Time Series/Data Processing and Analysis (MATH 587/GEOP 505)

Rick Aster and Brian Borchers

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Plotting Spectra Using Decibels

Because the amplitudes describing the spectral responses of physical systems in nature as well as filters and instruments, frequently span many orders of magnitude, amplitude responses are frequently plotted as a function of frequency using either as log-linear or log-log displays. The standard way to do this is using a *decibel* (dB) scale. The 'Bel' was originally a unit of sound intensity, after Alexander Graham Bell- a decibel is one tenth of a Bel.

The decibel relationship between two power levels is defined as

$$d = 10 \log_{10} \frac{P_1}{P_2} \tag{1}$$

In examining system responses, P_1/P_2 in (1) is commonly the ratio of output power level over input power level so that 0 dB corresponds to unit gain and amplification by a factor greater than one corresponds to $d > 0$.

Note that the definition we have given is for power ratios. In practice it is often necessary to consider voltage ratios as well. Assuming that the power is dissipated by a load of constant resistance R, the power is $P = V^2/R$, so

$$d = 10 \log_{10} \frac{P_1}{P_2} = 10 \log_{10} \frac{V_1^2/R}{V_2^2/R} = 10 \log_{10} \frac{V_1^2}{V_2^2} = 10 \log_{10} \left(\frac{V_1}{V_2}\right)^2 = 20 \log_{10} \frac{V_1}{V_2}.$$

Similarly, in seismic work power is proportional to velocity squared, so the factor of 20 is used. In general, the factor of 20 is used with amplitudes and the factor of 10 is used with power. An amplitude change of a factor of two is equal to about 6 dB, because $\log_{10} 2 = 0.3010$. An amplitude factor of $\sqrt{2}$ is equal to about 3 dB, and so forth.

An obvious question is what voltage to use for a time varying signal? The most common convention is to use the root mean square (RMS) average voltage.

$$V_{\mathrm{RMS}} = \sqrt{\frac{\int_0^T V(t)^2 dt}{T}}$$

If not specified you can reasonably assume that RMS voltage was intended.

Decibels are also conveniently used to express rates of exponential falloff in a system response. This is especially common in engineering specifications. For example, a response that is proportional to $1/f$ (e.g., a single-pole system with no zeros such as a simple RC low-pass analog filter) decays at (approximately) $6 \approx 20\log_{10}(2)$ dB per octave (per frequency doubling), or at $20 = 20\log_{10}(10)$ dB per decade (per 10-fold frequency increase.)

In a dB vs. log frequency plot, such asymptotic power law behavior is easy to predict (and sketch), because a falloff of f^{-n} is just a straight line with a slope of $20n$ dB/decade. One can thus approximately sketch the amplitude response of system as a set of simple lines with differing slopes (such plots are called *Bode plots*).

Recall from the lecture notes on linear time invariant systems in the frequency domain that the frequency response of an underdamped seismometer is given by

$$|\Phi(\omega)| = \frac{\omega^2}{\sqrt{(\omega^2-\omega_2^2)^2+4\xi^2\omega^2}}. \tag{2}$$

Expanding this in a Taylor series around $\omega = 0$, we get that

$$|\Phi(\omega)| = \frac{\omega^2}{\omega_s^2} + O(\omega^4). \tag{3}$$

Since $\omega = 2\pi f$,

$$|\Phi(f)| = O(f^2) \tag{4}$$

as f goes to 0. We would expect to see the frequency response drop off by 40 dB/decade or 12 dB/octave as f approaches 0.

Figures 1, 2, and 3 show the amplitude displacement-displacement response of an underdamped seismometer with $\zeta = \omega_s/\sqrt{2}$ in linear-linear, dB (log)-linear, and dB-log plots, respectively. Note that the Figure 3 plot is most easily interpretable, shows the essential characteristics of $|\Phi(f)|$ most clearly, and has the expected quadratic ($O(f^2)$) response fall-off of 40 dB/decade at low frequencies.

Note that we have looked the behavior of $|\Phi(f)|$ as f goes to 0. Some other authors consider the behavior of the power spectral density, $|\Phi(f)|^2$, as f goes to 0. Following that convention, the power spectral density is $O(f^4)$ as f approaches 0. However, in either case we would still see a decrease of 40 dB/decade on the plot.

So far, we have only used dB's as a measure of relative power. In situations where we want to establish an absolute scale, we must first pick a reference level for 0 dB. For example, in acoustics, a commonly used reference level for the sound pressure level (SPL) is an amplitude of 20 μPa in air. In this system, the unit is "dB(SPL)". In electronics, a commonly used reference level is one milliwatt. In this system the unit is "dBm". A voltage based scheme that is independent of the particular impedance uses a reference level of 0.775 V. This voltage happens to produce 1 milliwatt of power in a 600 ohm resistor (600 ohms

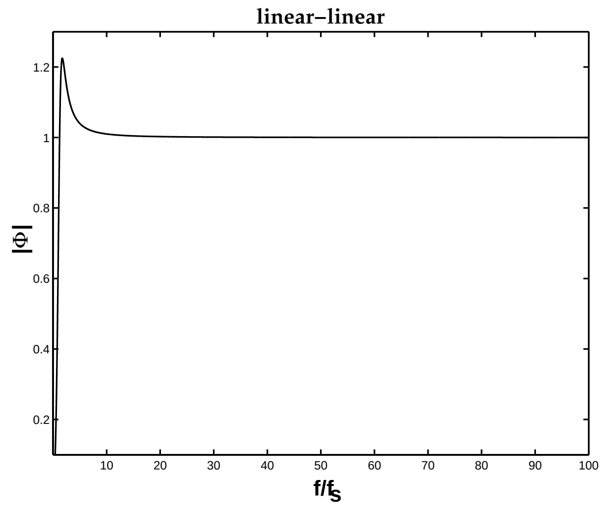

Figure 1: Linear-linear plot of the amplitude response of a seismometer, $\zeta = \omega_s/\sqrt{2}$.

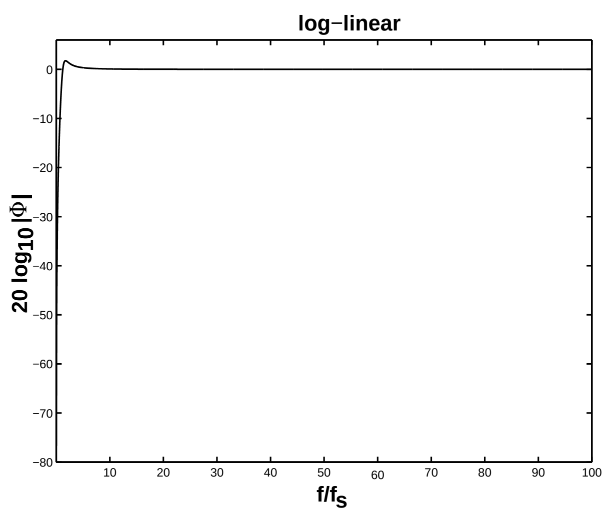

Figure 2: Log-linear plot of the amplitude response of a seismometer, $\zeta = \omega_s/\sqrt{2}$.

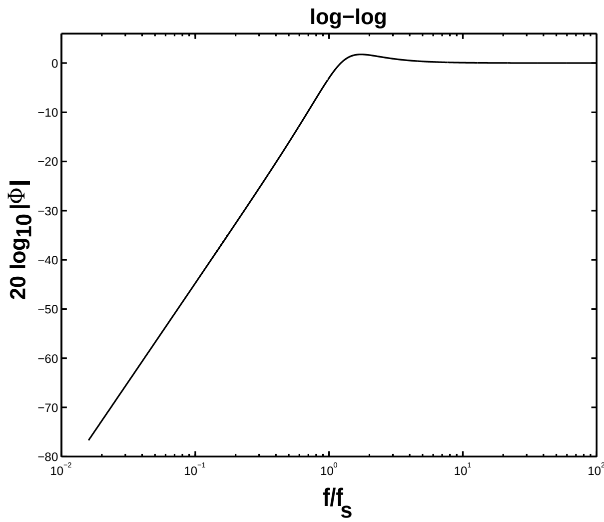

Figure 3: Log-log plot of the amplitude response of a seismometer, $\zeta = \omega_s/\sqrt{2}$.

is the standard impedance in audio work.) In this system, the unit is "dBu" for "unloaded". You'll also see "dBV" for voltage relative to 1 volt, and "dBv" which is an older name for "dBu".