Data Processing and Analysis

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Notes on Fourier Transforms and Convolution

These notes summarize some important facts about Fourier transforms and convolutions. Note that there are several different definitions of the Fourier transform in common use. If you refer to other books which use different definitions then of course the formulas will be different. These notes have been made consistent with the following definition of the Fourier transform and its inverse.

$$\mathcal{F}(\phi(t)) = \Phi(f) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ift}dt$$

$$\mathcal{F}^{-1}(\Phi(f)) = \phi(t) = \int_{-\infty}^{\infty} \Phi(f)e^{2\pi ift}df$$

The notation $\phi(t) \supset \Phi(f)$ is also used to indicate that $\Phi(f)$ is the Fourier transform of $\phi(t)$. Another common notational convention is that capital letters are used for the names of functions in the frequency domain, and corresponding lower case letters are used for functions in the time domain. In this notation, s is often used in place of f for frequency. Thus $F(s)$ denotes the Fourier transform of $f(t)$.

Because these two definitions are very nearly symmetric (note the sign change in the exponent of e), it is possible to use a Fourier transform pair $\phi(t) \supset \Phi(f)$ to construct a second Fourier transform pair $\Phi(t) \supset \phi(-f)$. For example, the transform

$$\mathcal{F}(\text{sgn } t) = \frac{1}{\pi if}$$

translates into

$$\mathcal{F}\left(\frac{1}{\pi it}\right) = \text{sgn} - f = -\text{sgn } f$$

The operation of convolution is defined by

$$\phi(t) * \psi(t) = \int_{-\infty}^{\infty} \phi(\tau)\psi(t-\tau)d\tau$$

Note that a simple change of variables shows that

$$\phi(t) * \psi(t) = \int_{-\infty}^{\infty} \phi(t-\tau)\psi(\tau)d\tau$$

Thus convolution is commutative. It turns out that convolution also satisfies the associative property and that convolution and addition have a distributive property.

General Properties

$\phi(t)$	$\Phi(f)$
$a\phi(t) + b\psi(t)$	$a\Phi(f) + b\Psi(f)$
$\phi(at)$	$\frac{1}{\lvert a \rvert}\Phi(f/a)$
$\phi'(t)$	$2\pi i f\Phi(f)$
$e^{2\pi iat}\phi(t)$	$\Phi(f-a)$
$\phi(t-a)$	$e^{-2\pi iaf}\Phi(f)$
$\phi(t)\cos(2\pi f_0 t)$	$(\Phi(f-f_0)+\Phi(f+f_0))/2$
$\phi(t) * \psi(t)$	$\Phi(f)\Psi(f)$
$\phi(t)\psi(t)$	$\Phi(f) * \Psi(f)$
$\int_{-\infty}^{t}\phi(\tau)d\tau$	$\frac{\Phi(f)}{2\pi if} + \frac{\delta(f)}{2}\int_{-\infty}^{\infty}\phi(\tau)d\tau$

Parseval's Theorem

$$\int_{-\infty}^{\infty}\phi(t)\phi^*(t)dt = \int_{-\infty}^{\infty}\Phi(f)\Phi^*(f)df.$$

Symmetry Properties

$\phi(t)$	$\Phi(f)$
even	even
odd	odd
real, even	real, even
real, odd	imaginary, odd
imaginary, even	imaginary, even
imaginary, odd	real, odd
complex, even	complex, even
complex, odd	complex, odd
real, asymmetrical	complex, Hermitian
imaginary, asymmetrical	complex, anti-Hermitian
Hermitian	real
anti-Hermitian	imaginary

Specific Fourier Transform Pairs

$\phi(t)$	$\Phi(f)$
$\delta(t)$	1
1	$\delta(f)$
$e^{-a\pi t^2}$	$\frac{1}{\sqrt{a}}e^{-a\pi f^2/a}$
$\mathrm{II}(t)$	$\text{sinc } f = \frac{\sin \pi f}{\pi f}$
$\mathrm{III}(t)$	$\mathrm{III}(f)$
$\frac{1}{t}$	$-\pi i \text{sgn } f$
$\text{sgn } t$	$\frac{1}{\pi i f}$
$\cos(2\pi f_0 t)$	$(\delta(f + f_0) + \delta(f - f_0))/2$
$\sin(2\pi f_0 t)$	$i(\delta(f + f_0) - \delta(f - f_0))/2$
$H(t)$	$\frac{1}{2\pi i f} + \frac{\delta(f)}{2}$
$e^{-t}H(t)$	$\frac{1}{1+2\pi i f}$

Other Definitions of the Fourier Transform

Much more extensive tables of Fourier transforms are available in various reference books. These tables are often based on slightly different definitions of the Fourier transform. One typical variation is using $+2\pi ift$ in the exponent in the Fourier transform formula instead of $-2\pi ift$. Another common variation involves pulling the 2π out of the exponential and putting a factor of 2π in front of the integral in either the forward or inverse transform or putting a factor of $\sqrt{2\pi}$ in both the forward and inverse transforms. Transforms based on these alternative definitions can be converted to our system without too much difficulty.

For example one common definition of the Fourier transform is

$$\Phi_{\text{alt}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-i\omega t} dt$$

Here Φ_{alt} denotes the Fourier transform under the alternate definition. Under this definition,

$$\Phi_{\text{alt}}(2\pi f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-i2\pi ft} dt.$$

Thus

$$\Phi_{\text{alt}}(2\pi f) = \frac{1}{\sqrt{2\pi}} \Phi(f)$$

where $\Phi(f)$ is the transform under our definition. Solving for $\Phi(f)$, we get

$$\Phi(f) = \sqrt{2\pi}\,\Phi_{\text{alt}}(2\pi f).$$