Data Processing and Analysis

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Introduction to Linear Systems, Part 2: The Frequency Domain

In Chapter 1, we examined signals in linear systems using time as the independent variable. We now address the fundamentals of Fourier theory, where the independent variable is frequency. The basic insight that leads to Fourier Theory is that linear systems, being subject to superposition and scaling, can be analyzed in terms of their *frequency response*, that is, in terms of their response to pure sinusoidal or exponential inputs.

Consider the response, $g(t)$ of a linear system with impulse response $\phi(t)$ to a unit-amplitude, complex input of frequency f, $e^{\imath 2\pi f t}$. The time domain response of any such system is given by the convolution of the input function and the impulse response

$$g(t) = \int_{-\infty}^{\infty} \phi(\tau) e^{\imath 2\pi f (t-\tau)} \, d\tau \ . \tag{1}$$

Because a time shift in the argument of an exponential is mathematically equivalent to multiplication by another exponential

$$g(t) = e^{\imath 2\pi f t} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau} \, d\tau \ = \ e^{\imath 2\pi f t} \cdot \Phi(f) \ . \tag{2}$$

(2) shows that the response of any linear system to a complex sinusoidal input is unchanged in functional form (a complex sinusoidal signal of the *same frequency*) and is only modified in amplitude and phase (by the complex factor $\Phi(f)$). The frequency, f, in (2) is arbitrary. Thus, if an arbitrary input $\psi(t)$ is decomposed into a sum of sinusoidal components, then, because of superposition, the relationship between $\psi(t)$ and $g(t) = \psi(t) * \phi(t)$ can be completely characterized by $\Phi(f)$, the *transfer function* of the system. $\Phi(f)$ is the *Fourier transform* (or *spectrum*) of the impulse response of the system, $\phi(t)$.

There are several conventions that are variously used in defining the Fourier transform. The definitions that we will use are those most commonly encountered in geophysics

$$\Phi(f) = \mathcal{F}[\phi(t)] = \int_{-\infty}^{\infty} \phi(t) e^{-\imath 2\pi f t} dt \tag{3}$$

$$\phi(t) = \mathcal{F}^{-1}[\Phi(f)] = \int_{-\infty}^{\infty} \Phi(f) e^{\imath 2\pi f t} df \tag{4}$$

where F denotes the Fourier transform operation, and $\mathcal{F}^{-1}$ denotes the *inverse Fourier transform* operation.

Be aware that in some other areas of physics and in exploration geophysics the sign convention on the complex exponentials of (5) and (6) is reversed, so that the forward transform has a plus sign in the exponent and the inverse transform has a minus sign in the exponent. This will of course not affect any fundamentals of the analysis, only the convention by which phase is measured. An alternative common formulation uses the radian frequency $\omega = 2\pi f$ rather than f to characterize the frequency. This introduces a factor of 2π into the scaling of the transform pair (as can be seen by a simple change of variables applied to the above pair) to produce

$$\Phi(\omega) = \mathcal{F}[\phi(t)] = \int_{-\infty}^{\infty} \phi(t) e^{-\imath\omega t} dt \tag{5}$$

$$\phi(t) = \frac{1}{2\pi}\mathcal{F}^{-1}[\Phi(\omega)] = \int_{-\infty}^{\infty} \Phi(\omega) e^{\imath\omega t} d\omega \ . \tag{6}$$

We'll occasionally employ the above transform pair here in cases when it saves us from having to write a large number of 2π factors in our expressions.

Differential equations and Fourier theory. A particularly tractable and not uncommon situation in the physical sciences occurs when a system relating two time functions, $x(t)$ and $y(t)$, is characterizable by a linear homogeneous differential equation with constant coefficients. For functions of a single variable, t, the general form of such a differential equation is

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \ . \tag{7}$$

As none of the coefficients (the a_i and b_i) depend on t, (7) describes a time-invariant system. Because all of the terms are linear (there are no powers or other nonlinear functions of x, y, or their derivatives), it is also a linear system, obeying superposition and scaling (note that differentiation itself is a linear operation). To obtain an expression for the transfer function corresponding to (7), substitute an exponential unit amplitude exponential of arbitrary frequency for the input, $x(t)$, and output, $y(t)$, so that

$$x(t) = e^{\imath 2\pi f t} \tag{8}$$

and, as must be the case for any linear, time-invariant system (2),

$$y(t) = \Phi(f) e^{\imath 2\pi f t} \ . \tag{9}$$

Substituting (8) and (9) into (7), dividing both sides by $e^{\imath 2\pi f t}$, and solving for $\Phi(f)$ produces the system transfer function, which is a ratio of two complex polynomials in f.

$$\Phi(f) = \frac{\sum_{j=0}^{m} b_j (2\pi\imath f)^j}{\sum_{k=0}^{n} a_k (2\pi\imath f)^k} \tag{10}$$

The values of f where the numerator is zero are referred to as *zeros* of $\Phi(f)$, as the response is zero at this frequency, regardless of the amplitude of the input signal. Conversely, frequencies for which the denominator is zero are called *poles*, as the response becomes very large at these frequencies. Note that we don't have to worry too much about any mysteries regarding $e^{\imath 2\pi ft}$ being a complex number, as

$$e^{\imath 2\pi ft} = \cos(2\pi ft) + \imath \sin(2\pi ft) \tag{11}$$

and we could almost have just as easily chosen to propagate the real or the imaginary part of the input signal alone through the system to reach an equivalent conclusion; in this case an input (cosine, sine) signal simply produces a scaled output (cosine, sin) with a phase shift. Note that frequencies for which we have zero or infinite response may be imaginary or complex, in which case the corresponding input function, (8) may be an increasing or decreasing exponential, or an increasing or decreasing exponentially damped sinusoid, respectively.

Example: Response of a seismometer. As an important example of such a linear system from geophysical instrumentation, consider (Figure 1) a damped vertical harmonic oscillator with a rigid case that is fixed to the Earth. A mass, M, is supported by a spring, in parallel with a damping or *dashpot* component that produces Newtonian damping (i.e., a retarding force that is proportional to velocity). Intuitively, it you may see that the motion of the mass relative to the Earth will provide some sort of representation of the true vertical ground motion. For example, if the mass were completely decoupled, so that it remained stationary in its inertial reference frame while the Earth moved, then the motion of the mass relative to its case (which is, recall, rigidly attached to the Earth) would be exactly the negative of the ground motion).

The differential equation of motion for the mass in such a *seismometer* can be derived using Newton's second law by equating the (upward) forces of the spring and damper acting on the mass with the (upward) acceleration times the mass, i.e.,

$$F_{up} = Ma_{up} \tag{12}$$

or

$$-D\frac{d\xi(t)}{dt} + K[\xi_0 - \xi(t)] \ = M\frac{d^2}{dt^2}[\xi(t) + u(t)] \tag{13}$$

which gives rise to the homogeneous differential equation

$$M\frac{d^2}{dt^2}[\xi(t) + u(t)] + D\frac{d\xi(t)}{dt} + K[\xi(t) - \xi_0] = 0 \ . \tag{14}$$

Here, u is the motion of the Earth (up positive), ξ is the position of the mass, which has an equilibrium position in the Earth's gravity field of ξ_0 (both measured up positive relative to the surface of the Earth), M is the mass of the inertial component, D is the dashpot constant (units of force per velocity), and K is the spring constant (units of force per distance).

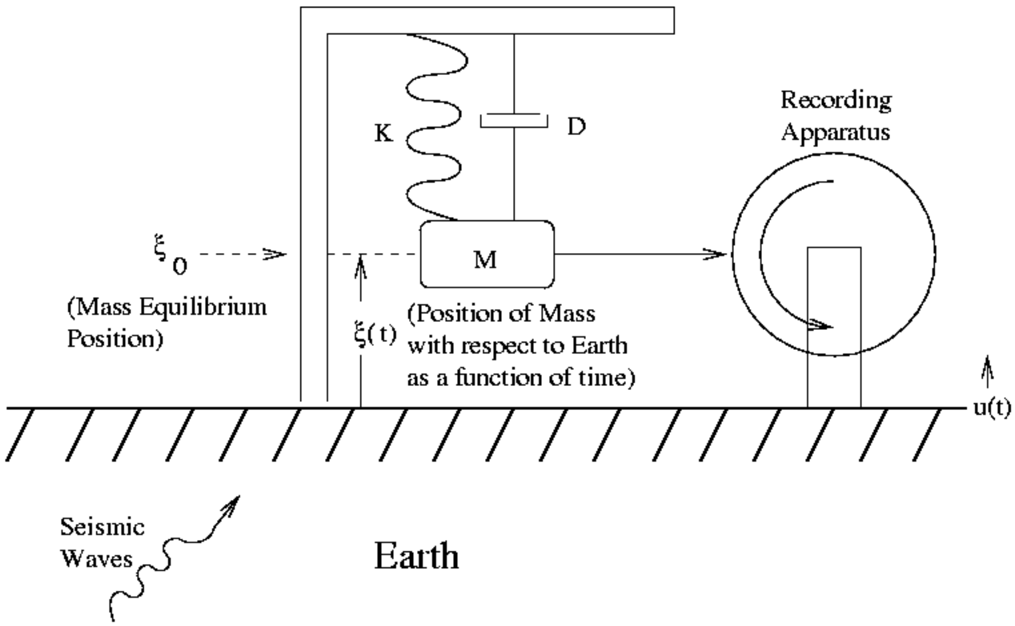

Figure 1: A Mechanical Seismometer

We can simplify (14) somewhat by writing the equation of motion for the mass in an upward positive coordinate system (z) where $z = 0$ is the equilibrium position in the Earth's gravitational field, so that $z(t) = \xi(t) - \xi_0$. This gives

$$\ddot{z} + 2\zeta\dot{z} + \omega_s^2 z = -\ddot{u} \tag{15}$$

where the *damping coefficient* is

$$2\zeta \;=\; D/M \tag{16}$$

and

$$\omega_s \;=\; (K/M)^{1/2} \tag{17}$$

is the angular undamped or *natural* frequency of the system. (15) is a linear homogeneous equation where the input, u, is the displacement of the Earth, and the output, z, is the deviation of the mass from its equilibrium position, relative to the seismometer frame.

Using (10), we now write the transfer function of the seismometer system (seismometer displacement response to a displacement of the Earth)

$$\Phi(\omega) = \frac{z(\omega)}{u(\omega)} = \frac{-(\imath\omega)^2}{(\imath\omega)^2 + 2\zeta(\imath\omega) + \omega_s^2} = \frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{18}$$

or, in terms of amplitude and phase

$$|\Phi(\omega)| = \frac{\omega^2}{[(\omega^2 - \omega_s^2)^2 + 4\zeta^2\omega^2]^{1/2}} \tag{19}$$

$$\theta = \arg[\Phi(\omega)] = \pi - \tan^{-1}\frac{-2\zeta\omega}{\omega^2 - \omega_s^2} \ . \tag{20}$$

At high frequencies ($\omega \gg \omega_s$), $|\Phi(\omega)| \approx 1$, and $\theta \approx \pi$, so the seismometer displacement from equilibrium is the negative of the Earth displacement, $z \approx -u$. In this case, the Earth moves so rapidly that the mass cannot follow the motion at all, and the position of the mass relative to the frame is thus just $-u$.

At low frequencies ($\omega \ll \omega_s$), $|\phi(\omega)| \approx \omega^2/\omega_s^2$, so that response amplitude falls off quadratically with decreased frequency. From the time domain representation (15), we see that this response is proportional to the negative of the Earth's acceleration, $z \propto -\ddot{u}$.

The mechanical seismometer, in displacement, thus acts like a displacement sensor at high frequencies and as an accelerometer at low frequencies. Around $\omega = \omega_s$, the system undergoes a transition between these two end-member behaviors. One can already see why very low frequency natural frequencies are desirable for seismometers; if ω_s is very small, the true displacement of the Earth is recoverable directly from the instrument response.

The frequency response for displacement input and displacement output [(19) and (20)] is plotted in Figure 2 for various damping factors, where the complex response is plotted in terms of its amplitude and phase.

In examining Figure 2, first consider the amplitude response when the damping, ζ, is small relative to ω_s In this case the system exhibits a large amplitude response for input frequencies near ω_s. This occurs because the system is excited near its natural resonant frequency and there is little energy loss via the dashpot. When ζ becomes larger than ω_s, the resonance peak in the amplitude response disappears, and the system no longer oscillates freely.

Next consider the phase response. At the undamped resonance period, the phase is $-90°$, implying that the output is phase-shifted by that amount (by -$\pi/2$ radians) relative to the input. A cosine Earth motion of frequency ω_s would be phase shifted into a sine mass displacement. Regardless of damping, the phase shift approaches zero at low frequencies and approaches π (a factor of -1) at high frequencies.

Purely mechanical seismometers such as that described above were among the first such instruments used to record accurate ground motion from earthquakes or other sources (they were first widely deployed starting in the 1890's). In most modern seismometers mass motion is sensed as a voltage which is proportional to the velocity of the mass using an inductive coil and magnetic field, a method pioneered by Prince Boris Galitzin of Russia around 1906. If the mass motion is small, the induction circuit is linear and, as a bonus, the induced current in the inductive coil produces an electromagnetic force that counteracts the motion of the mass and thus provides predictable and stable damping. In the electromagnetic seismometer the output is a voltage that is proportional to the velocity, $\dot{z}$, of the mass relative to its frame (or case), and is thus the time derivative of the displacement response. The system response of a differentiator, which is characterized by the differential equation

$$y(t) = \dot{x}(t) \ , \tag{21}$$

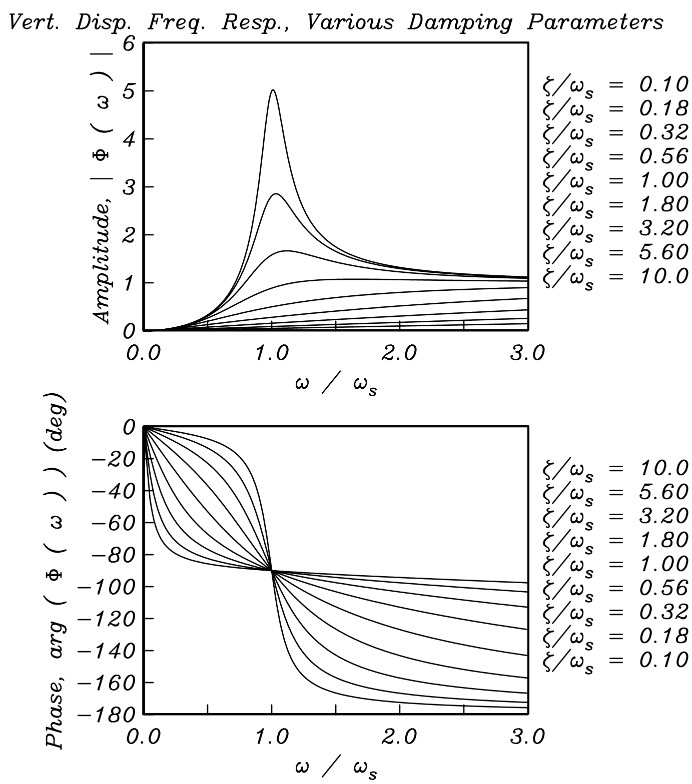

Figure 2: Frequency Response of the Mechanical Seismometer

can be trivially seen (10) to be just $i\omega$, so that the transfer function of an inductive seismometer system as voltage out versus Earth displacement is

$$\Phi_{induction,D}(\omega) = \frac{\dot{z}(\omega)}{u(\omega)} = \frac{-\imath\omega^3}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \ . \tag{22}$$

Note that if we consider the Earth *velocity*, $\dot{u}$ instead of the Earth displacement, u as the input signal, the response of the inductive seismometer can be written as

$$\Phi_{induction,V}(\omega) = \frac{\dot{z}(\omega)}{\dot{u}(\omega)} = \frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{23}$$

which is identical to (18), and the same response discussion as above applies, except that the output is in volts (practically speaking, with a gain constant controlled by the electronics of the data logging system) for a ground velocity input rather than output displacement for ground displacement. For this reason, such seismometers are sometimes referred to as *velocimeters*.

The inverse Fourier transform of a response function $\Phi(\omega)$ will give the time domain impulse response of the system. The following conditions can be shown to be sufficient for the existence of a Fourier transform:

1. $\phi(t)$ has only a finite number of maxima and minima in any finite time interval. This eliminates very wiggly functions (e.g., $\sin(1/x)$).

2. $\phi(t)$ has only a finite number of finite discontinuities in any finite time interval. Pathological functions such as 1 where the argument is rational and 0 where the argument is irrational won't work.

3. $\phi(t)$ is has finite "energy", so that

$$\int_{-\infty}^{\infty} |\phi(t)|^2 \, dt \tag{24}$$

   is bounded.

There are useful functions that do not satisfy (24), yet still have Fourier transforms (such transforms will have delta or other discontinuous functional components). Clearly, for example, (24) is not satisfied for the displacement transfer function (18) in the seismometer system. It is a little easier to obtain the displacement response to an impulsive Earth acceleration ($\ddot{u} = \delta(t)$) by the inverse Fourier transform method by solving

$$\ddot{a} + 2\zeta\dot{a} + \omega_s^2 a = -\delta(t) \tag{25}$$

which is shown in Figure 3 (we'll do the detailed calculation, and solve for the displacement response to Earth displacement later).

Energy in the Time and Frequency Domains; Parseval's theorem. The inverse Fourier transform says that time domain signals can be expressed as an infinite summation of complex exponentials. We might therefore expect a simple

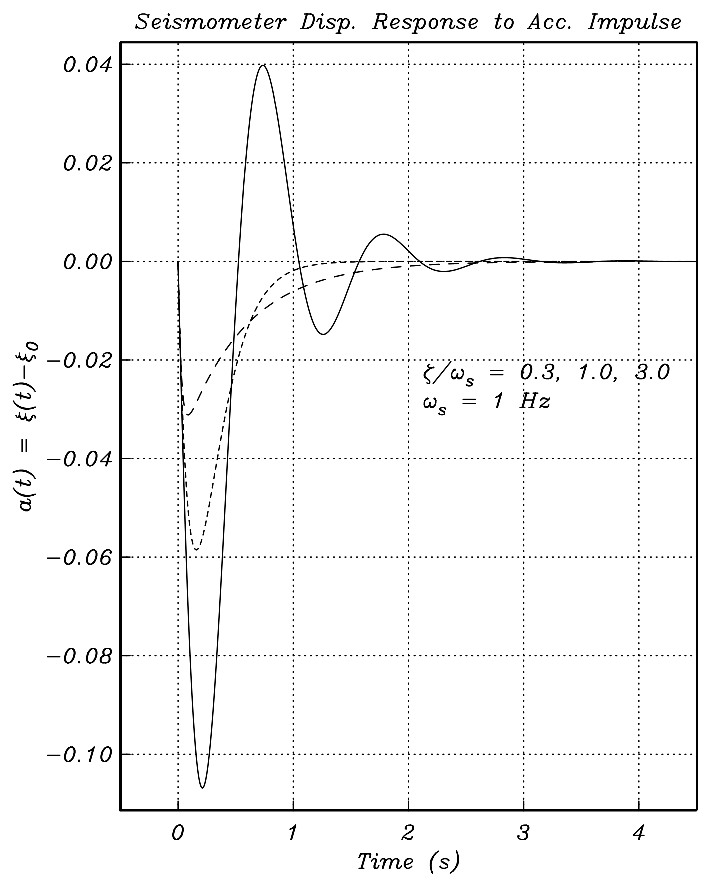

Figure 3: Response of the Mechanical Seismometer to an Acceleration Impulse

relationship between signal energy expressed in the time and frequency domains. Consider the total energy in a real (or complex) time domain signal, $\phi(t)$

$$E = \int_{-\infty}^{\infty} \phi(t)\phi^*(t)\ dt \tag{26}$$

where the asterisk denotes complex conjugation (which has no effect if $\phi(t)$ is real). Invoking (6), this can be written as

$$E = \int_{-\infty}^{\infty} \phi(t) \left(\int_{-\infty}^{\infty} \Phi^*(f) e^{-\imath 2\pi f t} df \right)\ dt\ . \tag{27}$$

Interchanging the order of integration, we get

$$E = \int_{-\infty}^{\infty} \Phi^*(f) \left(\int_{-\infty}^{\infty} \phi(t) e^{-\imath 2\pi f t} dt \right)\ df \tag{28}$$

which gives

$$E = \int_{-\infty}^{\infty} \Phi^*(f)\Phi(f)\ df = \int_{-\infty}^{\infty} \phi(t)\phi^*(t)\ dt\ . \tag{29}$$

Equation (29) is variously referred to as *Parseval's*, *Rayleigh's* or *Plancherel's* theorem. It says that one can evaluate the energy in a signal as either an integral of its amplitude squared time domain representation over all time, or as an integral across all of its amplitude squared frequency components over all frequencies. In a more general sense, Parseval's theorem says that the Fourier transform is *length preserving*, i.e., the "size" of the function (in the size-sense of the integral of the amplitude squared) is the same in the time and frequency domains.

Properties of the Fourier transform. We next consider the Fourier transforms of some canonical functions and discuss general symmetries and other properties. An important function in time series analysis which we saw in Chapter 1 is the boxcar function, $\Pi(t)$. The Fourier transform of the boxcar function is (Figure 4)

$$\mathcal{F}[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-\imath 2\pi f t}\ dt \tag{30}$$

$$= \int_{-1/2}^{1/2} e^{-\imath 2\pi f t} dt = \int_{-1/2}^{1/2} \cos(2\pi f t)\ dt \tag{31}$$

$$= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)\ . \tag{32}$$

The corresponding inverse transform is thus

$$\mathcal{F}^{-1}[\text{sinc}(f)] = \int_{-\infty}^{\infty} \text{sinc}(f) e^{\imath 2\pi f t} df = \Pi(t)\ . \tag{33}$$

Taking the complex conjugate and interchanging f and t, gives us the Fourier transform of sinc(t)

$$\Pi(f) = \int_{-\infty}^{\infty} \text{sinc}(t) e^{-\imath 2\pi f t} dt\ . \tag{34}$$

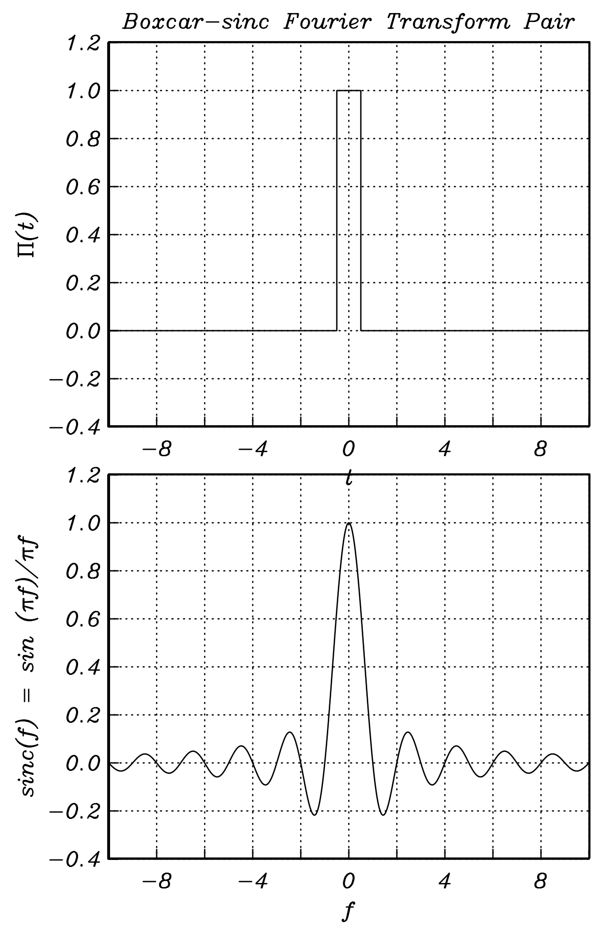

Figure 4: The Boxcar-Sinc Fourier Transform Pair

Note that (32) and (33) show, perhaps surprisingly, that we can get discontinuous functions by the integration smooth functions.

The Fourier transform of a delta function is easily seen to be

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-\imath 2\pi f t} dt = 1 \ . \tag{35}$$

So a delta function can be thought of as consisting of an equal weighting of $e^{-\imath 2\pi f t}$ functions across *all* frequencies, with no relative phase shifts. Going the other direction, from the frequency to the time domain, gives

$$\mathcal{F}^{-1}(1) = \int_{-\infty}^{\infty} e^{\imath 2\pi f t} df = \delta(t) \ . \tag{36}$$

One way to grasp (36) is to imagine the oscillating terms of the integrand all averaging out to zero, except exactly at $t = 0$, where they all have value one and will reinforce each other, i.e.,

$$\mathcal{F}^{-1}(1) = \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} e^{\imath 2\pi f t} df + \int_{\epsilon}^{\infty} e^{\imath 2\pi f t} \ df = 2 \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \cos(2\pi f t) \ df \ . \tag{37}$$

A very useful property of the Fourier transform is the *shifting property*; a simple time shift of a function only changes the phase (not the amplitude) of its Fourier transform. Furthermore the phase shift in this case is proportional to frequency.

Consider the Fourier transform of a general function

$$\mathcal{F}[\phi(t - t_0)] = \int_{-\infty}^{\infty} \phi(t - t_0) e^{-\imath 2\pi f t} dt \ . \tag{38}$$

Substituting $\tau = t - t_0$ gives

$$= \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f(\tau + t_0)} \ d\tau = e^{-\imath 2\pi f t_0} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau} \ d\tau \tag{39}$$

$$= e^{-\imath 2\pi f t_0} \Phi(f) \tag{40}$$

so that time shifts in the time domain correspond to linear (with respect to frequency) phase shifts in the frequency domain. You could also derive this relationship by considering what phase shifts all of the complex sinusoidal components in a general signal would have to accrue produce the same summation, only shifted in time.

Another important relationship is *time-frequency scaling* or *similarity*, consider

$$\mathcal{F}[\phi(\alpha t)] = \int_{-\infty}^{\infty} \phi(\alpha t) e^{-\imath 2\pi f t} dt \ . \tag{41}$$

For $\alpha > 0$, this gives

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} \phi(\tau) e^{-\imath 2\pi f \tau / \alpha} d\tau = \frac{1}{\alpha} \Phi\left(\frac{f}{\alpha}\right) \ , \tag{42}$$

using the substitution $\tau = \alpha t$. For $\alpha < 0$, the limits on the definite integral become reversed with the change of variable, so we get

$$\mathcal{F}[\phi(\alpha t)] = -\frac{1}{\alpha}\Phi\left(\frac{f}{\alpha}\right) \tag{43}$$

so that, in general

$$\mathcal{F}[\phi(\alpha t)] = \frac{1}{|\alpha|}\Phi\left(\frac{f}{\alpha}\right) \ . \tag{44}$$

Thus, when we "squeeze" a function in the time domain, its Fourier transform "spreads out" in the frequency domain (and vice-versa). An extreme end member showing this behavior is the delta function, which is an infinitely squeezed function in the time domain with an infinitely spread out transform (the 1 function; (35)) in the frequency domain.

As you have probably already suspected, there is a *duality* between the time and frequency domains, the precise relationship is

$$\mathcal{F}[\phi(t)] = \Phi(f) \tag{45}$$

$$\mathcal{F}[\Phi(t)] = \phi(-f) \ . \tag{46}$$

Any function can be decomposed into even and odd parts with respect to the origin

$$\phi(t) = \phi_e(t) + \phi_o(t) \tag{47}$$

$$= \frac{1}{2}[\phi(t) + \phi(-t)] + \frac{1}{2}[\phi(t) - \phi(-t)] \tag{48}$$

where $\phi_e(t) = \phi_e(-t)$ and $\phi_o(t) = -\phi_o(-t)$. This decomposition can be used to show that the Fourier transform exhibits various symmetry relations.

Consider the transform of a general real and even function, ϕ_e.

$$\mathcal{F}[\phi_e(t)] = \int_{-\infty}^{\infty} \phi_e(t) e^{-\imath 2\pi f t} dt \tag{49}$$

$$= \int_{-\infty}^{\infty} \phi_e(t)\cos(2\pi f t)\ dt - \imath \int_{-\infty}^{\infty} \phi_e(t)\sin(2\pi f t)\ dt \tag{50}$$

$$= 2\int_{0}^{\infty} \phi_e(t)\cos(2\pi f t) dt \tag{51}$$

which is even and is purely real. Similarly, for an odd, real function, ϕ_o, the Fourier transform

$$\mathcal{F}[\phi_o(t)] = \int_{-\infty}^{\infty} \phi_o(t) e^{-\imath 2\pi f t} dt \tag{52}$$

$$= \int_{-\infty}^{\infty} \phi_o(t)\cos(2\pi f t) dt - \imath \int_{-\infty}^{\infty} \phi_o(t)\sin(2\pi f t)\ dt \tag{53}$$

$$= -2\imath \int_{0}^{\infty} \phi_o(t)\sin(2\pi f t)\ dt \tag{54}$$

is odd and purely imaginary. Thus, the Fourier transform of an arbitrary real function containing both odd and even components may be evaluated as a superposition of (51) and (54), frequently referred to as the *cosine transform* and *sine transform*, respectively. Using superposition, one can derive a list of basic symmetry relationships between the time and frequency domains:

$\phi(t)$	$\Phi(f)$
even	even
odd	odd
real, even	real, even
real, odd	imaginary, odd
imaginary, even	imaginary, even
imaginary, odd	real, odd
complex, even	complex, even
complex, odd	complex, odd
real, asymmetrical	complex, Hermitian
imaginary, asymmetrical	complex, anti-Hermitian
Hermitian	real
anti-Hermitian	imaginary

where a *Hermitian* function has an even real part and an odd imaginary part ($\Phi(f) = \Phi^*(-f)$) and an *anti-Hermitian* function has an odd real part and an even imaginary part ($\Phi(f) = -\Phi^*(-f)$).

One of the most important conceptual and practical relationships between the time and frequency domains is embodied in the *convolution theorem.* Consider the Fourier transform of the convolution of two functions

$$\mathcal{F}[\phi_1(t) * \phi_2(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi_1(\tau)\phi_2(t-\tau)\ d\tau \right) e^{-\imath 2\pi f t} dt\ . \qquad (55)$$

Reversing the order of integration gives

$$\mathcal{F}[\phi_1(t) * \phi_2(t)] = \int_{-\infty}^{\infty} \phi_1(\tau) \left(\int_{-\infty}^{\infty} \phi_2(t-\tau) e^{-\imath 2\pi f t} dt \right)\ d\tau\ . \qquad (56)$$

However, by the time shift property (40), this is just

$$\int_{-\infty}^{\infty} \phi_1(\tau)\Phi_2(f) e^{-\imath 2\pi f \tau}\ d\tau = \Phi_1(f)\Phi_2(f) \qquad (57)$$

so that convolution in the time domain corresponds to multiplication in the frequency domain! Similarly, we can show that multiplication in the time domain corresponds to convolution in the frequency domain

$$\mathcal{F}[\phi_1(t)\phi_2(t)] = \Phi_1(f) * \Phi_2(f)\ . \qquad (58)$$

This can be understood intuitively based on what we know about the response of linear systems, as the response of a linear system at each frequency is just the complex amplitude of that frequency component in the input, times the complex value of the response function of the system at that frequency.

Recall that time differentiation has a remarkably simple form in the frequency domain

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}\int_{-\infty}^{\infty} \Phi(f)e^{\imath 2\pi f t}\, df \tag{59}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[\Phi(f)e^{\imath 2\pi f t}]\, df = \int_{-\infty}^{\infty} 2\pi\imath f\Phi(f)e^{\imath 2\pi f t}df = \mathcal{F}^{-1}[2\pi\imath f\Phi(f)] \tag{60}$$

taking the Fourier transform of both sides gives:

$$= \mathcal{F}[\frac{d}{dt}\phi(t)] = 2\pi\imath f\Phi(f)\ . \tag{61}$$

(61) clearly shows that differentiation amplifies high frequency signal components relative to those at low frequency, and thus belongs to a class of operators generally referred to as *high-pass filters*.

The situation for integration is somewhat more complex

$$\mathcal{F}\left(\int_{-\infty}^{t} \phi(\tau)d\tau\right) = \frac{\Phi(f)}{2\pi\imath f} + \frac{\delta(f)}{2}\int_{-\infty}^{\infty}\phi(t)dt \tag{62}$$

where the delta function term accommodates the contribution of a possible non-zero mean value in $\phi(t)$. A definite integrator is thus a *low-pass filter*, as it reinforces low frequencies relative to high frequencies.

(61) and (62) are helpful in computing some otherwise nonstraightforward Fourier transforms, especially for discontinuous functions. Consider the step function. Using (62) gives

$$\mathcal{F}[\mathrm{H}(t)] = \mathcal{F}\left(\int_{-\infty}^{t} \delta(\tau)d\tau\right) \tag{63}$$

$$= \frac{1}{2\pi\imath f} + \frac{\delta(f)}{2}\ . \tag{64}$$

The Fourier transform of the sign function is thus

$$\mathcal{F}[2\mathrm{H}(t) - 1] = \frac{1}{\pi\imath f}\ . \tag{65}$$

Example: Equilibrium elastic response of a loaded, buoyantly supported crust. The differentiation and integration properties of the Fourier transform provide a useful method for obtaining solutions to ordinary linear integrodifferential equations. An example of geophysical interest is the downward deflection of a rigid plate (such as the Earth's crust) buoyantly supported by an underlying liquid (to first order, the mantle) to a distributed load (such as an ice cap, volcano, or reservoir) (Figure 5.

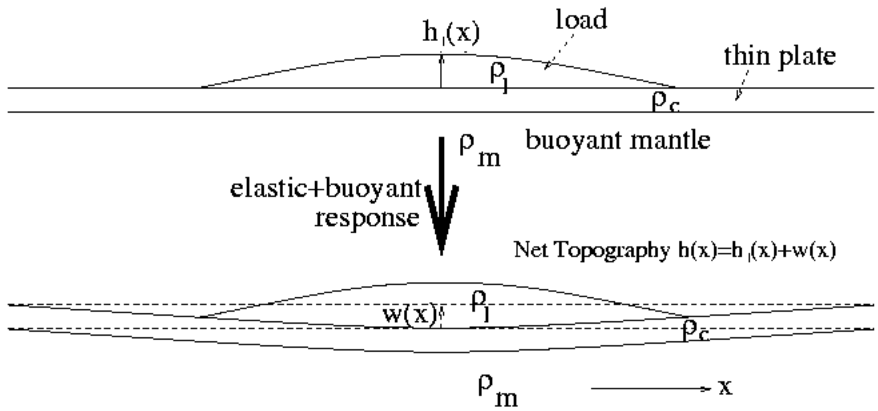

Figure 5: A Buoyant, Rigid Plate with a Spatial Load

The model for the small-deformation equilibrium of a deformed plate is a linear differential equation (e.g., Banks et al., *Geophys J.*, B51, 431-452, 1977; Turcotte and Shubert, *Geodynamics*, 1982)

$$D\bigtriangledown^4 w(r) = p(r) \tag{66}$$

where $w(r)$ is the upward deflection of the plate and $p(r)$ is the upward force per unit area. The forcing term, $p(r)$, arises from a topographic load, $h_l(r)$ and from a buoyancy term due to the displaced mantle. D is the *flexural rigidity*, which depends on the thickness and elastic moduli of the plate

$$D \ = \ \frac{E\tau^3}{12(1-\nu^2)} \tag{67}$$

where τ is the plate thickness, E is Young's Modulus, and ν is Poisson's Ratio.

In one spatial dimension, x, (66) becomes

$$D\frac{\partial^4 w(x)}{\partial x^4} = p(x) \ . \tag{68}$$

The total forcing function for a load of homogeneous density, ρ_l is the sum of the load and the opposite-directed buoyant compensation of the mantle

$$p(x) = -\rho_l g h_l(x) + B(x) \tag{69}$$

where ρ_l is the density of the added material, g is the acceleration of gravity, and $B(x)$ is the buoyancy term due to mantle material of density ρ_m,

$$B(x) = -\rho_m g w(x) \ . \tag{70}$$

We can thus write the forcing term in terms of the input load $h_l(x)$ as the linear homogeneous differential equation

$$p(x) = D\frac{\partial^4 w(x)}{\partial x^4} = -g(\rho_l h_l(x) + \rho_m w(x)) \ . \tag{71}$$

Now we can solve for the resulting crustal deformation, using the same method that we utilized for the seismometer system, by separating response, $w(x)$, and input, $h_l(x)$ related terms, and then taking spatial Fourier transforms to obtain

$$[(2\pi\imath k)^4 D + g\rho_m]W(k) = -\rho_l g H_l(k) \tag{72}$$

where k is the *spatial frequency* (units of 1/length), the spatial counterpart of f. Here, to keep our Fourier conventions unchanged from previous discussion, note that k is just $1/\lambda$, or the reciprocal wavelength (this is slightly different than the common use of k as the wavenumber, which is $2\pi/\lambda$, and is the spatial equivalent of the radian frequency $2\pi f$). Our definition of k here is utilized to make the mathematics a bit cleaner and is entirely analogous to the convention used in thes notes for t and f.

Note that $H_l(k)$ is the spatial Fourier transform of the input

$$H_l(k) = \int_{-\infty}^{\infty} h_l(x)e^{-\imath 2\pi kx}\ dx \tag{73}$$

(*not* the step function). The (spatial) frequency domain solution is thus

$$W(k) = -H_l(k)\frac{\frac{\rho_l}{\rho_m}}{1 + \frac{16\pi^4 k^4 D}{g\rho_m}} \ . \tag{74}$$

Note that (74) depends strongly on the reciprocal wavelength, k. For k large, the response of the system becomes becomes negligible. Conversely, for k small, the response becomes increasingly significant, reaching a maximum value of

$$W_{max} = W(0) = -H_l(0)\rho_l/\rho_m \tag{75}$$

as $k \to 0$. Thus, for long-wavelength (small k) spatial components of the landscape, we say that we have a large degree of buoyant *compensation*, as the topographic load is primarily supported by mantle buoyancy. At short spatial wavelengths, on the other hand (large k), the landscape is almost totally supported by the flexural rigidity of the crust. The degree of compensation for a spatial component of wavelength $\lambda = 1/k$, is the deflection of the system relative to W_{max}

$$C = \frac{W(k)}{W_{max}} \ . \tag{76}$$

We can evaluate the impulse response in the x domain by taking the inverse Fourier transform of $W(k)/H_l(k)$ (preferably with the assistance of a table of integral transforms), to obtain

$$q(x) = \mathcal{F}^{-1}[W(k)/H_l(k)] \tag{77}$$

or

$$q(x) = \frac{-2g\rho_l}{D} \int_0^\infty \frac{\cos(2\pi kx)\ dk}{\alpha^4 + (2\pi k)^4} \tag{78}$$

where

$$\alpha = \left(\frac{g\rho_m}{D}\right)^{1/4} \tag{79}$$

so that (e.g., Erdelyn *et al.*, *Tables of Integral Transforms*, Volume 1, 1954):

$$q(x) = \frac{-\sqrt{2}g\rho_l}{4\alpha^3 D}\ e^{\frac{-\alpha|x|}{\sqrt{2}}} \left(\sin\frac{\alpha|x|}{\sqrt{2}} + \cos\frac{\alpha|x|}{\sqrt{2}}\right)\ . \tag{80}$$

This function is plotted in Figure 6 and consists a central depression and an outboard peripheral upwarp. Note that (80) is the impulse response of this system, as $W(k)$ is the response for $H_l(k) = 1$ (74), or $h_l(x) = \delta(x)$, so that *any* 1-d deformation of a rigid plate to a load (assumed to be infinitely extending in the out-of plane direction) can thus be calculated by convolving $q(x)$ and the specific linear load distribution.

Note also that the net topography for the system in equilibrium is given by the sum of the input load topography and the system response

$$h(x) = h_l(x) + w(x)\ . \tag{81}$$

Example: Time domain seismometer response. We can use Fourier tools to obtain a result for the displacement response of the vertical seismometer in the time domain by noting, as above, that the time domain response to an impulsive acceleration characterized by $\ddot{u} = \delta(t)$ is characterized by

$$\ddot{a} + 2\zeta\dot{a} + \omega_s^2 = -\delta(t)\ . \tag{82}$$

Taking the Fourier transform of both sides and solving for $a(\omega)$, the displacement response to an acceleration impulse input, gives the frequency domain expression

$$a(\omega) = \frac{1}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{83}$$

Note that this is just the response of the seismometer system to the displacement impulse (18), divided by $-\omega^2$. This is appropriate, as the input function has been twice differentiated in the time domain and the response of a differentiator is $\imath\omega$ (61).

The time domain displacement response to an acceleration impulse input is therefore

$$\phi(t) = \mathcal{F}^{-1}\left(a(\omega)\right) = \mathcal{F}^{-1}\left(\frac{1}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}\right) \tag{84}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\ d\omega}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\ d\omega}{(\omega - \omega_1 - \imath\zeta)(\omega + \omega_1 - \imath\zeta)} \tag{85}$$

where

$$\omega_1 = \sqrt{\omega_s^2 - \zeta^2} \tag{86}$$

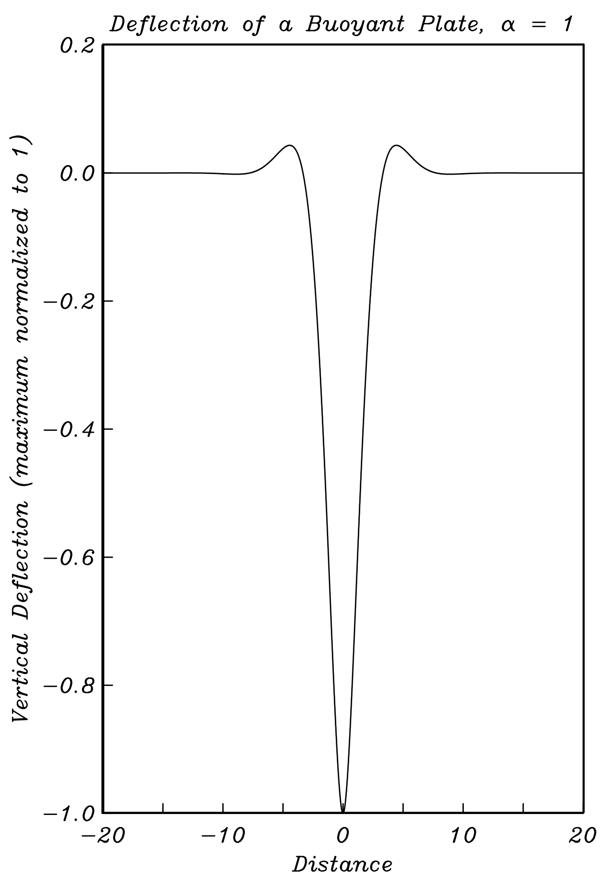

Figure 6: Response of a Buoyant, Rigid Plate to an Spatial Impulse Load

Solving this integral is relatively straightforward using the residue theorem from complex analysis and separation into three cases. For $\omega_s > \zeta$, the system exhibits a distinct resonance near $\omega = \omega_s$ (as we have already seen from examining the frequency response; Figure 2) and is referred to as *underdamped.* In this case, the poles of the integrand in (85) lie at $(\omega_1, \imath\zeta)$ and $(-\omega_1, \imath\zeta)$ in the complex ω plane. The time domain solution is found from the residues of the two complex poles of the integrand to be

$$a(t) = \frac{-\mathrm{H}(t)}{\omega_1} e^{-\zeta t} \sin(\omega_1 t) \ . \tag{87}$$

When $\omega_s < \zeta$, the system does not resonate, the complex poles of the integrand lie on the positive imaginary axis of the complex ω plane, and the system is referred to as being *overdamped.* ω_1^2 is negative in this case, and the result is an impulse response that is a sum of real exponentials

$$a(t) = \frac{-\mathrm{H}(t)}{2(\zeta^2 - \omega_s^2)^{1/2}} \left(e^{-(\zeta - (\zeta^2 - \omega_s^2)^{1/2})t} - e^{-(\zeta + (\zeta^2 + \omega_s^2)^{1/2})t} \right) \ . \tag{88}$$

The case $\omega_s = \zeta$ is a transition between the underdamped and overdamped case,s referred to as *critically damped.* Because there is a double pole, a special case of the residue theorem must be applied to obtain the impulse response, which is

$$a(t) = -\mathrm{H}(t) t e^{-\zeta t} \ . \tag{89}$$

These time domain responses are shown in Figure 3.

How do we evaluate the displacement impulse response of the system to Earth displacement? One way is to reexpress the integrand in the inverse transform of (18) to strip off a delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\omega^2 e^{\imath\omega t} \, d\omega}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 \ + \ \frac{2\imath\zeta\omega + \zeta^2 + \omega_1^2}{(\omega - \omega_1 - \imath\zeta)(\omega + \omega_1 - \imath\zeta)} \right) e^{\imath\omega t} \, d\omega \tag{90}$$

$$= -\delta(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(2\imath\zeta\omega + \zeta^2 + \omega_1^2) e^{\imath\omega t} \, d\omega}{(\omega - \omega_1 - \imath\zeta)(\omega + \omega_1 - \imath\zeta)} \tag{91}$$

and then evaluate the remaining integral using the residue theorem. Another way to solve (91) is to note that $a(t)$ is the time domain solution for the system response to an Earth acceleration of $a_0(t) = \delta(t)$. Because the seismometer system and the differentiation operation are linear, we can evaluate the seismometer displacement response from a displacement impulse by twice differentiating $a(t)$ with respect to time. For the underdamped system, for example, this gives

$$d(t) = \frac{d^2 a(t)}{dt^2} = \frac{d^2}{dt^2} \left(\frac{-\mathrm{H}(t)}{\omega_1} e^{-\zeta t} \sin(\omega_1 t) \right) \tag{92}$$

$$= -\frac{1}{\omega_1} \left(\mathrm{H}''(t) e^{-\zeta t} \sin(\omega_1 t) - \mathrm{H}'(t) \zeta e^{-\zeta t} \sin(\omega_1 t) \right.$$

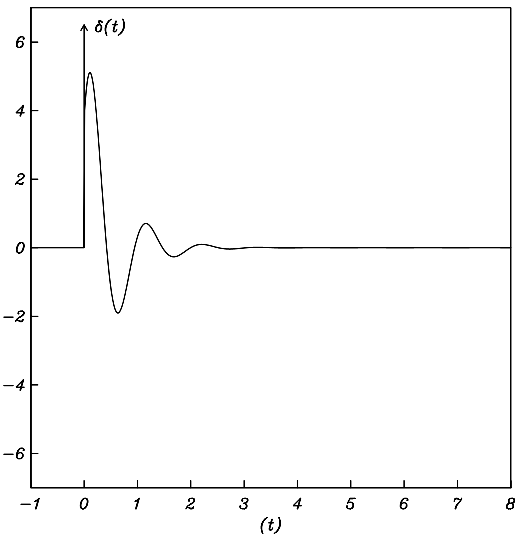

Figure 7: Displacement Output/Displacement Input Response of an Underdamped Seismometer ($\zeta = 0.3\omega_s$; $\omega_s = 2\pi$ Hz) to a Displacement Impulse

$$+\mathrm{H}'(t)e^{-\zeta t}\omega_1\cos(\omega_1 t) - \mathrm{H}'(t)\zeta e^{-\zeta t}\sin(\omega_1 t)\mathrm{H}(t)\zeta^2 e^{-\zeta t}\sin(\omega_1 t) -$$
$$+\mathrm{H}(t)\zeta e^{-\zeta t}\omega_1\cos(\omega_1 t) + \mathrm{H}'(t)e^{-\zeta t}\omega_1\cos(\omega_1 t)$$
$$-\mathrm{H}(t)\zeta e^{-\zeta t}\omega_1\cos(\omega_1 t) - \mathrm{H}(t)e^{-\zeta t}\omega_1^2\sin(\omega_1 t)\big) \ . \tag{93}$$

Using $\mathrm{H}'(t) = \delta(t)$ and $\mathrm{H}''(t) = \delta'(t)$, and noting that $\delta'(t)\sin(\omega_1 t)e^{-\zeta t} = -\delta(t)\omega_1$, and $\delta(t)e^{-\zeta t}\omega_1\cos(\omega_1 t) = \delta(t)\omega_1$ gives

$$d(t) = -\frac{1}{\omega_1}\left(\delta(t)\omega_1 - 2\mathrm{H}(t)\omega_1\zeta e^{-\zeta t}\cos(\omega_1 t) + \mathrm{H}(t)\zeta^2 e^{-\zeta t}\sin(\omega_1 t) - \mathrm{H}(t)\omega_1^2 e^{-\zeta t}\sin(\omega_1 t)\right) \ . \tag{94}$$

or

$$d(t) = -\delta(t) + \mathrm{H}(t)e^{-\zeta t}\left(-2\zeta\cos(\omega_1 t) + \left(\zeta^2/\omega_1 - \omega_1\right)\sin(\omega_1 t)\right) \ . \tag{95}$$

In the limit as $\omega_s \to 0$, and for an undamped ($\zeta = 0$) seismometer, we obtain

$$d(t) = -\delta(t) \ . \tag{96}$$

Note that as the resonant frequency, ω_1, becomes small (the resonant period becomes large), (96) and Figure 7 approach the ideal instrument response of a delta function (with a trivial minus sign). Because seismologists frequently want to know the true ground displacement (its long-period asymptotic spectral level is proportional to the seismic moment, among other reasons), seismometers with very long natural periods are desirable and constitute the instrumental backbone of much of modern seismology. In practice, most seismometers have an output that is proportional to velocity, but if they have suitably low noise at long periods the native output can be stably integrated to produce a displacement seismogram.

Moment-spectral relationships. As an additional perspective on the rich mathematics of Fourier theory can be obtained by noting that all of the moments of the time domain function, $\phi(t)$, can be expressed in terms of the behavior of $\Phi(f)$ at the origin. Consider the n^{th} moment

$$\phi_n = \int_{-\infty}^{\infty} t^n \phi(t) dt \ . \tag{97}$$

The n^{th} derivative of $\Phi(f)$ with respect to f is

$$\frac{\partial^n \Phi(f)}{\partial f^n} = \int_{-\infty}^{\infty} (-2\pi\imath t)^n \phi(t) e^{-\imath 2\pi f t} dt \tag{98}$$

so that

$$\frac{1}{(-2\pi\imath)^n} \left(\frac{\partial^n \Phi(f)}{\partial f^n} \right) = \int_{-\infty}^{\infty} t^n \phi(t) e^{-\imath 2\pi f t} dt \ . \tag{99}$$

Evaluating both sides at $f = 0$ gives

$$\frac{1}{(-2\pi\imath)^n} \left(\frac{\partial^n \Phi(0)}{\partial f^n} \right) = \int_{-\infty}^{\infty} t^n \phi(t) \ dt = \phi_n \ . \tag{100}$$

Thus, we can now see that the 0^{th} moment of $\phi(t)$, the total area under $\phi(t)$, is just $\Phi(0)$. Similarly, the 1^{st} moment of $\phi(t)$ is just

$$\int_{-\infty}^{\infty} t\phi(t) dt = -\frac{1}{2\pi\imath} \left(\Phi'(f) \right)_{f=0} \tag{101}$$

where

$$\Phi'(f) = \frac{\partial \Phi(f)}{\partial f} \tag{102}$$

so that the slope of $\Phi(f)$ at the origin is proportional to the expectated value of t

$$< t >_{\phi(t)} = \frac{\int_{-\infty}^{\infty} t\phi(t) \ dt}{\int_{-\infty}^{\infty} \phi(t) \ dt} \ . \tag{103}$$

Time functions which which are symmetrical must therefore have Fourier transforms with zero slope at $f = 0$ (we can also see this from the aforementioned symmetry relations).

The 2^{nd} moment is

$$\int_{-\infty}^{\infty} t^2 \phi(t) dt = -\frac{1}{4\pi^2} \left(\Phi''(f) \right)_{f=0} \tag{104}$$

so that the curvature of $\Phi(f)$ at the origin is proportional to the second moment of $\phi(t)$. For functions which have an infinite second moment, the Fourier transform has a cusp at the origin, for example,

$$F \left(\frac{1}{\alpha^2 + t^2} \right) = \frac{e^{-\alpha |f|}}{2\alpha} \ . \tag{105}$$

Next, consider the variance of $\phi(t)$

$$\sigma^2[\phi(t)] = <(t- <t>)^2>_{\phi(t)} = \frac{\int_{-\infty}^{\infty}(t^2 - 2t <t> + <t>^2)\phi(t)\ dt}{\int_{-\infty}^{\infty}\phi(t)\ dt} \tag{106}$$

$$= \frac{1}{\Phi(0)}\left(\frac{\Phi''(0)}{(-2\pi\imath)^2} - 2\frac{\Phi'(0)}{-2\pi\imath}\cdot\frac{\Phi'(0)}{-2\pi\imath\Phi(0)} + \frac{[\Phi'(0)]^2}{(-2\pi\imath)^2}\cdot\frac{\Phi(0)}{\Phi(0)^2}\right) \tag{107}$$

$$= \frac{1}{4\pi^2\Phi(0)}\left(-\Phi''(0) + \frac{[\Phi'(0)]^2}{\Phi(0)}\right)\ . \tag{108}$$

What is the variance, then, of $\phi_1(t) * \phi_2(t)$? Using the convolution theorem (57) makes this straightforward, as

$$\sigma^2[\phi_1(t) * \phi_2(t)] = \frac{1}{4\pi^2\Phi_1(0)\Phi_2(0)}\left(-(\Phi_1\Phi_2)''(0) + \frac{[(\Phi_1\Phi_2)'(0)]^2}{\Phi_1(0)\Phi_2(0)}\right) \tag{109}$$

$$= \frac{1}{4\pi^2}\left[-\frac{\Phi_1''(0)}{\Phi_1(0)} - \frac{\Phi_2''(0)}{\Phi_2(0)} + \left(\frac{\Phi_1'(0)}{\Phi_1(0)}\right)^2 + \left(\frac{\Phi_2'(0)}{\Phi_2(0)}\right)^2\right] = \sigma^2[\phi_1(t)] + \sigma^2[\phi_2(t)] \tag{110}$$

which gives the important result that the variance of a convolution result is just the sum of the variances of the two constituent functions. This is a quantitative measure of the amount of "spreading" that occurs in the convolution operation. Unless one or both of the constituent functions in the convolution has zero variance, the convolution result will always have greater variance than either of the two input functions.

Causal systems and the Hilbert transform. An important relationship exists between the real and imaginary parts of the Fourier transform of a real causal function, $\phi_c(t)$, that is, a real function that is zero for all $t < 0$. To see this, we first decompose $\phi_c(t)$ into its even and odd parts

$$\phi_c(t) = \phi_e(t) + \phi_o(t) = 1/2(\phi_c(t) + \phi_c(-t)) + 1/2(\phi_c(t) - \phi_c(-t)). \tag{111}$$

For the causal function, we can express $\phi_o(t)$ in terms of $\phi_e(t)$, as:

$$\phi_o(t) = \phi_e(t)\quad (t > 0) \tag{112}$$

and

$$\phi_o(t) = -\phi_e(t)\quad (t < 0) \tag{113}$$

Thus

$$\phi_c(t) = [1 + \mathrm{sgn}(t)]\phi_e(t)\ . \tag{114}$$

By superposition, using the frequency domain convolution theorem (58),

$$\Phi_c(f) = \Phi_e(f) + \mathcal{F}[\mathrm{sgn}(t)] * \Phi_e(f)\ , \tag{115}$$

and using the Fourier transform of the sign function (65), we obtain the Fourier transform of $\phi_c(t)$ explicitly in terms of the Fourier transform of $\phi_e(t)$

$$\Phi_c(f) = \Phi_e(f) + \frac{-\imath}{\pi f} * \Phi_e(f)\ . \tag{116}$$

Note that because $\phi_e(t)$ is real and even, so is $\Phi_e(f)$. Thus, the real and imaginary parts of $\Phi_c(f)$ are related to each other by the real convolution operator $(-1/\pi f)$. This relationship can be summarized by

$$\Im[\Phi_c(f)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Re[\Phi_c(\xi)]}{\xi - f}\, d\xi = \Re[\Phi_c(f)] * \frac{-1}{\pi f} \;=\; \mathrm{H}[\Re[\Phi_c(f)]] \; . \tag{117}$$

and conversely,

$$\Re[\Phi_c(f)] = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im[\Phi_c(\xi)]}{\xi - f}\, d\xi = \Im[\Phi_c(f)] * \frac{1}{\pi f} \;=\; \mathrm{H}^{-1}[\Im[\Phi_c(f)]] \; . \tag{118}$$

One can confirm (118) by showing that

$$-\frac{1}{\pi f} * \frac{1}{\pi f} = \delta(f) \; . \tag{119}$$

(117) is the *Hilbert transform* and (118) is the *inverse Hilbert transform* operator, acting on $\Re[\Phi_c(f)]$ and $\Im[\Phi_c(f)]$, respectively. This relationship puts constraints on the frequency response of all physical (causal) transfer functions. If we take the Hilbert transform of a *time* function, we get the associated *quadrature* function.

$$\mathrm{H}[\phi(t)] = \hat{\phi}(t) \; . \tag{120}$$

The Fourier transform of the quadrature function has the same amplitude information as the original function, but its phase is multiplied by $\imath$ sgn(f), so that it is phase shifted by $-\pi/2$ for negative frequencies and by $\pi/2$ for positive frequencies.

An *analytic signal* is one in which the real and imaginary parts are related by the Hilbert transform (so that its Fourier transform is zero for all negative frequencies)

$$A(t) = \phi(t) - \imath\hat{\phi}(t) \; . \tag{121}$$

Among its other uses, the analytic time series formulation is useful in evaluating the amplitude envelope, $|A(t)|$, of a function.

An important example of a causal physical system is the attenuation which occurs when a wave propagates through a lossy medium. In seismology, such media (which of course include all real materials) are referred to as *anelastic*. The loss mechanisms need not concern us in detail here, but they include work done at grain boundaries and other irreversible changes in the material. The observational result of attenuation is that the energy arriving at the receiver is less than that which one would expect from considering the effects of geometrical spreading and other ray path effects alone.

For the idealized case of a one-dimensional plane wave propagating through a lossless medium (e.g., an electromagnetic wave propagating through a perfect vacuum, or a seismic wave propagating through a perfectly elastic medium) the signal, β, at position x and time t is simply the signal at the source delayed by the propagation time x/v

$$a(x,t) = a(t - x/v) \tag{122}$$

where v is the phase velocity. If the time function at the source is $a(t)$, then we can express the signal at an arbitrary time and place as

$$a(x,t) = a_0(t) * \delta(t - t_0) \tag{123}$$

where $t_0 = x/v$ and $a_0(t)$ is the signal at $x = 0$. We are assuming here that all frequency components propagate at a single velocity, v. Such a medium is referred to as *nondispersive*. The transfer function of a lossless, nondispersive system is therefore that of a time delay. Consider an exponential input at some frequency, f, the output of the delay system is

$$a(x,f) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-i2\pi f t}\ dt = e^{-\imath 2\pi f t_0} = e^{-\imath 2\pi f x/v}\ . \tag{124}$$

The quality factor, Q, of an oscillating system is given by

$$\frac{1}{Q(f)} = \frac{\delta E}{2\pi E} \tag{125}$$

where E is the peak energy of the system and δE is the energy lost in each cycle, assuming $Q \gg 1$. For a propagating sinusoidal disturbance, then, the loss relationship as a function of x is

$$\delta E = \frac{dE}{dx}\lambda \tag{126}$$

as the field goes through one oscillation in a wavelength, $\lambda = v/f$. Combining (126) and (125), we have

$$\frac{2\pi E}{Q} = \frac{dE}{dx}\lambda \tag{127}$$

which has a solution for propagating energy of

$$E(x,\ f) = E_0(t) e^{-2\pi f x/Qv} \tag{128}$$

or for propagating amplitude of

$$b(x,\ f) = b_0(t) e^{-\pi f x/Qv}\ . \tag{129}$$

The combined transfer function for the system is thus, by the convolution theorem (57)

$$c(x,\ f) = F\left(\frac{1}{a_0(t)} a(x,t)\ *\ \frac{1}{b_0(t)} b(x,t)\right) \tag{130}$$

$$\frac{1}{a_0} a(x,\ f)\ \cdot\ \frac{1}{b_0} b(x,\ f) = e^{-\imath 2\pi f x/v}\ \cdot\ e^{-\pi f x/Qv}\ . \tag{131}$$

Taking the inverse Fourier transform of $c(x,\ f)$ to obtain the impulse response of the system, we have (taking the absolute value of f so that negative and positive frequencies are treated equally)

$$c(x,t) = \int_{-\infty}^{\infty} e^{2\pi(-|f|t_0/2Q + \imath f(t-t_0))}\ df \tag{132}$$

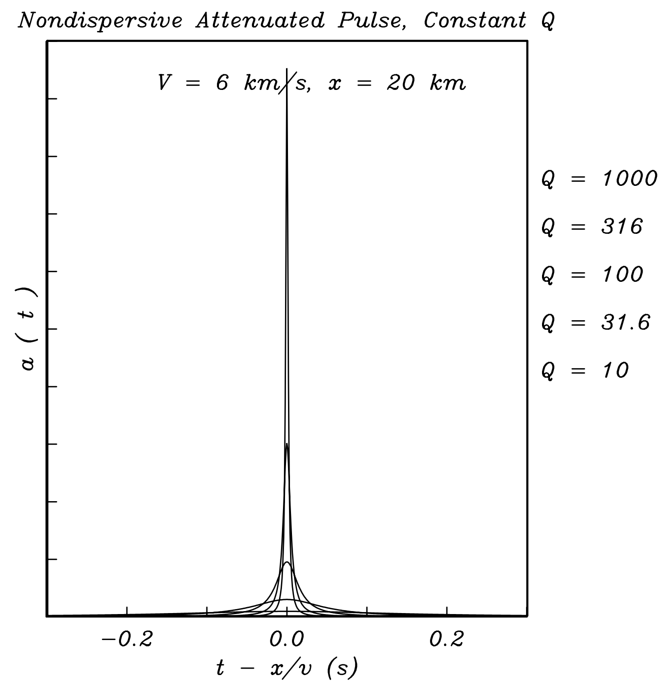

Figure 8: Attenuated Pulses, Constant Q

$$= \int_0^\infty e^{2\pi(-ft_0/2Q+\imath f(t-t_0))}\ df + \int_{-\infty}^0 e^{2\pi(ft_0/2Q+\imath f(t-t_0))}\ df \tag{133}$$

$$= -\frac{1}{2\pi}\left[\frac{1}{(\imath t-(\imath+1/2Q)t_0} - \frac{1}{(\imath t-(\imath-1/2Q)t_0}\right] \tag{134}$$

$$= \frac{1}{\pi}\left(\frac{(t_0/2Q)}{(t-t_0)^2+(t_0/2Q)^2}\right) \tag{135}$$

which is plotted in Figure 8

(135) is a symmetrical pulse with a maximum at $t = t_0$. Note, however, that $c(x,\ t)$ is not zero for $t < t_0$. This solution is therefore acausal and cannot correspond to the behavior of the real world. Reexamining our assumptions, we find that we must reassess both the nondispersiveness of the medium and the constancy of Q with frequency. A moment's reflection reveals that we cannot get an asymmetrical, causal pulse by simply allowing Q to vary as an even function of frequency, as the Q operator will affect positive and negative frequencies equally and hence will not alter the symmetry of the pulse. Thus, we are led to the conclusion that all lossy media must be dispersive!

The general transfer function for a wave propagating towards positive x is thus a generalization of (131)

$$c(x,\ f) = e^{-\pi|f|x/Q(f)v(f)}\ \cdot\ e^{-\imath 2\pi fx/v(f)} \tag{136}$$

where v and Q are now functions of f. We can write this as

$$c(x,\ f) = e^{-2\pi\imath Kx} \tag{137}$$

if we define the complex wavenumber, K as

$$K = \frac{-\imath|f|}{2Q(f)v(f)} + \frac{f}{v(f)} \ = \ \frac{f}{v(f)} + \imath\alpha(f) \tag{138}$$

where $\alpha(f)$ is the attenuation factor. The impulse response is thus the inverse Fourier transform of this

$$c(x,\ t) = \int_{-\infty}^{\infty} e^{\imath 2\pi(-Kx+ft)}\ df \tag{139}$$

It can be shown (e.g., Aki and Richards, v. 1, 1980) that constraining $c(x,\ t)$ to be causal, i.e., equal to zero for $t < t_1 = x/v_\infty$ places the following constraint on the dispersive velocity function

$$\frac{f}{v(f)} = \frac{f}{v_\infty} + \mathrm{H}[\alpha(f)] \tag{140}$$

where v_∞ is the phase velocity at infinite frequency and H is the Hilbert transform. Finding solutions to (140) is non-trivial, and there is no solution for constant Q. If we take Q to be constant over the seismic frequency range, however, we can arrive at the useful solution proposed by Azimi *et al.* (*Izvestiya, Physics of the Solid Earth*, pp. 88-93, 1968), where the phase velocity is approximately given by

$$\frac{1}{v(f)} = \frac{1}{v_\infty} + \frac{2\alpha_0}{\pi} \ln\left(\frac{1}{2\pi f \alpha_1}\right) \tag{141}$$

where α_0 and α_1 are constants. Using

$$\alpha_0 \approx (2v_\infty Q)^{-1}. \tag{142}$$

and

$$\alpha_1 = 0.01 \text{ s} \tag{143}$$

Figure 9 showns the results of numerically integrating (141) for various values of Q to obtain attenuation pulses which are asymmetrical and exhibit a much better approximation to causal behavior than the nondispersive pulses of Figure 8.

The effect of feedback on the transfer function. An important engineering concept is the effect of *feedback* on the transfer function of a system. Figure 10 shows the situation where a filtered portion of an output signal, modified by the feedback transfer function Φ_2 is subtracted from the input signal (*negative feedback*). The effect of feedback can alter the system response significantly and, in the case of engineering applications, can do so in several highly desirable ways. The net transfer function for the system of Figure 10 is

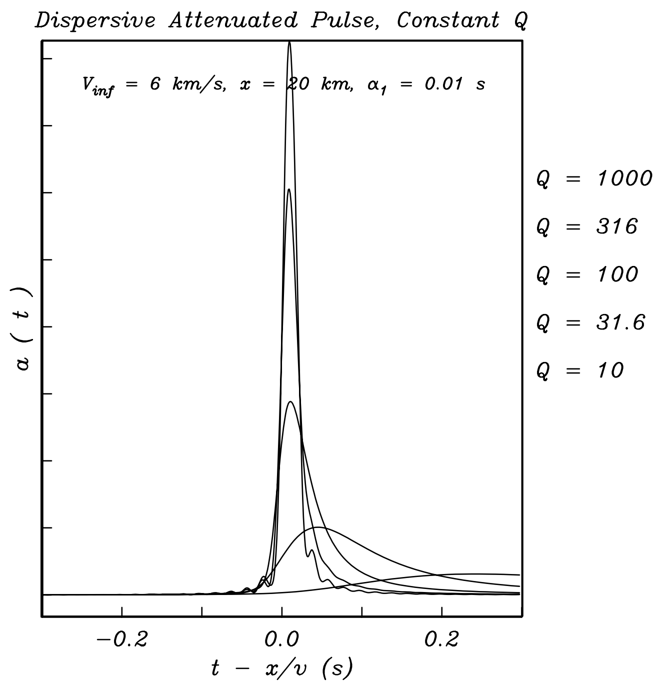

Figure 9: Attenuated Pulses, Quasi-Causal Q

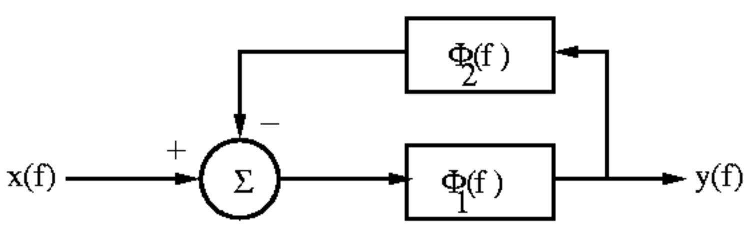

Figure 10: A linear system with feedback

$$Y(\omega) = (X(\omega) - \Phi_2(\omega)Y(\omega))\Phi_1(\omega) \tag{144}$$

which gives

$$\Phi_{fb}(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\Phi_1(\omega)}{1 + \Phi_1(\omega)\Phi_2(\omega)} \; . \tag{145}$$

For example, consider Φ_1 to be the displacement transfer function for a seismometer (18) with damping ζ and natural frequency ω_s, and the feedback component transfer function being a constant $\Phi_2 = k$. In this case the transfer function of the fed back system is

$$\Phi_{fb}(\omega) = \frac{\frac{-\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}}{1 - \frac{k\omega^2}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}} = \frac{-\omega^2}{(1-k)\omega^2 - 2\imath\zeta\omega - \omega_s^2} \tag{146}$$

which has poles at

$$\omega_{fb} = \frac{\imath\zeta \pm \sqrt{(1-k)\omega_s^2 - \zeta^2}}{1-k} \tag{147}$$

instead of the original poles at

$$\omega = \imath\zeta \pm \sqrt{\omega_s^2 - \zeta^2} = \imath\zeta \pm \omega_1 \; . \tag{148}$$

A plot of the poles of the function in $z = \imath\omega_{fb}$ complex plane (Figure 11); see the ancillary Poles and Zeros notes), shows the system behavior as k is increased from zero for an initially $\omega_s = 2\pi$ rad/s underdamped seismometer with $\zeta = 0.1\omega_s$. The damping increases as k increases (the ratio (real(z)/imag(z) increases), and the system response approaches critical damping. As the amount of feedback is increased, the system response approaches that of a very-long period seismometer.

Feedback is the essence of modern broadband seismometer design; the feedback makes it possible to build portable stable, low noise instruments with periods $T = 2\pi/\omega_{fb}$ as long as several hundred seconds. An added technical advantage associate with feedback is, if there is enough gain in Φ_2 so that (145) becomes

$$\Phi_{fb}(\omega) \approx \frac{y(\omega)}{x(\omega)} \approx \frac{1}{\Phi_2(\omega)} \; . \tag{149}$$

This is a remarkable, and very important result, because in this case the system response can be designed to be effectively completely dependent on the feedback components of the system, Φ_2 (which are typically electronic), rather than on less controllable mechanical seismometer components inherent in Φ_1.

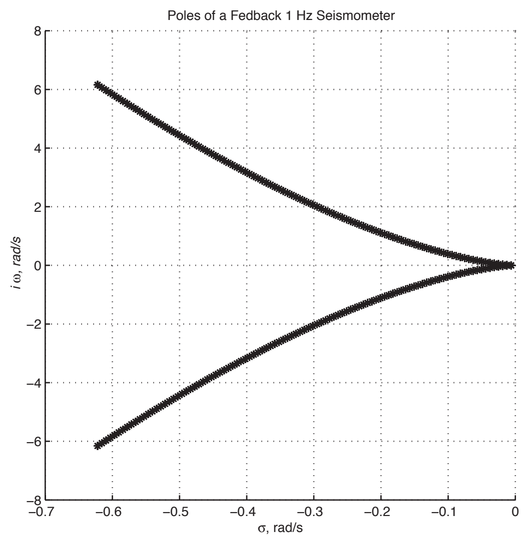

Figure 11: Poles for a seismometer with simple feedback ($0.01 \leq k \leq 0.99$).

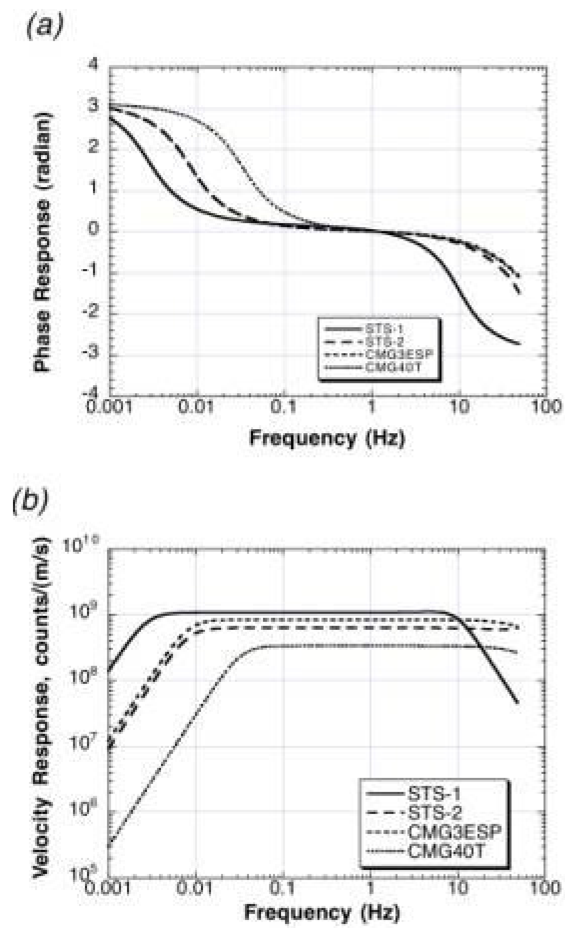

Figure 12: Phase and Velocity responses of some modern seismograph systems used in PASSCAL, GSN, and other networks, c/o the Incorporated Research Institutions for Seismology (IRIS).

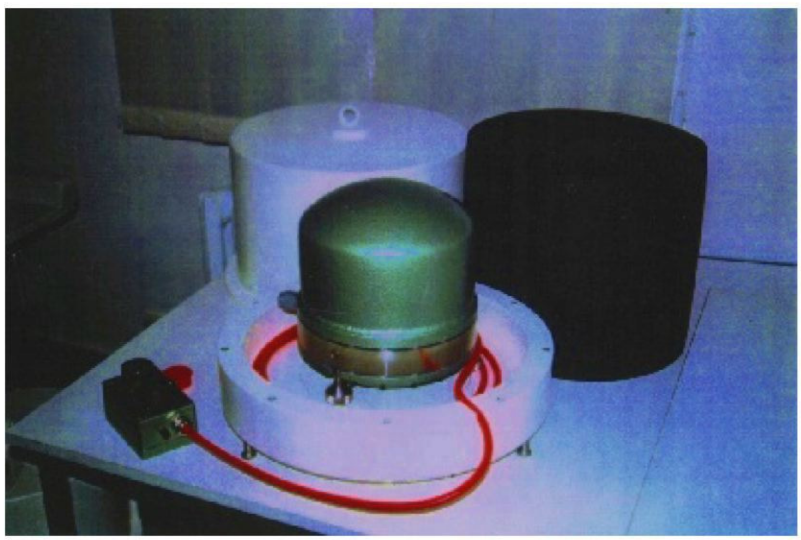

Figure 13: An STS2 seismometer.