessel function of the first kind and 0^{th} order. The response of the Kaiser-Bessel windowed low pass filter is superior in smoothness and in attenuation (reduction of spectral leakage) to the rectangular window, as shown in the plots on the following pages.

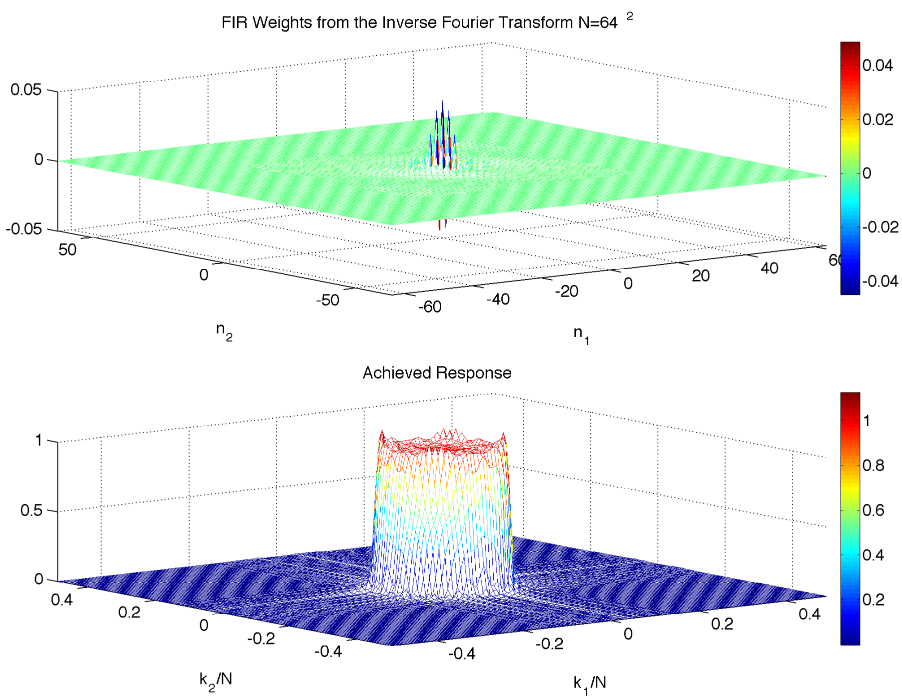


Figure 3: A 64 by 64 truncated FIR realization of the ideal low pass filter response.

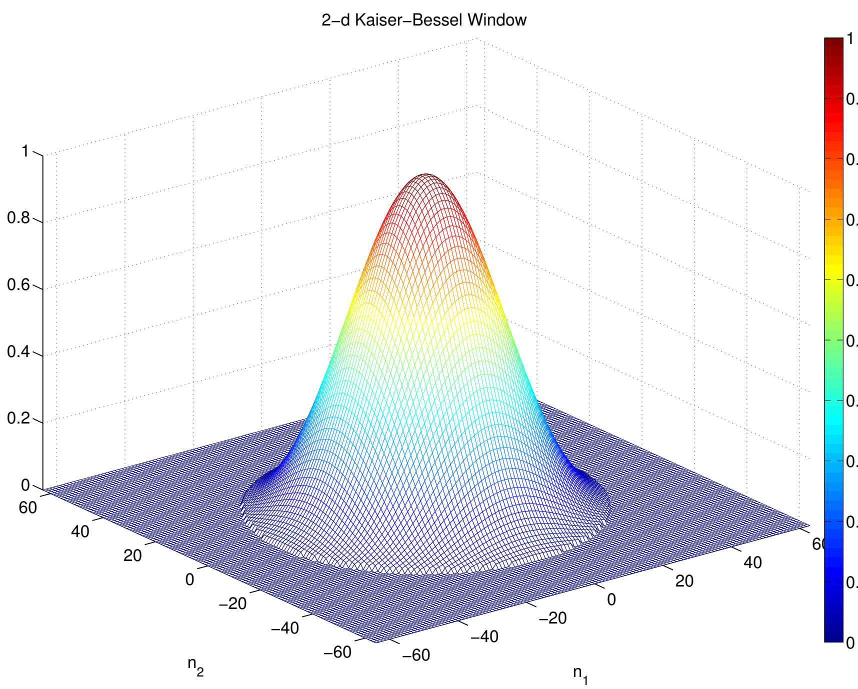
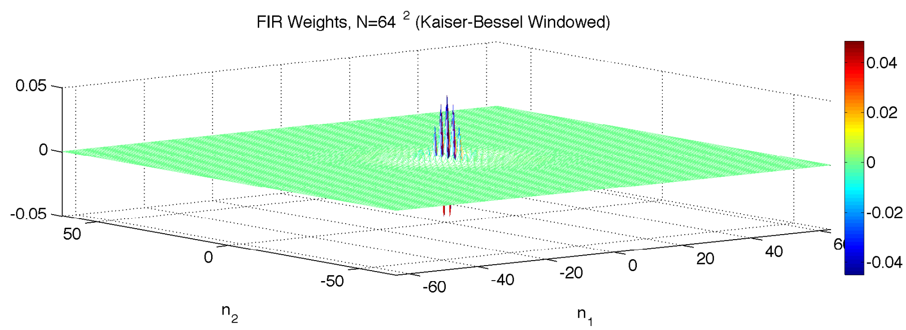


Figure 4: A Kaiser Bessel window in 2 dimensions.



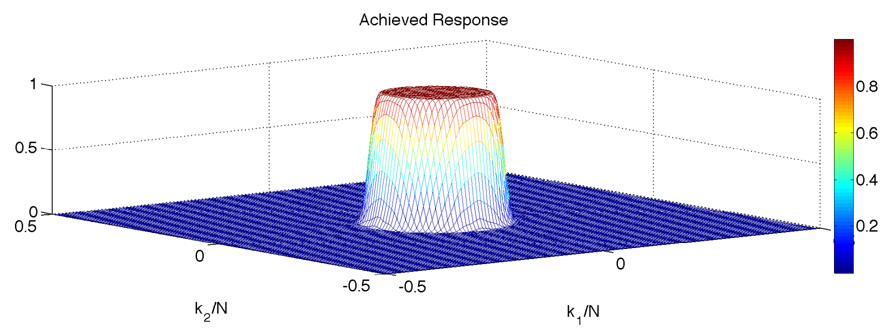


Figure 5: A 64 by 64 Kaiser Bessel-windowed FIR realization of the ideal low pass filter response.

Frequency-Wavenumber Filtering

We next consider some aspects of filtering in a two-dimensional system where the two-dimensions do not have the same units. Consider a linear array of seismometers or antennae deployed in the $\hat{x}$ direction with a constant spacing. Signals from such an array can be displayed in a 2-dimensional *record section*, where we have t as the ordinate and channel number, or x, as the abscissa (or vice-versa). The response of such a system to a traveling, sinusoidal plane wave of frequency f_0

$$\phi(t,x) = e^{\imath 2\pi f_0 (t - x/v_0)} \tag{48}$$

where v_0 is the *apparent phase velocity* of the wave across the array, is of particular interest, as such signals impinge upon the array at specific angles given by

$$\theta = \sin^{-1}(c/v_0) \tag{49}$$

where c is the true wave velocity in the medium and θ is the angle between the planar wavefront and the $\hat{x}$ direction. Thus, when $\theta = 0$, the apparent phase velocity $v_0 = \infty$, as the wavefront strikes all of the sensors simultaneously. Conversely, when $\theta = 90$, $v_0 = c$, as the plane wave is propagating directly along the array axis (in the $\hat{x}$ direction).

If we arrange the data in (t, x)-space to form a 2-dimensional array (practically speaking, we may have to resample the traces to form an evenly-spaced array in the sampled case), we can take a 2-d Fourier transform of (48) as

$$\Phi(f,k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x) e^{-\imath 2\pi f t} e^{\imath 2\pi x f/v}\, dt\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x) e^{-\imath 2\pi f t} e^{\imath 2\pi k x}\, dt\, dx \tag{50}$$

where the *wavenumber* (or *spatial frequency*) is, here, defined as the reciprocal length

$$k = 1/\lambda = f/v \; . \tag{51}$$

The $f-k$ transform of the plane wave evaluated using (48)) is thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\imath 2\pi f_0 t} e^{-\imath 2\pi x k_0} e^{-\imath 2\pi f t} e^{\imath 2\pi x k}\; dt\; dx = \delta(f - f_0, k - k_0) \tag{52}$$

so that every traveling sinusoidal wave of a given frequency and wavenumber in (x,t)-space maps to a delta function in (f,k)-space!

Note that we have chosen a mixed exponential sign convention for the $f - k$ transform, where the frequency portion has a minus sign in the exponent, consistent with our previous convention for 1- and 2-dimensional transforms, but the wavenumber transform exponent has a plus sign. We do this so that waves propagating towards increasing x for increasing t (like 48) will map into the first quadrant of the $f-k$ plane. Of course there are three other conventions of exponent signs which could be chosen here.

In $f-k$ space, arbitrary signals of a given apparent phase velocity, v_0 are specified by (51), so that such signals lie along lines which intersect the $f-k$

origin and have slopes of v_0 in an f vs. k presentation. Now suppose that we wish to selectively resolve waves within a range of apparent velocities. This procedure is called *beam forming*, as it was first developed in radar and radio transmission applications. In seismological applications, because of Snell's law, the horizontal phase velocity of a signal remains constant throughout a given ray path in a horizontally homogeneous medium. Thus, beam forming using seismic array data selectively examines waves which turn within a particular depth range (as our array is generally deployed horizontally). For a simple 1-d array of sensors we can preferentially extract signals with a specific phase velocity above some cutoff value, v_0, by using a filter with an $f-k$ response given by

$$Y(f,k) = \begin{cases} 1 & -|f|/v_0 \le k \le |f|/v_0 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

It's instructive to examine the impulse response of (e.g., Kanasewich, 1975), given by the inverse $f-k$ transform

$$y(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(f,k) e^{\imath 2\pi f t} e^{-\imath 2\pi k x}\, dk\, df \ . \tag{54}$$

Of course, in practical situations, x and t are both discrete variables, so that, for unit time sampling interval, $\Delta t = 1$ and unit spatial sampling interval, $\Delta x = 1$

$$y(n\Delta t, (m+1/2)\Delta x) = y(n, m+1/2) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} Y(f,k) e^{\imath 2\pi f n} e^{-\imath 2\pi k(m+1/2)}\, dk\, df \tag{55}$$

where we have assumed that there are an even number of receivers in the array, so that the half-integer spatial index, $m+1/2$ gives a symmetric deployment relative to the x origin. Evaluating the integral over k for $Y(f,k)$ gives

$$\int_{-1/2}^{1/2} e^{\imath 2\pi f n} \left(\frac{e^{-\imath 2\pi k(m+1/2)}}{-\imath 2\pi(m+1/2)} \right)\Bigg|_{k=-|f|/v_0}^{|f|/v_0} df \tag{56}$$

$$= \frac{1}{\pi(m+1/2)} \int_{-1/2}^{1/2} e^{\imath 2\pi f n} \sin(2\pi(m+1/2)|f|/v_0)\, df \tag{57}$$

$$= \frac{2}{\pi(m+1/2)} \int_{0}^{1/2} \cos 2\pi f n \sin(2\pi(m+1/2) f/v_0)\, df \tag{58}$$

using unity apparent velocity as the cutoff value for the sake of illustration gives

$$v_0 = \Delta x/\Delta t = 1 \tag{59}$$

so that

$$y(n, m+1/2) = \frac{2}{\pi(m+1/2)} \int_{0}^{1/2} \sin(2\pi f(m+1/2)) \cos(2\pi f n)\, df \ . \tag{60}$$

Because, for $m^2 \neq n^2$,

$$\int \sin(mx)\cos(nx)dx = \frac{-\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \tag{61}$$

we have

$$y(n, m+1/2) = \frac{2}{\pi(m+1/2)} \left(\frac{-\cos(2\pi f(n+m+1/2))}{4\pi(n+m+1/2)} - \frac{\cos(2\pi f(-n+m+1/2))}{4\pi(-n+m+1/2)} \right) \Bigg|_0^{1/2} \tag{62}$$

$$= \frac{2}{\pi(m+1/2)} \times \tag{63}$$

$$\left(\frac{-\cos(\pi(n+m+1/2))}{4\pi(n+m+1/2)} - \frac{\cos(\pi(-n+m+1/2))}{4\pi(-n+m+1/2)} + \frac{1}{4\pi(n+m+1/2)} + \frac{1}{4\pi(-n+m+1/2)} \right) . \tag{64}$$

As m and n are integers, the cosine terms are zero, so that

$$y(n, m+1/2) = \frac{1}{2\pi^2(m+1/2)} \left(\frac{1}{(n+m+1/2)} + \frac{1}{(-n+m+1/2)} \right) \tag{65}$$

or

$$y(n, m+1/2) = \frac{1}{\pi^2 \left[(m+1/2)^2 - n^2\right]} . \tag{66}$$

As is usual in FIR filter design problems, the weights are nonzero for large indices (n and m) and we are forced into a truncation procedure to produce a finite set of filter weights. As in our previous examples, the Kaiser Bessel window provides a good choice for truncating the 2-d weights. Rectangular and Kaiser-Bessel windowed realizations of the velocity filter (66) for 64 channels of 64 sample data are shown on the following page.

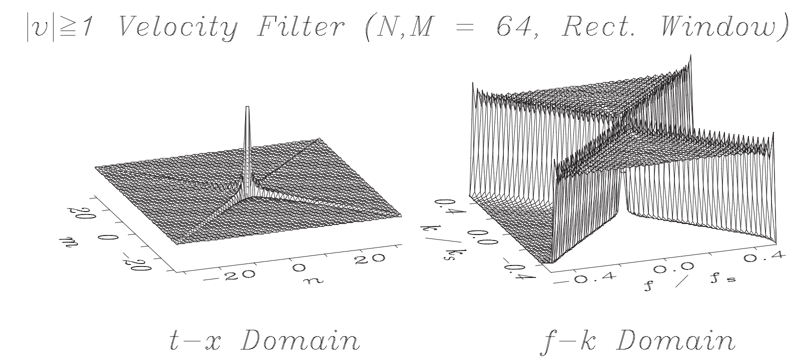


Figure 6: A rectangular-widowed velocity filter.

As the 3-d perspective plots make it difficult to see the $x-t$ domain impulse response, we also show a plot of the impulse response traces for 16 traces of 64 samples. Each time series in the impulse response consists of a simple convolving kernel. The response of the filter, $r(n, m+1/2)$ to an arbitrary input, $\phi(n, m+1/2)$, is thus given by the 2-d convolution of (66) with the input traces

$$r(n, m+1/2) = \sum_{i=1}^{N} \sum_{j=-M/2}^{M/2-1} \phi(i, j+1/2) y(n-i, m+1/2-j) \tag{67}$$

$r(n, 1/2)$ is thus obtainable by convolving each time series in the input with the corresponding time series in the impulse response (66), followed by a summation (stack) of the resultant M convolutions all m.

A particularly simple $f-k$ filter has weights given by

$$y(n, m) = \delta(n = 0) \tag{68}$$

The $m = 0$ output of such a filter is just a zero-lag stack of the input traces. The $f-k$ impulse response of such a system is just the Discrete Fourier transform of $y(n, m)$

$$Y(\nu, \mu) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \delta(n = 0) e^{-\imath 2\pi \nu n/N} e^{\imath 2\pi \mu m/M} \tag{69}$$

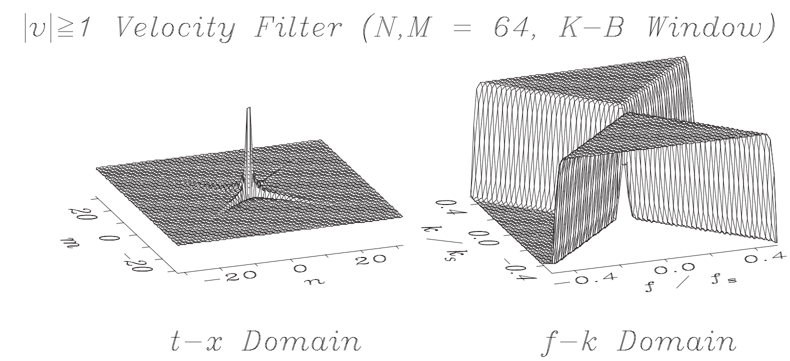


Figure 7: A Kaiser-Bessel-windowed velocity filter.

where our frequency-wavenumber indices are the integers (ν, μ).

$$= \sum_{m=0}^{M-1} e^{\imath 2\pi\mu m/M} = \frac{1 - e^{\imath 2\pi\mu}}{1 - e^{\imath 2\pi\mu/M}} \tag{70}$$

which has the amplitude response given by the Dirichlet kernel

$$|Y(\nu, \mu)| = \frac{\sin(\pi\mu)}{\sin(\pi\mu/M)} \tag{71}$$

which is independent of the Nyquist-normalized frequency, ν. The $t - x$ and $f - k$ plots are shown on the following page. The zero-lag stack, then, acts like a low pass filter in k and a high pass filter in v, so that waves with large k (short wavelengths) and low v (less vertical ray paths) will be attenuated, while those with small k (long wavelengths) will be relatively unaffected.

Consider now what happens if we stack the time series with some time lag, Δ, imposed between the channels, so that the impulse response is now

$$y(n, m) = \delta(n + \Delta m). \tag{72}$$

Such a system is called a *phased array* and has many applications in geophysics, optics, and electromagnetics (e.g., RADAR). The $f - k$ response then becomes

$$Y(\nu, \mu) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \delta(n + \Delta m) e^{-\imath 2\pi\nu n/N} e^{\imath 2\pi\mu m/n} \tag{73}$$

$$= \sum_{m=0}^{M-1} e^{\imath 2\pi\nu\Delta m/N} e^{\imath 2\pi\mu m/M} \tag{74}$$

for the symmetric case $N = M$, we have

$$= \sum_{m=0}^{M-1} e^{\imath 2\pi m(\nu\Delta + \mu)/N} \tag{75}$$

which gives the amplitude response

$$|Y(\nu, \mu)| = \frac{\sin(\pi(\nu\Delta + \mu))}{\sin(\pi(\nu\Delta + \mu)/M)} \tag{76}$$

which is shown on the following page for $\Delta = 1$, along with the response of the unlagged series. Rotating the impulse response in the $t - x$ domain has thus simply rotated the Fourier Transform by the same angle (in this case, 45°). We now know how to modify the velocity filter to enclose some other hourglass-shaped swath of the $f - k$ plane – we simply must impose a linear lag between the initial time traces to rotate the response function to the desired angle.

An important application of phased arrays is to receive or transmit narrow frequency band energy preferentially from a small range of azimuths. Consider a linear hydrophone array trailed from a ship with an array element spacing of $\Delta x = 30$ m and a length of 3600 m ($M = 121$ elements in all). If such an array is receiving energy from a narrow-band source (so that we are only interested in a small range of frequencies), we can calculate the width of the main lobe of the Dirichlet kernel response if we know the sound speed (about 1500 m/s in water).

For a $f_1 = 50$ Hz source, the wavelength is thus about 30 m. The $f - k$ response of the streamer for stacked traces is

$$|Y(\nu, \mu)| = \frac{\sin(\pi\mu)}{\sin(\pi\mu/M)} \tag{77}$$

where we can convert a general discrete $f-k$ transform to a function of Nyquist-normalized wavenumber, k, and Nyquist-normalized frequency, f, using the transformations

$$\mu = Mk/k_s \tag{78}$$

$$\nu = Nf/f_s \tag{79}$$

where k_s is the spatial sampling frequency

$$k_s = 1/(\Delta x) = 1/30 \text{ m}^{-1} \tag{80}$$

and f_s is the time sampling frequency to obtain

$$|Y(f, k)| = \frac{\sin(M\pi k/k_s)}{\sin(\pi k/k_s)} \ . \tag{81}$$

The first zero of this function occurs at $k = k_1$, defined by

$$\sin(M\pi k_1/k_s) = 0 \ (k_1 \neq 0) \tag{82}$$

or where

$$k_1 = k_s/M \approx 2.75 \times 10^{-4} \text{ m}^{-1} \tag{83}$$

which occurs at a plane wave emergence angle of

$$\theta = \sin^{-1}(c/v_1) = \sin^{-1}(ck_1/f_1) = \sin^{-1}(1500 \cdot 2.75 \times 10^{-4}/50) \approx 0.47^\circ \tag{84}$$

(corresponding to a phase lag of 2π between the first and last hydrophones) so that the total width of the main lobe is $\pm\theta$, or about 1°. The second major maximum occurs when the contributions of the plane wave are again in phase at all of the receivers, where $k = k_s$ and

$$\theta = \sin^{-1}(ck_s/f_1) = \sin^{-1}(1500/(30 \cdot 50)) = 90^\circ. \tag{85}$$

If frequency is doubled to $f_2 = 100$ Hz, then the wavelength is halved, and the main lobe becomes narrower, with the first zero now occurring at

$$\theta = \sin^{-1}(ck_1/f_2) = \sin^{-1}(1500 \cdot 2.75 \times 10^{-4}/100) \approx 0.24^\circ \ . \tag{86}$$

The second major maximum now occurs at only

$$\theta = \sin^{-1}(ck_s/f_2) = \sin^{-1}(1500 \cdot 2/(60 \cdot 100)) = 30^\circ. \tag{87}$$

so that the main beam has become narrower, but we now have a second maximum to contend with at 30° from normal incidence.

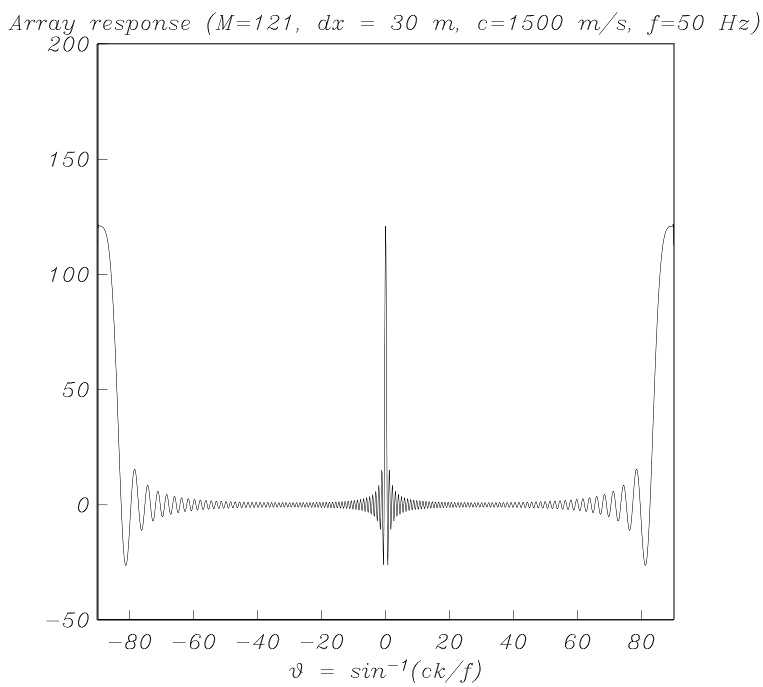


Figure 8: A linear array response as a function of incident angle.

Frequency-Wavenumber Filtering with 2-dimensional arrays

Next, we consider data from a 2-dimensional array of instruments. Again, we can decompose incident energy into a superposition of traveling waves, but we now have an additional spatial dimension to contend with because our signals now have two spatial dimensions.

A particular wave field sampled by a two-dimensional array can be decomposed into plane waves

$$\phi(t,x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(f,k_x,k_y)e^{\imath 2\pi f t}e^{-\imath 2\pi k_x x}e^{-\imath 2\pi k_y y}\; df\; dk_x\; dk_y \quad (88)$$

where k_x and k_y are the wavenumbers in the x and y directions and $\Phi(k_x, k_y, f)$ is a 3-dimensional frequency-wavenumber spectrum. A particular plane wave propagates at an azimuth, ϕ, specified by

$$\phi = \tan^{-1}(k_y/k_x) \quad (89)$$

k_x and k_y are thus not independent, but are related by the Pythagorean theorem

$$k_x^2 + k_y^2 = f^2/v^2. \quad (90)$$

The $f - k$ spectrum of a 2-dimensional time signal is thus

$$\Phi(f,k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x,y)e^{-\imath 2\pi f t}e^{\imath 2\pi k_x x}e^{\imath 2\pi k_y y}dtdxdy \quad (91)$$

and its discrete counterpart is

$$\Phi(\nu,\mu_x,\mu_y) = \sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} \phi(l,n,m)e^{-\imath 2\pi n\nu/N}e^{\imath 2\pi l\mu_x/L}e^{\imath 2\pi m\mu_y/M}\;. \quad (92)$$

As in the case of an ideal 1-dimensional array, we can (theoretically at least) calculate a frequency-wavenumber spectrum from real data using (92) to determine the nature of the incident energy in terms of a plane wave decomposition. Unfortunately, this is not usually the case in seismology, particularly at high frequencies, as spatial heterogeneity induces scattering which fragments the wavefront near the array, reducing the signal coherence from sensor to sensor. One can improve the situation somewhat by introducing station corrections (e.g., Aki and Richards, *Theoretical Seismology,* p. 610), so that the wavefront is best reconstructed (this procedure is analogous to the adaptive optical techniques used in modern large telescopes).

Upward and Downward Continuation of Remotely Sensed Data

Consider a point mass, m, located at the origin, which produces a gravitational field

$$\vec{g}(\hat{r}) = \frac{-mG\hat{r}}{r^2} = \frac{-mG(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2+y^2+z^2)^{3/2}} \tag{93}$$

where G is Newton's gravitational constant. At a general position (x, y, z), the vertical (z component) of the gravity field will thus be

$$g_z = \hat{z} \cdot \vec{g} = \frac{-mGz}{(x^2+y^2+z^2)^{3/2}} \tag{94}$$

The integral of g_z over the xy plane is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_z \; dx \; dy = -mGz \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2+z^2)^{-3/2} \; dx \; dy \tag{95}$$

$$= -2\pi mGz \int_0^{\infty} \frac{r \; dr}{(z^2+r^2)^{3/2}} \tag{96}$$

$$= -2\pi mGz \left(\frac{-1}{(z^2+r^2)^{1/2}} \right) \Bigg|_0^{\infty} = -2\pi mG \tag{97}$$

which, interestingly, does not depend on z. If we take the output of our system to be the vertical field at $z = 0$, then we clearly have a delta function at the origin with a magnitude given by (94), as the field has no vertical component except exactly at the origin. Next consider a surface at a height h above the xy plane. The vertical field there is just

$$g_z(h) = -\frac{h}{2\pi(x^2+y^2+h^2)^{3/2}} = -\frac{h}{2\pi(r^2+h^2)^{3/2}} \tag{98}$$

where we have normalized the response by (94). As field quantities obey superposition and linearity, vertical field measurements of a general field obtained at an arbitrary height $z = h$ are thus specified by the 2-dimensional convolution of (98) with the field at $z = 0$.

We can examine the frequency response of this filter by taking the Fourier transform of (98)

$$g(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{h}{2\pi(x^2+y^2+h^2)^{3/2}} e^{-\imath 2\pi k_x x} e^{-\imath 2\pi k_y y} \; dx \; dy \tag{99}$$

which can be solved to obtain

$$g(k_x, k_y) = e^{-2\pi h (k_x^2 + k_y^2)^{1/2}} \tag{100}$$

which is the frequency response of the upward continuation filter. Note that (100) is thus a low pass filter – as we move away from the ($z = 0$) plane, we loose the high frequencies in our survey. Conversely, if we wish to extrapolate downwards to the earth's surface, we need to implement the (unstable) inverse filter, $g^{-1}(k_x, k_y)$. This 2-dimensional deconvolution can be achieved in a stable way by a regularized (e.g., 2-d water level) deconvolution in the frequency domain.

Multi-dimensional filtering in MATLAB

Basic filtering operations can be done with the functions *filter2* and *conv2*. There are also 2-dimensional DFT operations (*fft2* and *ifft2*), as well as a routine (*fftn* and *ifftn*) for arbitrary dimensionality. 2-dimensional FIR filter design programs are also available using the windowing and frequency sampling methods (*fwind1/fwind2*, *fsamp2*, respectively) in the image processing toolbox. This toolbox also has two-dimensional functions (*fspecial*) and many, many other useful functions for operating on 2-dimensional arrays.