The continuous wavelet transform is a transform with two independent variables, scale *a*, and time lag **, that produces a map of amplitude at scale vs time lag for the signal. The Morlet-Grossmann definition (Grossmann et al., 1989; Starck et al., 2010) for the CWT is



where the \* represents the complex conjugate of the function. The CWT is simply a correlation of the signal, *f(t)*, with a scaled basis function *.* In general, the basis function is complex and is



termed the “mother wavelet”. The wavelet coefficient, *W(a,)* is also complex and can be represented in the Fourier domain as

.



The CWT is a linear operation and has an exact inverse transform given by the double integral



where C is found from a Parseval-like integral



that requires the basis function to have zero mean for this integral to be bounded at . We use Morlet’s wavelet (Goupillaud, Grossmann, & Morlet, 1985) in which



.



Morlet’s wavelet is a heavily damped sinusoid around t=0 allowing the scale parameter to be interpreted as Fourier period (inverse frequency) in the CWT domain since we use. Other wavelets may be used and the techniques outlined here will work in a similar way. However, the Morlet wavelet is harmonic-like so that the continuous wavelet transform can be interpreted by relating scale to Fourier period.



“Hard Thresholding” is the non-linear process of keeping wavelet coefficients if they are greater than a threshold criterion, , otherwise they are set to zero (Donoho, Johnstone, Kerkyacharian, & Picard, 1995). Mathematically, this is represented by



Alternatively, thresholding can be done in a somewhat less severe manner by modifying the coefficients that survive by the inferred noise level. This is called soft thresholding (Weaver, Yansun, Healy, & Cromwell, 1991) and is given by



where

.



Soft thresholding minimizes outliers in the noise better than hard thresholding.

The threshold function, , is determined based on the statistics of the absolute value of the noise estimate and is determined for each wavelet scale, *a*. If the statistics are Gaussian, then the thresholding can be computed using the mean and standard deviation of the absolute value of the wavelet transform at each scale:



.



where



and *c* is a constant that controls the threshold. The limits *t2* and *t1* represent the limits of the noise time lag window.

There are a number of ways to estimate the threshold coefficient, *c*, contained in equation . If the wavelet coefficients follow a normal distribution, then setting *c* = 3 will yield a signal at the 99.7% confidence level (Starck et al., 2010). Donoho and Johnstone(1994) suggest a somewhat less stringent criterion called the “universal” threshold that is related to the number of noise samples, *N*, at each scale. The universal threshold is given as

.



Normally, *c* from this relation is close to the value 3 for practical problems.

Since the distribution for real data is often unpredictable, we take the approach of empirically estimating the cumulative distribution of noise in the time window at each scale then calculating the 99% confidence value for the distribution. The empirical cumulative distribution function (ECDF*a*) is determined by ordering the N noise values and then assigning a probability jump of 1/N when a value is attained, starting with the smallest value. The threshold function becomes:



 (13)

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