Data Processing and Analysis

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Introduction to Linear Systems, Part 1: The Time Domain

Our primary goal in this course is to understand methods of analyzing temporal and spatial series, especially as applied to *linear systems*, both in continuous and sampled (discrete) time, and to demonstrate applications to important problems in geophysics and other physical sciences. Much of the demonstration and homework in this course will be done using MATLAB. You are thus encouraged to demo and/or refamiliarize yourself with this package at the earliest opportunity. There is also a MATLAB primer on the class web page.

We will be mostly concerned with an important class of physical situations that can be adequately characterized by *linear systems.* A linear system is a functional transformation, ϕ, which converts an *input* signal, $x(t)$ to an *output* signal, $y(t)$

$$y(t) = \phi[x(t)] \tag{1}$$

and which follows the principles of superposition

$$\phi[x_1(t) + x_2(t)] = \phi[x_1(t)] + \phi[x_2(t)] \tag{2}$$

and scaling

$$\phi[\alpha x(t)] = \alpha\phi[x(t)] \tag{3}$$

where α is a scalar. Note that for positive integer values of α (3) is equivalent to (2). (3) also implies that the output of the system is zero when there is no input

$$\phi[0] = 0\ . \tag{4}$$

Many of the phenomena which we wish to study in geophysics and other areas of science are linear. Sometimes we study very weak perturbations to a physical system (e.g., small gravity variations, seismic disturbances far away from the source; effects due to small fluctuations in the magnetic field) and the linear approximation is valid because the system is not tweaked very far from equilibrium. Common situations where linearity does not hold up are generally instances of large amplitude (e.g., high strain elastic waves near an underground nuclear explosion or earthquake; ocean waves breaking at a shoreline). In these cases the physics of the problem depends strongly on the amplitude of the perturbation, so that superposition (2) and scaling (3) do not hold (and aren't even acceptable approximations).

Many interesting systems are also *time-invariant*, i.e., the functionality of ϕ is not time dependent. In some situations, of course we intentionally look for gradual time variations in a system response, but these usually take place on time scales greater than the duration of our signals of interest. For example, earthquake prediction researchers hope that this is not the case for some aspect of evolving earth response in an incipient mainshock region.

A linear system is said to be *causal* if the output at time t_0 depends only on values of the input for $t \leq t_0$. Note that all physical processes are causal (as acausal systems propagate information backwards in time!). It is very easy mathematically, to construct non-causal mathematical systems, and these formulations may be useful in processing stored information. Also keep in mind that physical spatial phenomena (e.g. spatial filters) need not obey "causality" constraints.

A linear system is said to be *stable* if every noninfinite input produces a noninfinite output. While obvious for systems in the physical world (which will become non-linear in some manner rather than produce an infinite output) stability is important consideration in mathematical models of active systems (i.e., systems that have feedback between output and input).

The simple rules defining linear systems provide far-ranging and very useful constraints on the mathematical characterization of the system. Most importantly, linear systems are especially tractable, and very useful analysis tools, embodied in *Fourier Theory* describes their behavior complementary domains of time and frequency.

It may at first appear remarkable that the input to output transformation of *any* linear, time-invariant system can be characterized by a general integral relation (a *convolution*). To derive this result, we must first define the *Dirac delta* or *impulse function*. The delta function is discontinuous; it is nonzero only exactly where its argument is zero, where it is infinite. One way of conceptualizing the delta function (and to make it mathematically rigorous) is to define it as a limiting set of functions. One definition (e.g., Bracewell) is:

$$\delta(t) = \lim_{\tau \to 0} \ \tau^{-1} \Pi(t/\tau) \tag{5}$$

where $\tau^{-1}\Pi(t/\tau)$ is the unit-area rectangle or *boxcar* function of height τ^{-1} and width τ. The limit of (5) as τ approaches zero is an infinitesimally narrow pulse of infinite amplitude centered on $t = 0$, and having unit area. It can be shown that one need not start with the rectangle function to obtain the same functional limit, we could just as easily have considered a limit of any set of unit-area functions (e.g., an appropriately scaled set of Gaussians). Although the delta function may seem outrageously artificial, it actually has a plethora of analytical uses in the theory of physical and theoretical system behavior.

The usefulness of $\delta(t)$ in our present context arises from its *sifting property*, whereby it can retrieve a functional value at a particular argument from within an integral

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0) \tag{6}$$

$$= \; f(t_0) \quad a \;\le\; t_0 \;\le\; b \tag{7}$$

$$= \; 0 \quad \text{elsewhere} \tag{8}$$

for any $f(t)$ continuous at finite $t = t_0$.

The delta function is one of several related discontinuous functions which will be of use to us. Another is the *step function*

$$\mathrm{H}(t-t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau \tag{9}$$

which is 0 for $t < t_0$, 1 for $t > t_0$, and takes a discontinuous step at $t = t_0$. The step function is a useful mathematical construction for "turning on" a system at $t = t_0$.

We can define the *boxcar function*, $\Pi(t)$, and *sign function*, $\mathrm{sgn}(t)$, in terms of $\mathrm{H}(t)$

$$\Pi(t) = \mathrm{H}(t+1/2) - \mathrm{H}(t-1/2)\ . \tag{10}$$

$$\mathrm{sgn}(t) = \frac{|t|}{t} = 2\mathrm{H}(t) - 1\ . \tag{11}$$

$\mathrm{sgn}(t)$ is also sometimes referred to as the *signum* function.

The *impulse response* of a system is the output produced by an impulse function input

$$h(t) = \phi[\delta(t)]\ . \tag{12}$$

We will now show the important result that the response of a linear, time-invariant system to an arbitrary input is characterizable as a convolution. First, note that any input signal, $f(t)$, can be written as a summation of impulse functions because of the sifting property (8) of the delta function

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\ d\tau\ . \tag{13}$$

Thus, for a general linear system characterized by an operator, ϕ, the response, $g(t)$, to an arbitrary input, $f(t)$, is just that operator acting on (13)

$$g(t) = \phi[f(t)] = \phi\left[\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau\right] \tag{14}$$

or, from the definition of the integral,

$$g(t) = \phi\left[\lim_{\Delta\tau\to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)\delta(t-\tau_n)\Delta\tau\right]\ . \tag{15}$$

If ϕ characterizes a linear process, we can move it inside of the summation using the superposition (2) and scaling (3) relations, where the $f(\tau_n)$ become scalar weights

$$g(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} f(\tau_n)\phi[\delta(t-\tau_n)]\Delta\tau \ . \tag{16}$$

If ϕ is time-invariant, then $\phi[\delta(t-\tau_n)]$ is just the time-lagged impulse response, $h(t-\tau_n)$ (12). Making this substitution, (16) then converges to the integral

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \ . \tag{17}$$

Equation (17) is the *convolution integral*, or simple the *convolution*, of $f(t)$ and $h(t)$, often written in shorthand as

$$g(t) = f(t) * h(t) \ . \tag{18}$$

Thus, convolution of a general input signal with an appropriate impulse response exactly describes the corresponding output signal, and this is true for *any* linear time-invariant system. An important observation regarding (17) is that convolution is a "smearing" action where the input function is typically broadened by the impulse response function. For example, a linear time-invariant measurement apparatus which records signals from the outside world exactly would have a delta function impulse response (so that its output, given by the convolution of an impulse and the real world signal would exactly match the desired observable). To see this, note that (13) is itself a convolution; convolution with a delta function simply returns the input signal, shifted in time (delayed or advanced) by the delta function's origin time

$$f(t) * \delta(t-t_0) \ = \int_{-\infty}^{\infty} f(\tau)\delta(t-t_0-\tau) \ d\tau = f(t-t_0) \ . \tag{19}$$

A "perfect" recording instrument would thus have a delta function impulse response.

As all functions can be thought of as continuous integral superpositions of delta functions (13) it is clear that a necessary and sufficient condition for system stability is that the impulse response be bounded for all t.

Convolution with a step function

$$\int_{-\infty}^{\infty} f(\tau)\mathrm{H}(t-\tau) \ d\tau = \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{t} \delta(\xi-\tau)d\xi \ d\tau \tag{20}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t} f(\tau)\delta(\xi-\tau)d\xi \ d\tau \tag{21}$$

$$= \int_{-\infty}^{t} \int_{-\infty}^{\infty} f(\tau)\delta(\xi-\tau) \ d\tau d\xi = \int_{-\infty}^{t} f(\xi) \ d\xi \tag{22}$$

is the definite integral of f from $t = -\infty$ up to time t. Thus, while convolution with a delta function returns the system impulse response, convolution with a step function performs the definite integration operation.

$\delta(t)$ can usefully be regarded as the time derivative of $\mathrm{H}(t)$. The significance of convolution with the time derivative of $\delta(t)$ is left as an exercise.

Another useful function for the analysis of linear systems is the *sampling function* (Bracewell's *shah* function)

$$r\text{Ш}(rt) = \sum_{n=-\infty}^{\infty} r\delta(rt - n) \ . \tag{23}$$

Multiplication by $\text{Ш}(rt)$ produces a continuous time representation of a *sampled* time series, with nonzero weighted impulses at $t = (..., -2/r, -1/r, 0, 1/r, 2/r, ...)$, where the weights are the values of the original function at those points. r is referred to as the *sampling rate* (the additional factor of r in (23) is required to maintain unit-area delta functions). Sampled time series (not necessarily in one dimension, but frequently in 2 or more dimensions, and usually uniformly sampled in time or space) make up the vast majority of geophysical and other types of scientific data.

Time domain interpretation of convolution. A way to develop further insight into convolution is to graphically examine the operation of the convolution integral

$$c(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) \ d\tau \ . \tag{24}$$

The procedure is as follows:

1. Plot both $f_1(\tau)$ and $f_2(t-\tau)$ on the τ-axis. Note that this operation flips the function $f_2(\tau)$ about the τ-axis and shifts it by an amount t (which is the independent variable of the output function $c(t)$).

2. Visualize that as t advances, $f_2(t - \tau)$ slides along the τ-axis.

3. For each t, the convolution integral (24) gives the area of the product $f_1(\tau) \cdot f_2(t - \tau)$.

As an example, consider the convolution of $\Pi(t)$ (10) and a truncated exponential, $e^{-t}\mathrm{H}(t)$.

$$c(t) = \int_{-\infty}^{\infty} \Pi(\tau)\mathrm{H}(t - \tau)e^{-(t-\tau)} \ d\tau \ . \tag{25}$$

Because of the discontinuities in $\Pi(t)$, the solution is found by examining three cases:

- Case (a) $t \ \leq \ -1/2$

  The nonzero portions of the functions do not overlap, and $c(t) = 0$.

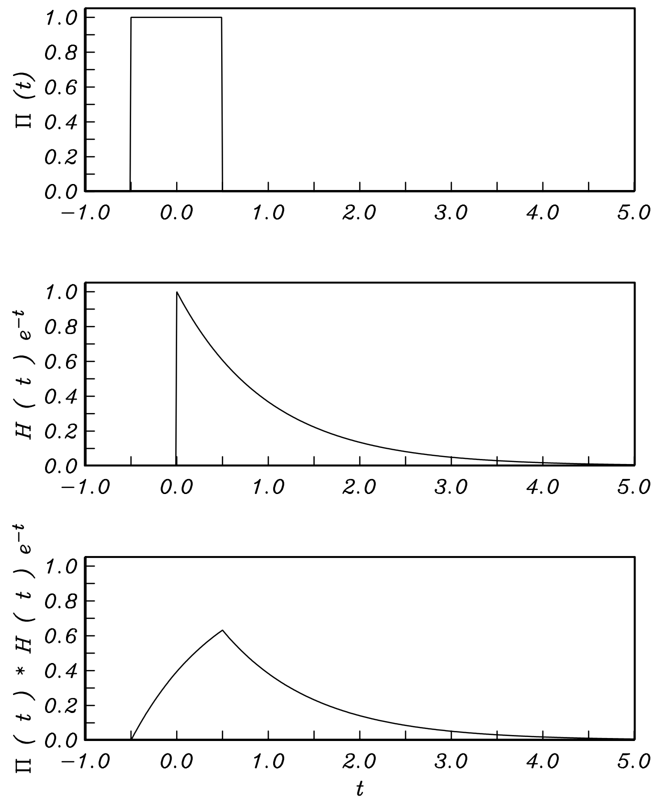

Figure 1: Convolution Example.

- Case (b) $-1/2 \ \leq \ t \ \leq 1/2$

  The sliding exponential partially overlaps the boxcar function. The appropriate integral is

$$c(t) = \int_{-1/2}^{t} 1 \cdot e^{-(t-\tau)} \ d\tau = 1 - e^{-(t+1/2)} \ . \tag{26}$$

- Case (c) $t \ \geq \ 1/2$

  The sliding exponential completely overlaps the boxcar function. The integral is

$$c(t) = \int_{-1/2}^{1/2} 1 \cdot e^{-(t-\tau)} \ d\tau = e^{-(t-1/2)} - e^{-(t+1/2)} \ . \tag{27}$$

The result of this convolution is plotted in Figure 1. Note that we could have equivalently written the convolution as

$$c(t) = \int_{-\infty}^{\infty} \Pi(t-\tau)\mathrm{H}(\tau)e^{-\tau}\; d\tau \; . \tag{28}$$

This produces the same answer with somewhat different integrals. A more efficient and elegant way of evaluating convolutions will become apparent after we learn how to examine functions in the *frequency domain*, rather than the *time domain*, as we have done here.

Autocorrelation and crosscorrelation. Several other integral operations, commonly used in time and spatial series analysis are closely related to convolution.

Autocorrelation is similar to *autoconvolution*

$$f(t) * f(t) = \int_{-\infty}^{\infty} f(\tau)f(t-\tau)\; d\tau \tag{29}$$

except that one of the functional components in the τ-domain is *not* time-reversed. The autocorrelation of a real function, $f(t)$, is

$$A(t) = \int_{-\infty}^{\infty} f(\xi)f(\xi-t)\; d\xi = \int_{\infty}^{-\infty} f(\xi-t)f(\xi)\; (-d\xi) \tag{30}$$

which is, if we let $\xi - t = -\tau$,

$$= \int_{-\infty}^{\infty} f(-\tau)f(t-\tau)\; d\tau = f(-t) * f(t) = f(t) * f(-t) \; . \tag{31}$$

If $f(t)$ is symmetric in time (an *even function*; $f(t) = f(-t)$), then the autoconvolution and autocorrelation are equal. Also, because the autocorrelation integral (31) is unchanged when we interchange $\pm t$, we see that autocorrelation always produces an even function.

It is often convenient to divide (31) by the signal energy to obtain a normalized autocorrelation form

$$a(t) = \frac{A(t)}{\int_{-\infty}^{\infty} f^2(\tau)\; d\tau} \; . \tag{32}$$

(32) is bounded on the interval $[-1,\; 1]$. Note that for (32) and (31) to converge, the *signal energy*

$$E = A(0) = \int_{-\infty}^{\infty} f^2(\tau)\; d\tau \tag{33}$$

must be finite. It is thus necessary for $f^2(t)$ to have finite area (zero mean alone is not sufficient).

The *crosscorrelation* of two functions, $f_1(t)$ and $f_2(t)$ (often referred to simply as the *correlation*) is

$$C(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(\tau-t)d\tau = \int_{-\infty}^{\infty} f_1(\tau+t)f_2(\tau)d\tau = f_1(t) \star f_2(t) \tag{34}$$

If (34) is divided by the *cross-signal energy* we have a normalized version of the crosscorrelation corresponding to (32)

$$c(t) = \frac{C(t)}{\sqrt{\int_{-\infty}^{\infty} f_1^2(\tau)d\tau \cdot \int_{-\infty}^{\infty} f_2^2(\tau)d\tau}} \tag{35}$$

produces a value of one at zero time lag when the two functions are identical. Autocorrelation and correlation have important applications in power spectra, coherency, signal detection and timing, and array processing.

Correlations and Crosscorrelations in MATLAB. MATLAB has built in convolution (*conv*), and crosscorrelation (*xcorr*) functions in the Signal Processing toolbox. The numerical operations of MATLAB, of course, only operates on finite time series (or *sampled*) representations of functions stored as vectors or arrays of numbers. In time series analysis, such vectors or arrays are used to represent continuous functions. We will examine the issues associated with sampled functions in considerable detail detail later in the course. The MATLAB *conv* function thus calculates a sample-by-sample moving dot-product rather than an integral. The *xcorr* function includes a *coeff* option that performs the normalization of (32) or (35) Note that, because these MATLAB operations are simply moving vector dot products as functions of lag between the two functions, you will have to scale the results by the sampling interval to get results that agree with continuous integral values (i.e., there is no corresponding $d\tau$ factor in these calculations). You are encouraged to experiment with these MATLAB functions. Note that if you operate on two MATLAB time series, a_1 and a_2, which are of length n_1 and n_2 samples, respectively, then the convolution output from *conv*, $a_1 * a_2$ will be of length $(n_1 + n_2 + 1)$.

Here is MATLAB code that performs the above convolution example (Figure 1):

```
%MATLAB demonstration of example convolution in notes, part 1
%clear any old variables
clear

%total length of f1, f2 time series in seconds
N=10;

%time step size in seconds
dt=0.02;

%length of vectors to create
M=N/dt;

%zero time reference point
ztime=M/4;
```

```
%here is the boxcar function
%initialize f1
f1=zeros(M,1);
%insert ones into the correct elements
f1((ztime-0.5/dt):ztime+(0.5/dt))=ones(1+1/dt,1);

%here is the decaying exponential function (starting at zero time);
%initialize f2
f2=zeros(M,1);
%insert a decaying exponential into the correct elements
f2(ztime:M)=exp(-dt*(0:M-ztime));

%create the time axis vector for plotting f1 and f2
taxis = ((1:M)-ztime)*dt;

%plot f1
figure(1)
plot(taxis,f1)
grid
title('f_1(t)')
xlabel('time')

%plot f2
figure(2)
plot(taxis,f2)
grid
title('f_2(t)')
xlabel('time')

%do the convolution (multiplied by dt to make the sum an approximate integral)
c=conv(f1,f2)*dt;

%create the appropriate time axis vector for the convolution (which has length(c)=2*M-1) for
taxisc=((1:length(c))-2*ztime)*dt;

%plot the convolution
figure(3)
plot(taxisc,c)
grid
title('c(t)=f_1(t)*f_2(t)')
xlabel('time')
```