Data Processing and Analysis (GEOP 505)

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Finding an Impulse Response via Contour Integration

In the Chapter 2 notes, we noted that the time domain displacement response to an acceleration impulse input is

$$\phi(t) = F^{-1}\left(\frac{1}{\omega^2 - 2\imath\zeta\omega - \omega_s^2}\right) \tag{1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\, d\omega}{\omega^2 - 2\imath\zeta\omega - \omega_s^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath\omega t}\, d\omega}{(\omega - \omega_1 - \imath\zeta)(\omega + \omega_1 - \imath\zeta)} \tag{2}$$

where

$$\omega_1 = \sqrt{\omega_s^2 - \zeta^2}\ . \tag{3}$$

To solve (2), we utilize a remarkable and useful theorem from complex analysis, the *residue theorem*. Succinctly stated, the residue theorem says that, for a complex function in the complex plane that is defined and differentiable with a region except at an isolated singularity at a finite point z_0 (e.g., a pole in a transfer function), then, for a closed path, or contour, C, encompassing the singularity

$$\oint_C f(z)dz = 2\pi\imath a \tag{4}$$

where a is called the *residue* of $f(z)$ at z_0, where, for a pole of order m,

$$a = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]_{z=z_0}\ . \tag{5}$$

for a single pole at z_0, the residue is simply

$$a = [(z - z_0)f(z)]_{z=z_0}\ . \tag{6}$$

Evaluation of a contour integral in the complex plane thus involves evaluating the integrand at $z = z_0$ with the pole "removed" by first multiplying by the

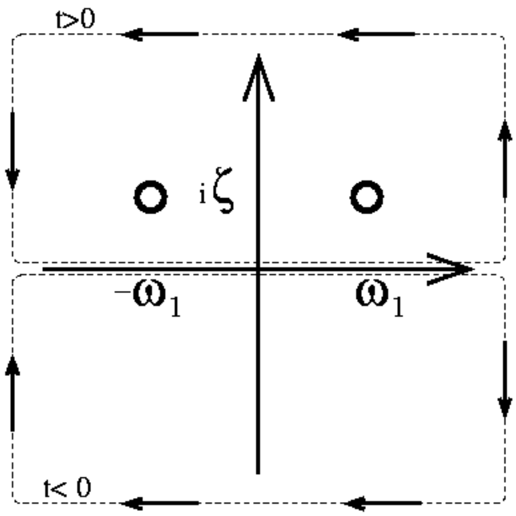

Figure 1: Contour integration in the complex ω plane.

factor $(z - z_0)$ More generally, if more than one distinct pole is enclosed by the integration path

$$\oint_C f(z)dz = 2\pi\imath \sum_i a_i \tag{7}$$

where the a_i are the residues at the encosed poles. If there are no poles enclosed by C, the integral will be zero (this is the same as saying that the function is *conservative*, or that the integral of $f(z)$ between two complex points doesn't depend on the integration path).

We can now use the residue theorem to evaluate the inverse Fourier transform (2). The poles of the integrand lie at $(\pm\omega_1, \imath\zeta)$. We conceptualize the inverse Fourier transform as a contour integration by integrating in the complex ω plane along the ω axis from $-\infty$ to ∞, and then closing the countour at $|z = \infty|$ (where the value of the integrand is zero). For $t < 0$ the contour is clockwise because of the $e^{\imath\omega t}$ factor and encompasses no poles (Figure 1). Thus

$$\phi(t) = 0 \ (t < 0) \ . \tag{8}$$

For $t > 0$ the contour is clockwise and encompases poles, so that the residue theorem gives

$$\phi(t) = \frac{\imath}{\omega_1}\left(\frac{e^{-\imath\omega_1 t}}{-2} + \frac{e^{\imath\omega_1 t}}{2}\right) e^{-\zeta t} \ . \tag{9}$$

For the underdamped case, where $\omega_2 > \zeta$, ω_1 is real, so that (setting the function to be zero for $t < 0$ with a step function) we have the impulse response

$$\phi_{underdamped} = \mathrm{H}(t)\frac{-1}{\omega_1}e^{-\zeta t}\sin(\omega_1 t)\ . \tag{10}$$

For the overdamped case, where $\omega_s < \zeta$, $\omega_1 = \imath\sqrt{\zeta^2 - \omega_s^2}$, and the poles lie on the negative real axis at $-\zeta \pm \sqrt{\zeta^2 - \omega_s^2}$. The impulse response function in this case can be written entirely with real exponentials as

$$\phi_{overdamped}(t) = \frac{-\mathrm{H}(t)}{2(\zeta^2 - \omega_s^2)^{1/2}}\left(e^{-(\zeta-(\zeta^2-\omega_s^2)^{1/2})t} - e^{-(\zeta+(\zeta^2+\omega_s^2)^{1/2})t}\right)\ . \tag{11}$$

For the critically damped case, $\omega_1 = 0$, and we have a repeated (order 2) pole at $\omega = \imath\zeta$. Application of (5) for $t > 0$ gives

$$\phi_{critical}(t) = \imath\frac{d}{d\omega}\mathrm{H}(t)e^{\imath\omega t}|_{\omega=\imath\zeta} = -\mathrm{H}(t)te^{-\zeta t}\ . \tag{12}$$