Cagniard-deHoop Solution for an Atmospheric Line Source

An isotropic line source is situated at a height x_3=-h in an inviscid fluid (Figure 1). This source is an approximation of a very high Mach number horizontally propagating space vehicle or meteor in the high atmosphere that produces a thin conical Mach cone.

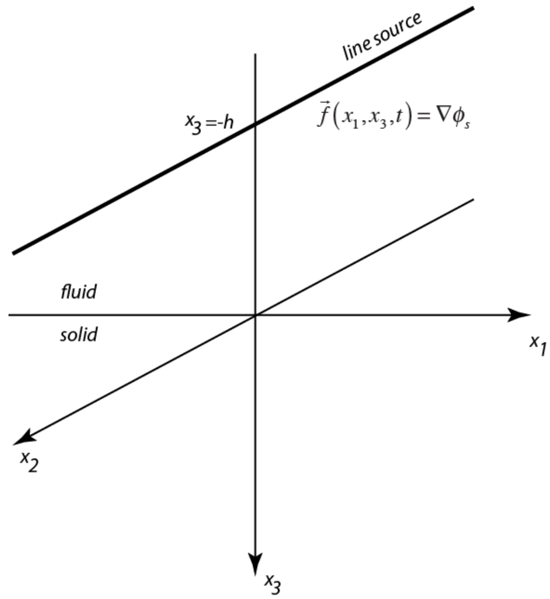

Figure 1 - Cartesian coordinate system for an isotropic line source situated in a fluid halfspace over an elastic solid halfspace.

The isotropic line source is represented by a body force given in terms of the gradient of scalar source potential.

$$\vec{f}(x_1,x_3,t)=\nabla\phi_s \ . \tag{1.1}$$

The wave equation for the P wave potential in the fluid halfspace is

$$\nabla^2\phi^f-\frac{1}{\alpha_f^2}\frac{\partial^2\phi^f}{\partial t^2}=-\frac{1}{\alpha_f^2}\phi_s \tag{1.2}$$

and the wave equations for the P and SV wave potentials in the solid halfspace are

$$\begin{aligned}\nabla^2\phi-\frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}=0\\ \nabla^2\psi-\frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}=0\end{aligned} \ . \tag{1.3}$$

Displacements can be found through the Helmholtz decomposition

$$\vec{u}=\nabla\phi+\nabla\times\vec{A} \tag{1.4}$$

which, for this two dimensional problem, reduces to

$$
\begin{aligned}
u_1 &= \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} \\
u_3 &= \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1}
\end{aligned} \quad . \tag{1.5}
$$

By inspection, the solution for the P wave potential in the fluid halfspace for a receiver in the fluid halfspace is

$$
\bar{\phi}^f\left(x_1, x_3, s\right) = \frac{F(s)}{2\pi\alpha_f^2}\left\{ \operatorname{Im}\int_0^{+i\infty} \frac{e^{-s\left(px_1 + \left|x_3 + h\right|\eta_{\alpha_f}\right)}}{\eta_{\alpha_f}} dp + \operatorname{Im}\int_0^{+i\infty} \frac{R_{PP} e^{-s\left(px_1 - \left(x_3 - h\right)\eta_{\alpha_f}\right)}}{\eta_{\alpha_f}} dp \right\} . \tag{1.6}
$$

The solution for the P wave potential in the solid halfspace is

$$
\bar{\phi}\left(x_1, x_3, s\right) = \frac{F(s)}{2\pi\alpha_f^2} \operatorname{Im}\int_0^{+i\infty} \frac{T_{pp} e^{-s\left(px_1 + h\eta_{\alpha_f} + x_3\eta_\alpha\right)}}{\eta_{\alpha_f}} dp \tag{1.7}
$$

and for the SV wave potential

$$
\bar{\psi}\left(x_1, x_3, s\right) = \frac{F(s)}{2\pi\alpha_f^2} \operatorname{Im}\int_0^{+i\infty} \frac{T_{ps} e^{-s\left(px_1 + h\eta_{\alpha_f} + x_3\eta_\beta\right)}}{\eta_{\alpha_f}} dp \ . \tag{1.8}
$$

The displacements on the surface of the solid halfspace can be found by evaluating (1.5) and letting $x_3 \to 0$ for the solid halfspace solutions for the potential (1.7) and (1.8):

$$
\begin{aligned}
\bar{u}_1\left(x_1, 0, s\right) &= \frac{sF(s)}{2\pi\alpha_f^2} \operatorname{Im}\int_0^{+i\infty} \frac{\left(pT_{pp} - \eta_\beta T_{ps}\right) e^{-s\left(px_1 + h\eta_{\alpha_f}\right)}}{\eta_{\alpha_f}} dp \\
\bar{u}_3\left(x_1, 0, s\right) &= \frac{sF(s)}{2\pi\alpha_f^2} \operatorname{Im}\int_0^{+i\infty} \frac{\left(\eta_\alpha T_{pp} - pT_{ps}\right) e^{-s\left(px_1 + h\eta_{\alpha_f}\right)}}{\eta_{\alpha_f}} dp
\end{aligned} \quad . \tag{1.9}
$$

Cagniard’s contour is found by setting

$$
t = px_1 + h\eta_{\alpha_f} \tag{1.10}
$$

which, for $t \geq \frac{r}{\alpha_f}$ and $r = \left(h^2 + x_1^2\right)^{\frac{1}{2}}$,

$$
p = \frac{x_1}{r^2} t + i\left(t^2 - \frac{r^2}{\alpha_f^2}\right)^{\frac{1}{2}} \tag{1.11}
$$

and

$$\frac{dp}{dt} = \frac{+i\eta_{\alpha_f}}{\left(t^2 - \frac{r^2}{\alpha_f^2}\right)^{\frac{1}{2}}} \quad . \tag{1.12}$$

Letting $f(t) = H(t)$, the time domain solution is

$$\begin{aligned} u_1(x_1,0,t) &= \frac{1}{2\pi\alpha_f^2} \frac{\mathrm{Re}\left[pT_{pp} - \eta_\beta T_{ps}\right]_C}{\left(t^2 - \frac{r^2}{\alpha_f^2}\right)^{\frac{1}{2}}} H\left(t - \frac{r}{\alpha_f}\right) \\ u_3(x_1,0,t) &= \frac{1}{2\pi\alpha_f^2} \frac{\mathrm{Re}\left[\eta_\alpha T_{pp} + pT_{ps}\right]_C}{\left(t^2 - \frac{r^2}{\alpha_f^2}\right)^{\frac{1}{2}}} H\left(t - \frac{r}{\alpha_f}\right) \end{aligned} \quad . \tag{1.13}$$

It is understood that the functions of p are evaluated along Cagniard's contour, C, equation (1.11).

Figure 2 shows two example calculations for an impulse source time history. The P and S wave velocities of the solid halfspace are very important in determining the nature of the acoustic-seismic response. The common case of a solid substrate with higher P and S wave velocities (Figure 2a) gives rise to P and S head waves along with a very impulsive, acoustically coupled Rayleigh wave before the direct acoustic wave arrives at the station. In comparison, a solid substrate with P and S wave velocities lower than atmospheric acoustic wave velocity (Figure 2b) only shows the direct acoustic wave from the source. The large, acoustically coupled Rayleigh wave is a common hazard in explosion work since coupling can cause strong damaging ground motions away from the source.

A counter-intuitive characteristic of the solution with low solid halfspace velocities is the unusual particle direction of the direct arrival. A compressional acoustic wave gives rise to downward particle displacement at the halfspace boundary as expected. However, the radial component is polarized back towards the source due to the P-to-S conversion set up at the fluid/solid boundary.

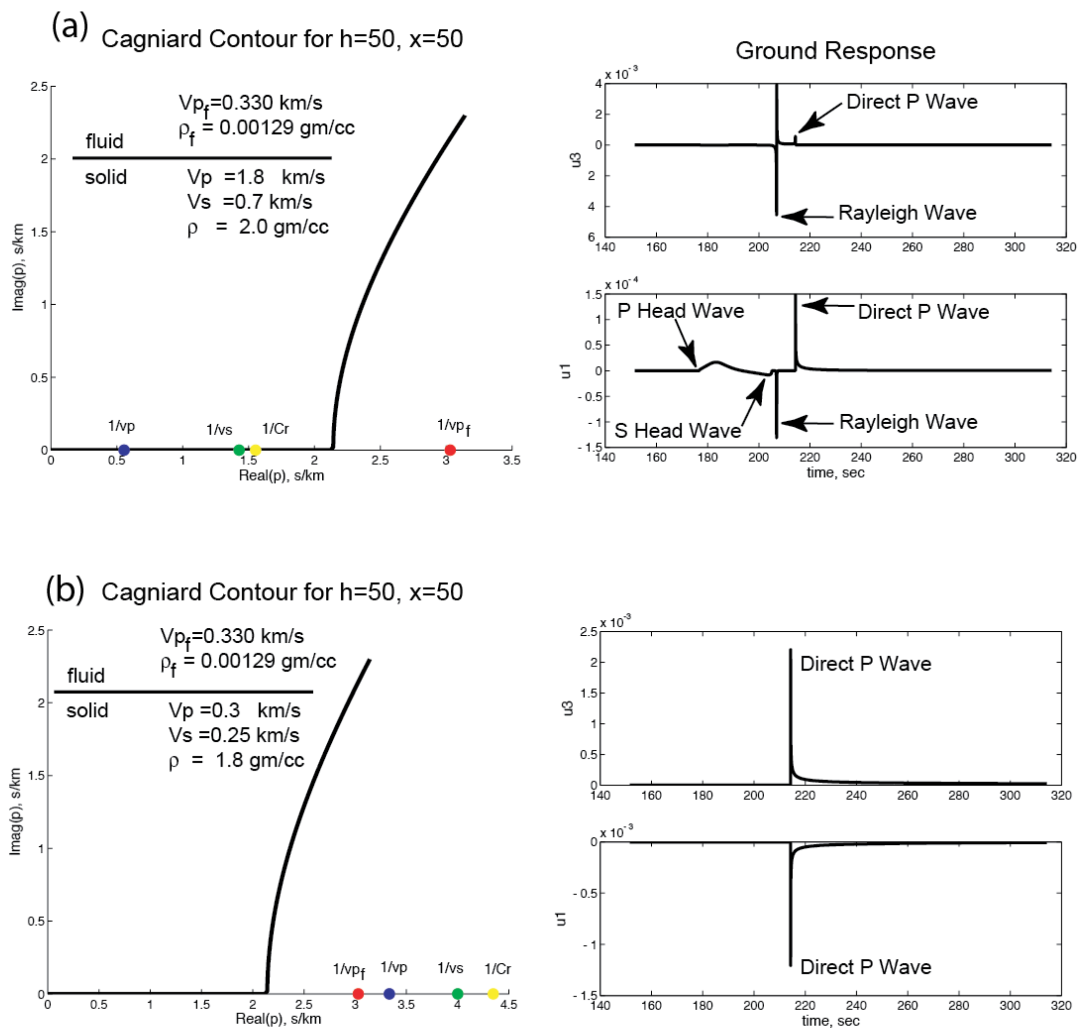

Figure 2 – Examples of the impulse response of an acoustic line source 50 km high in a halfspace atmosphere model. In (a) and (b), the receiver is located 50 km from the origin where the origin is located on the fluid/solid boundary. Top panels (a) show the path of the Cagniard contour (*left*) and the radial and vertical ground motion responses (*right*) for a relatively high velocity solid halfspace with average velocities similar to those found for Mississippi embayment sediments. Bottom panels (b) are arranged similarly but are for a low velocity halfspace more appropriate for the velocity of near-surface unconsolidated sediments. For (a), the location of the branch cuts for solid halfspace velocities predict P and S head waves along with the Rayleigh wave before the direct wave arrives from the acoustic source. The branch cuts for (b) are all to the right of p_0. Only the geometric direct arrival is predicted in the response. Recall that p_0 is the geometrical ray parameter for the generalized ray, here the direct acoustic arrival, and occurs at the point where the contour leaves the real axis.