Frequency/Wavenumber Analysis

1D Case

Assume that we have a continuous wavefield in t and x, $f(t,x)$. A seismic array consists of discrete spatial measurements of the wavefield at N locations in x. This spatially sampled wavefield can be represented by

$$f(x,t)=\sum_{j=1}^{N} f(x_j,t)\delta(x-x_j) \quad . \qquad (1)$$

The temporal and spatial Fourier transform of the wavefield is given by

$$\hat{\hat{F}}(k,\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,t)e^{-i\omega t}e^{-ikx}\,dt\,dx \quad . \qquad (2)$$

Substitute (1) into (2) and evaluate the integrals

$$\begin{aligned}
\hat{\hat{F}}(k,\omega)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sum_{j=1}^{N} f(x_j,t)\delta(x-x_j)e^{-i\omega t}e^{-ikx}\,dt\,dx \\
&=\sum_{j=1}^{N}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(x_j,t)e^{-i\omega t}\,dt\right\}\delta(x-x_j)e^{-ikx}dx \\
&=\sum_{j=1}^{N}\int_{-\infty}^{+\infty}\hat{F}(x_j,\omega)\delta(x-x_j)e^{-ikx}dx \\
&=\sum_{j=1}^{N}\hat{F}(x_j,\omega)e^{-ikx_j}
\end{aligned} \quad . \quad (3)$$

The "array response", $\hat{\hat{I}}(k,\omega)$, is defined when

$$f(x,t)=\delta(t) \qquad (4)$$

so that

$$\hat{\hat{I}}(k,\omega)=\sum_{j=1}^{N} e^{-ikx_j} \qquad (5)$$

and since $k=\omega p$, we can find the frequency-horizontal slowness response as

$$\hat{\hat{I}}_p(p,\omega)=\hat{\hat{I}}(\omega p,\omega)=\sum_{j=1}^{N} e^{-i\omega p x_j} \quad . \quad (6)$$

A "broadband" array response could also be defined as an average over a frequency band

$$\begin{aligned}\hat{\hat{I}}_{BB}(p,\omega) &= \int_{\omega_1}^{\omega_2} \hat{\hat{I}}_p(p,\omega) \\ &= \sum_{j=1}^{N} \frac{e^{-i\omega_2 p x_j} - e^{-i\omega_1 p x_j}}{-ipx_j}\end{aligned} \quad . \quad (7)$$

Here is an example of producing a 1D slowness-frequency spectrum for 24 channels of refraction data at 2m spacing. The source was a vertical hammer blow (Figure 1).

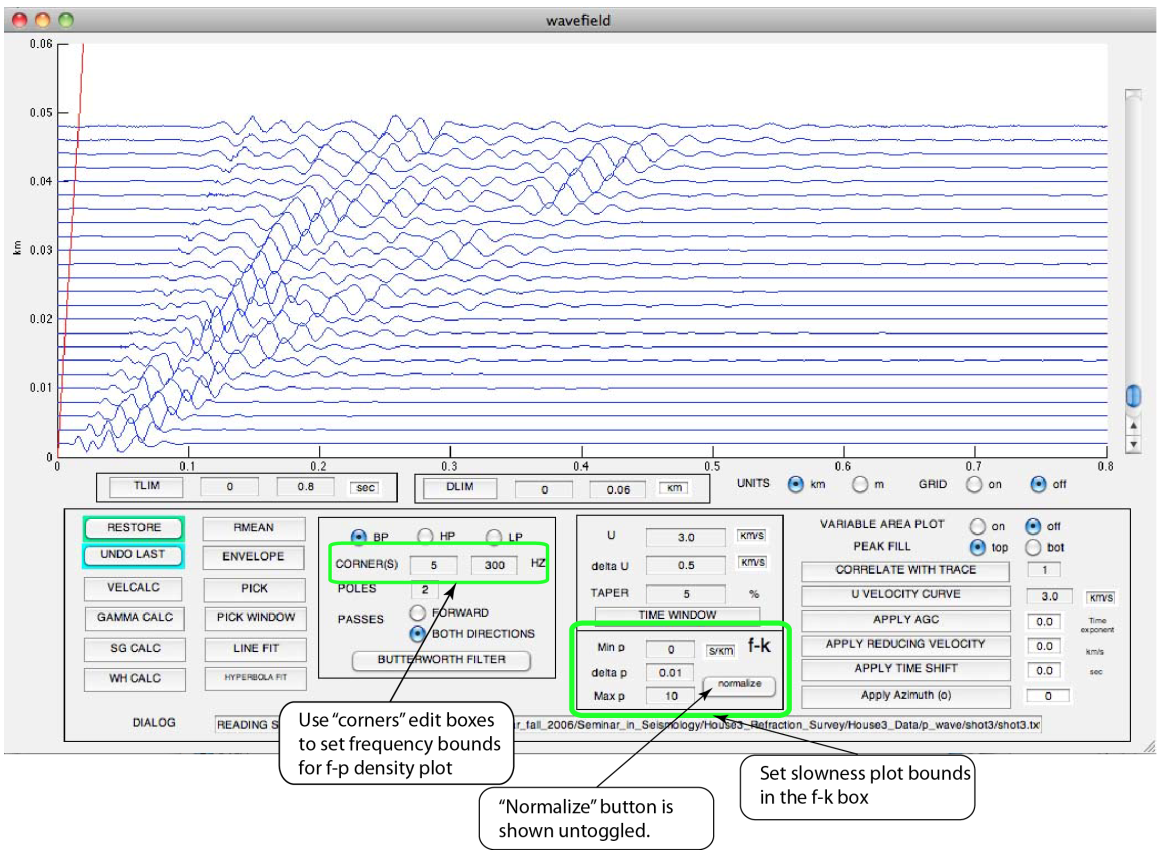

Figure 1 – Refraction data taken in the front yard of CERI in 2006. Shown are vertical component waveforms using the Matlab program "wavefield".

Here is the resulting 1D frequency-slowness spectra of the raw data without correcting for geometric spreading of the arrivals with distance (Figure 2).

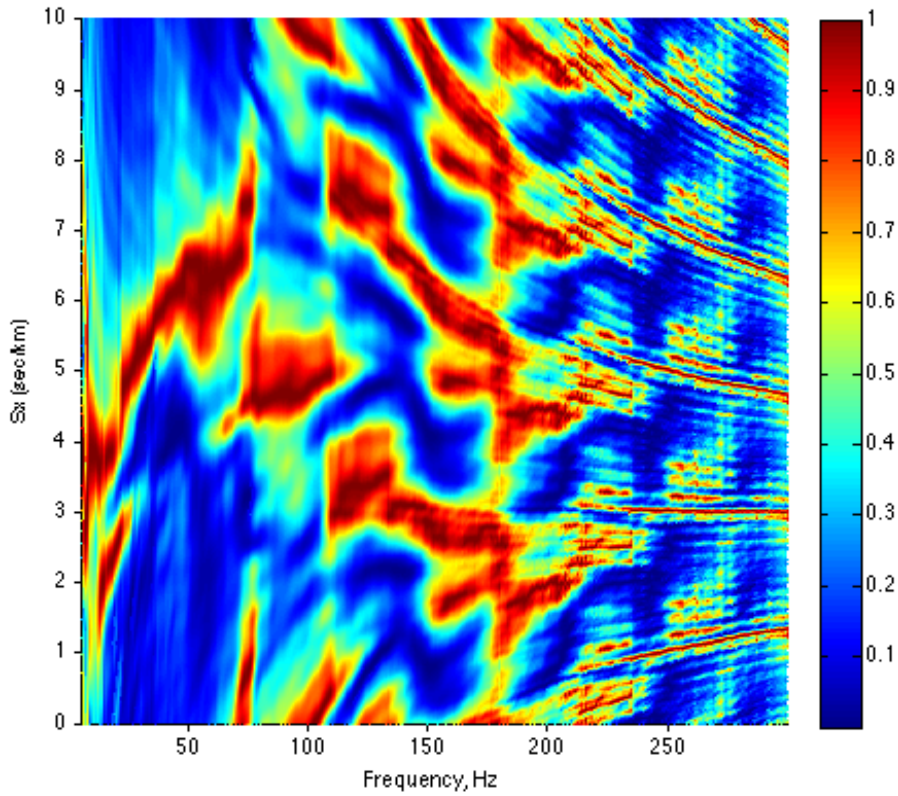

Figure 2 – 1D frequency-slowness spectra of the raw refraction data of Figure 1 without correcting for geometrical spreading. As a result, the spectra are dominated by large arrivals at relatively close distance to the source effectively making the array much shorter in aperture. The slowness spectra at each frequency are relatively wide except beyond 200 Hz where the source airwave dominates the power spectrum. The amplitude scale is for normalized power spectra at each frequency.

Because the data at greater distance has less amplitude from simple geometrical spreading, the frequency-slowness spectra are of relatively low resolution. This happens since data at greater range is weighted less in the summation in equation (3). This can be remedied by simply normalizing traces at each range to have the same maximum amplitude. After normalization, the frequency-slowness spectra appear more distinct (Figure 3). Note that the “plane wave” approximation was not applied to these data in that there are several kinds of waves in the data that have different dispersion characteristics.

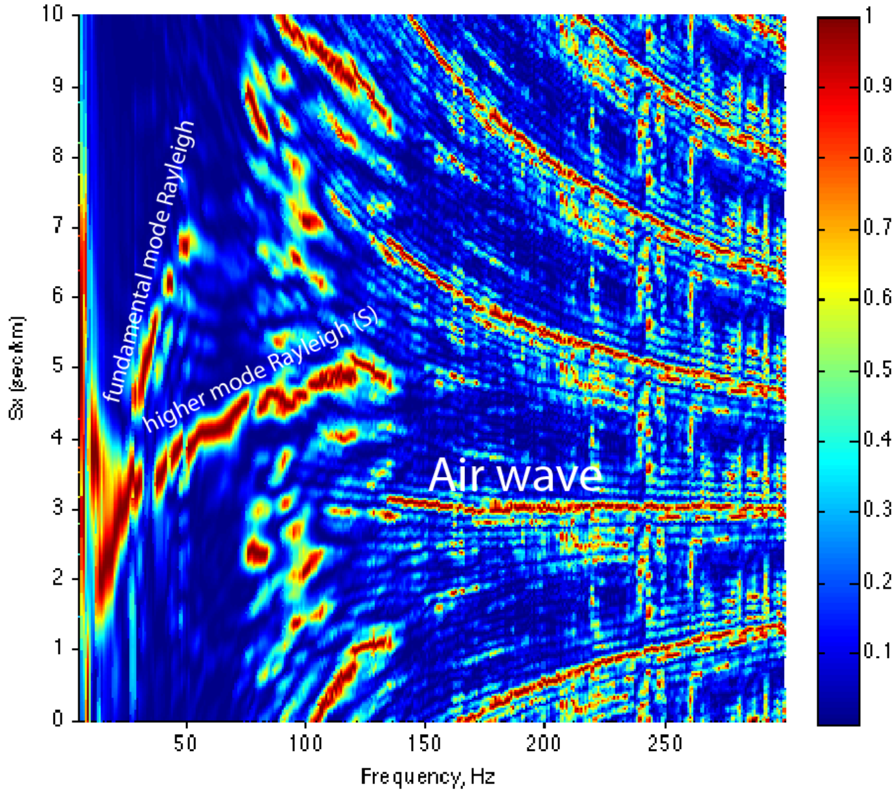

Figure 3 – 1D frequency-slowness spectra after normalizing each time domain waveform to effectively correct for geometrical spreading. Note the highly aliased Air wave spectra for frequencies greater than 125 Hz.

2D Case

2D frequency-wavenumber analysis works the same way as 1D except there is one more Fourier transform. Assume a two dimensional wavefield, $f(x, y, t)$. A two dimensional array of seismometers discretely samples the wavefield in space. Using the same kind of representation as (1), we write

$$f(x,y,t) = \sum_{j=1}^{N} f(x_j, y_j, t)\delta(x - x_j)\delta(y - y_j) \quad (8)$$

where we have used a 2D irregular "comb" function to isolate measurements at array element locations (x_j, y_j). Fourier transforming (8) in space and time gives

$$\hat{\hat{\hat{F}}}\left(k_x,k_y,\omega\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\left(x,y,t\right)e^{-i\omega t}e^{-ik_x x}e^{-ik_y y}\,dt\,dx\,dy\ . \quad (9)$$

Substituting (8) into (9) gives

$$\hat{\hat{\hat{F}}}\left(k_x,k_y,\omega\right)=\sum_{j=1}^{N}\hat{F}\left(x_j,y_j,\omega\right)e^{-ik_x x_j}e^{-ik_y y_j}\ . \quad (10)$$

Writing k_x and k_y in terms of the *x* and *y* horizontal slownesses:

$$\begin{aligned} k_x &= \omega p_x \\ k_y &= \omega p_y \end{aligned} \qquad (11)$$

gives

$$\hat{\hat{\hat{F}}}_p\left(p_x,p_y,\omega\right)=\sum_{j=1}^{N}\hat{F}\left(x_j,y_j,\omega\right)e^{-i\omega p_x x_j}e^{-i\omega p_y y_j}\ . \quad (12)$$

As in the 1D case the array response is given for an impulse wavefield ($F\left(x,y,t\right)=\delta\left(t\right)$) so that

$$\hat{\hat{\hat{I}}}_p\left(p_x,p_y,\omega\right)=\sum_{j=1}^{N}e^{-i\omega p_x x_j}e^{-i\omega p_y y_j}\ . \quad (13)$$

Likewise, a "broadband" array response can be obtained by averaging (integrating) the response over a frequency band as in (7) to get

$$\begin{aligned}\hat{\hat{\hat{I}}}_{BB}\left(p_x,p_y,\omega\right)&=\int_{\omega_1}^{\omega_2}\hat{\hat{\hat{I}}}_p\left(p_x,p_y,\omega\right)\\ &=\sum_{j=1}^{N}\frac{e^{-i\omega_2(p_x x_j+p_y y_j)}-e^{-i\omega_1(p_x x_j+p_y y_j)}}{-i(p_x x_j+p_y y_j)}\end{aligned}\ . \quad (14)$$

Figure 4 shows a theoretical calculation of the array response and co-array for a "Golay 3x6" array design. This is an unusual array built from 6 small tripartite arrays. The co-array diagram shows that the interstation distances and azimuths uniformly sample the *xy* plane, a surprising result considering the sparseness of the actual array design.

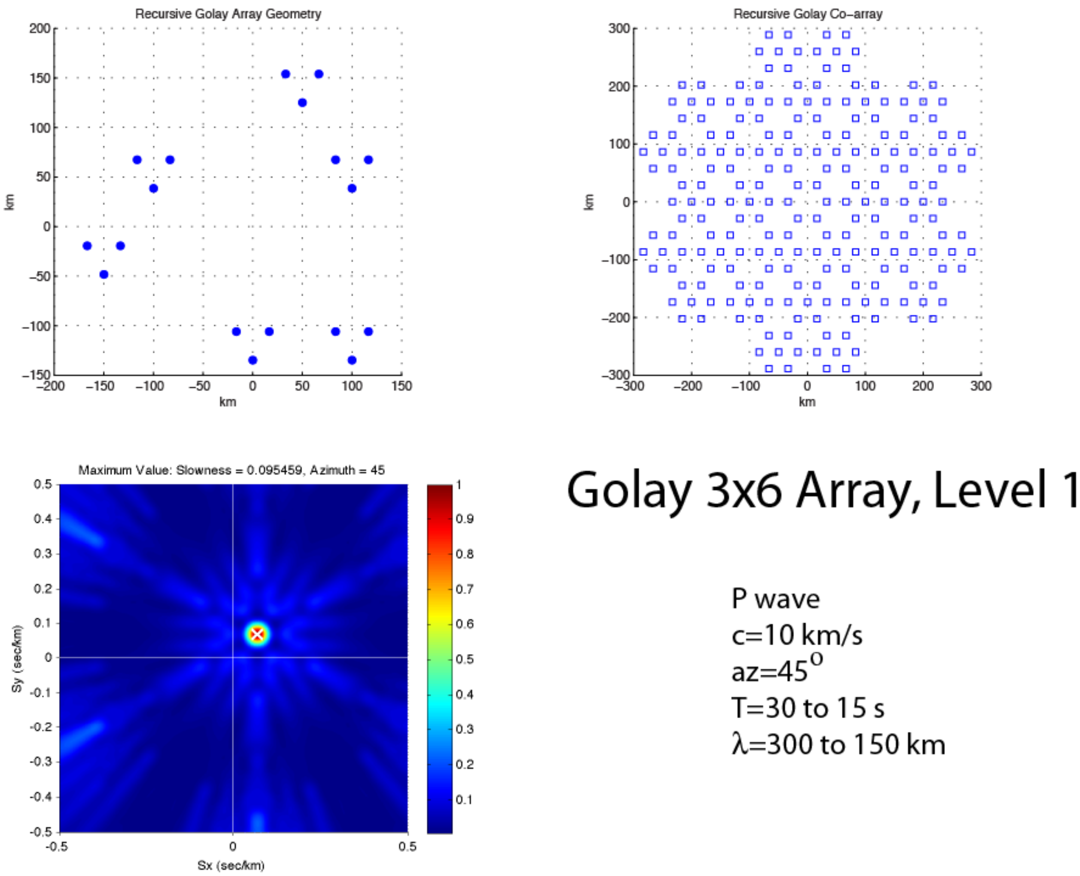

Figure 4 – Array geometry (*upper left*), co-array (*upper right*), and the broadband array response (*lower left*) for a bandpass between 15s to 30s period for a plane wave traveling from an azimuth of 45 degrees at a horizontal phase velocity of 10km/s. The co-array is a plot of all possible distances and azimuths between pairs of stations of the array.